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THE PRIMITIVE COHOMOLOGY OF THE THETA DIVISOR OF AN ABELIAN FIVEFOLD

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Dedicated to Herb Clemens

Abstract

The primitive cohomology of the theta divisor of a principally polarized abelian variety of dimension g contains a Hodge structure of level $g-3$ which we call the primal cohomology. The Hodge conjecture predicts that this is contained in the image, under the Abel-Jacobi map, of the cohomology of a family of curves in the theta divisor. In this paper we use the Prym map to show that this version of the Hodge conjecture is true for the theta divisor of a general abelian fivefold.

Contents

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Introduction

Let A be a principally polarized abelian variety (*ppav*) of dimension $g \geq 4$, let Θ be a symmetric theta divisor in A, and assume that Θ is smooth. The cohomology group

$$H^{g-1}(\Theta, \mathbb{Z})$$

contains a natural sublattice of rank $g! - \frac{1}{g+1}\binom{2g}{g}$ (see [IvS, p. 561]),

$$\mathbb{K} := \mathrm{Ker}(H^{g-1}(\Theta, \mathbb{Z}) \xrightarrow{j_*} H^{g+1}(A, \mathbb{Z})),$$

which we call the primal cohomology of Θ. There is also a Hodge structure $\mathbb{H} \subset H^{g-1}(\Theta, \mathbb{Z})$ which fits in an exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{H} \longrightarrow H^{g-3}(A, \mathbb{Z}) \longrightarrow 0.$$

By [IvS, p. 562], these Hodge structures are all of level $g-3$. For a rational Hodge structure $V := (V_{\mathbb{Q}}, V_{\mathbb{Q}} \otimes \mathbb{C} = \bigoplus_{p+q=n} V^{p,q})$ of weight n, the level $l(V)$ of V is defined as the positive integer

$$l(V) := \max\{\, |p-q| \mid V^{p,q} \neq 0 \,\}.$$

Grothendieck's version of the Hodge conjecture states that if $H^{g-1}(\Theta, \mathbb{Q})$ contains a Hodge substructure of level $g-3$, then it is contained in the image, under Gysin push-forward, of the cohomology of a smooth (possibly reducible) variety of dimension $g-2$. After tensoring with $\mathbb{Q}$ we have

$$\mathbb{H}_{\mathbb{Q}} := \mathbb{H} \otimes \mathbb{Q} = \mathbb{K}_{\mathbb{Q}} \oplus \theta \cdot H^{g-3}(A, \mathbb{Q})$$

where $\theta := [\Theta]$ is the cohomology class of Θ and $\theta \cdot H^{g-3}(A, \mathbb{Q})$ is the image of $H^{g-3}(A, \mathbb{Q}) \cong H^{g-3}(\Theta, \mathbb{Q})$ in $H^{g-1}(\Theta, \mathbb{Q})$. The subspace $\theta \cdot H^{g-3}(A, \mathbb{Q})$ is also a Hodge substructure of level $g-3$ and satisfies the Hodge conjecture since it is in the image, for instance, of the cohomology of an intersection of a translate of Θ with Θ. Therefore the Hodge conjecture for $\mathbb{H}_{\mathbb{Q}}$ is equivalent to the Hodge conjecture for $\mathbb{K}_{\mathbb{Q}}$.

An equivalent formulation of the Hodge conjecture is that $\mathbb{H}_{\mathbb{Q}}$ or $\mathbb{K}_{\mathbb{Q}}$ is contained in the image, under the Abel-Jacobi map, of the cohomology of some family of curves in Θ (see e.g. [I, pp. 492-493] for a proof of this elementary fact). For $g = 4$, it was proved in [IvS] that the family of Prym-embedded curves in Θ is a solution to this problem for $\mathbb{H}_{\mathbb{Q}}$.

For $g = 5$ a general ppav is again a Prym variety. However, in this case, every component of the family of Prym-embedded curves in Θ parametrizes curves that are translates of a single curve. Therefore the image of the cohomology of any of these components is contained in $\theta \cdot H^{g-3}(A, \mathbb{Q})$. Hence the family of Prym-embedded curves in Θ cannot be a solution to the Hodge conjecture for the primal cohomology $\mathbb{K}_{\mathbb{Q}}$.

Denote by $\mathcal{A}_g$ the coarse moduli space of principally polarized abelian varieties of dimension g. Representing (A, Θ) as a Prym variety and using some interesting geometric constructions, we construct a different family of curves in Θ which is a solution to the Hodge conjecture for $\mathbb{H}_{\mathbb{Q}}$ for (A, Θ) general in $\mathcal{A}_5$.

Theorem 1. *For (A, Θ) in a nonempty Zariski open subset of $\mathcal{A}_5$, the general Hodge conjecture holds for the Hodge structure $\mathbb{H}_{\mathbb{Q}} \subset H^4(\Theta, \mathbb{Q})$ and hence $\mathbb{K}_{\mathbb{Q}} \subset H^4(\Theta, \mathbb{Q})$.*

As the rational cohomology of Θ is the sum of $\mathbb{K}_{\mathbb{Q}}$ and the rational cohomology of A, our result, together with the main result of [H], implies

Corollary 2. *For (A, Θ) in the complement of countably many proper Zariski closed subsets of $\mathcal{A}_5$, the general Hodge conjecture holds for Θ.*

Note that there are relatively few examples of lower level Hodge substructures of the cohomology of algebraic varieties that are not already contained in the images of the cohomologies of subvarieties for trivial reasons. Some of the most interesting such examples are provided by abelian varieties, such as abelian varieties of Weil type (see [I]) and the primal cohomology of theta divisors. In fact we are not aware of any nontrivial examples that do not involve abelian varieties in some way. As far as we are aware, the primal cohomology of the theta divisor of an abelian fivefold is the first nontrivial case of a proof of the Hodge conjecture for a family of fourfolds of general type. The proof was considerably more difficult than the case of the theta divisor of the abelian fourfold worked out in [IvS] and required a difficult degeneration argument and nontrivial and interesting geometric constructions. As is often the case with deep conjectures such as the Hodge conjecture, the level of difficulty goes up rapidly with the dimension of the varieties concerned or, perhaps more accurately, with their Kodaira dimension.

It would be interesting to know whether the Hodge structure $\mathbb{K}$ is irreducible. This would considerably simplify our computation of the Abel-Jacobi map as in that case we would only have to prove that its image intersects $\mathbb{K}$ nontrivially.

Letting $\mathcal{R}_6$ denote the moduli space of étale double covers of curves of genus 6, further note that the monodromy group of the Prym map $\mathcal{R}_6 \to \mathcal{A}_5$ is the Weyl group $W(E_6)$ of the exceptional Lie algebra E_6 (see [Do, Theorem 4.2]). Also, the lattice $\mathbb{K}$ has rank 78 for $g = 5$, which is equal to the dimension of E_6. So one might wonder whether it is possible to define a natural isomorphism between $\mathbb{K}_{\mathbb{C}} := \mathbb{K} \otimes \mathbb{C}$ and E_6.

We now explain the general outline of our proof.

A general ppav of dimension 5 is the Prym variety of an étale double cover of smooth curves $\widetilde{X} \to X$ with X general of genus 6.

Using the 5-gonal construction (see [ILS]), we construct a family of curves in Θ (see Section 1)

$$\begin{array}{ccc} F_r & \xrightarrow{\rho_2} & \Theta \\ \downarrow{\scriptstyle \rho_1} & & \\ \widetilde{G}^1_5 & & \end{array}$$

dependent on the choice of a general point $r \in \widetilde{X}$. Here $\widetilde{G}^1_5$ is an étale double cover of the variety $G^1_5(X)$ parametrizing pencils of degree 5 on X ($\cong W^1_5(X)$ if X is not a plane quintic), which is a smooth irreducible surface for X sufficiently general. The Abel-Jacobi map for this family of curves is, by definition,

$$\rho_{2*}\rho_1^* \colon H^2(\widetilde{G}^1_5) \to H^4(\Theta).$$

The image of the Abel-Jacobi map defines a Hodge substructure of level ≤ 2 of the cohomology of Θ. Theorem 1 is a direct consequence of

Theorem 3. *The Hodge structure $\mathbb{H}_{\mathbb{Q}}$ is the sum of $\theta \cdot H^2(A, \mathbb{Q})$ and the image of $\rho_{2*}\rho_1^*$.*

We prove Theorem 3 by specializing the étale double cover $\widetilde{X} \to X$ to a Wirtinger cover. To define a Wirtinger cover, choose two general points p and q on a general curve C of genus 5 and let C_{pq} be the nodal curve of genus 6 obtained from C by identifying p and q. The Wirtinger cover $\widetilde{C}_{pq}$ is obtained as the union of two copies C_1 and C_2 of C, with the copy of p on each curve identified with the copy of q on the other. The Prym variety of the Wirtinger cover $\widetilde{C}_{pq} \to C_{pq}$ is naturally isomorphic to the polarized Jacobian $(J(C), \Theta_C)$ of the curve C (see e.g. Section 2.4 below).

In most of the paper we work with a one-parameter family $\mathcal{X} \to T$ of curves of genus 6 over an analytic disc T with smooth total space, with general fiber X_t a general curve of genus 6 and special fiber $X_0 = C_{pq}$ at $0 \in T$ a general one-nodal curve of genus 6. We also assume given an étale double cover $\widetilde{\mathcal{X}} \to \mathcal{X}$ whose special fiber $(\widetilde{X}_0 \to X_0) = (\widetilde{C}_{pq} \to C_{pq})$ is the Wirtinger cover described above. To this family one associates the family of polarized Prym varieties $(\mathcal{A}, \Theta) \to T$ with special fiber (A_0, Θ_0).

The plan of the paper is as follows.

In Section 1 we construct the family of curves F_r in the general case. In Section 2 we describe the family of curves in the Wirtinger double cover case. We also explicitly describe the flat limit G_0 of the base $G_t := \widetilde{G}^1_5(X_t)$ of the family. This is the transverse union of two smooth isomorphic surfaces. We prove that the total space $\mathcal{G} \to T$ of the family of the G_t is smooth.

In Section 3 we describe the total space of the family of theta divisors $\Theta \to T$. The singular locus of Θ_0 is a translate of the smooth genus 11 curve $W^1_4 \subset \operatorname{Pic}^4 C \cong JC$ parametrizing pencils of degree 4. We prove that the total space Θ has ten ordinary double points corresponding to the five $g_i \in W^1_4$, $i = 1, \ldots, 5$, such that $h^0(g_i - p - q) > 0$, and their residuals $h_i := |K_C - g_i|$.

In Section 4 we construct a semistable reduction $\widetilde{\Theta}$ of the family $\{ \Theta_t \}$. The central fiber $\widetilde{\Theta}_0$ of the new family has two components M_1 and M_2, where M_1 is a resolution of Θ_0 and M_2 is the exceptional divisor. During this process T is replaced by a double cover ramified only at 0, and we also replace the family $\mathcal{G}$ by $\widetilde{\mathcal{G}}$, which is a resolution of the base change of $\mathcal{G}$ to this double cover.

In Section 5 we recall the necessary background material about the Clemens-Schmid exact sequence and limit mixed Hodge structures.

In Section 6 we compute the limit mixed Hodge structure induced by the family $\widetilde{\Theta}$ on the cohomology of Θ_t. The weight filtration is nonzero only in weights 3, 4, 5 with associated graded pieces as follows:

$$\operatorname{Gr}_3 H^4(\Theta_t) \cong \operatorname{Gr}_5 H^4(\Theta_t) \cong \mathbb{Q}^{12}$$

and

$$\operatorname{Gr}_4 H^4(\Theta_t) \cong \mathbb{Q}^{264}.$$

To extend the family of curves to the central fiber we first assume we are given a section $r\colon T \to \widetilde{\mathcal{X}}, t \mapsto r_t$ of the family of curves $\widetilde{\mathcal{X}}$. Next we replace the families F_{r_t} by their images in the products $G_t \times \Theta_t$. The Abel-Jacobi map on the fiber at t can then be described as the map induced by the cycle $(\rho_1, \rho_2)_*[F_{r_t}] \in H^6(G_t \times \Theta_t)$:

$$H^2(G_t) \xrightarrow{\rho_1^*} H^2(G_t \times \Theta_t) \xrightarrow{\cup(\rho_1,\rho_2)_*[F_{r_t}]} H^8(G_t \times \Theta_t) \xrightarrow{\rho_{2*}} H^4(\Theta_t).$$

To compute the limit of these maps at 0, we need a semistable reduction of the fiber product $\widetilde{\mathcal{G}} \times_T \widetilde{\Theta}$. This is constructed in Section 7. The resulting space $\mathcal{P}$ is a small resolution of the fiber product $\widetilde{\mathcal{G}} \times_T \widetilde{\Theta}$.

In Section 8 we show how the computation of the Abel-Jacobi map on the general fiber can be reduced to computing it on (the strata of) the special fiber. We summarize the latter computations in Propositions 8.1-8.4 and show how Theorem 3 follows from them.

Sections 9 and 10 describe the limit families of curves at $t = 0$.

In Section 11 we prove Propositions 8.1-8.4. In other words, we compute the image of the Abel-Jacobi map AJ on the graded level with respect to the weight filtration:

$$\mathrm{Gr}_2\, H^2(\widetilde{\mathcal{G}}) \to \mathrm{Gr}_4\, H^4(\widetilde{\Theta}) \tag{0.1}$$

and

$$\mathrm{Gr}_1\, H^2(\widetilde{\mathcal{G}}) \to \mathrm{Gr}_3\, H^4(\widetilde{\Theta}). \tag{0.2}$$

Finally, in the Appendix (Section 12) we gather some technical results needed in the rest of the paper.

Remark 4. (1) For $g \leq 2$, $g! - \frac{1}{g+1}\binom{2g}{g} = 0$ so $\mathbb{K} = 0$. For $g = 3$, the lattice $\mathbb{K}$ has rank 1 and level 0; i.e., it is generated by a Hodge class. The abelian variety $(A, \Theta) = (JC, \Theta_C)$ is the Jacobian of a curve of genus 3. The theta divisor is isomorphic to the second symmetric power $C^{(2)}$ of C, and $\mathbb{K}$ is generated by the class $\theta - 2\eta$ where η is the cohomology class of the image of C in $C^{(2)}$ via addition of a point p of C:

$$\begin{array}{ccc} C & \hookrightarrow & C^{(2)} \\ t & \mapsto & t + p. \end{array}$$

(2) The primitive cohomology, in the sense of Lefschetz, is the subspace

$$H^4_{pr}(\Theta, \mathbb{Q}) := \mathrm{Ker}\left(H^4(\Theta, \mathbb{Q}) \xrightarrow{\cup\theta|_\Theta} H^6(\Theta, \mathbb{Q})\right).$$

The relation between the primitive and the primal cohomology is

$$H^4_{pr}(\Theta, \mathbb{Q}) = \mathbb{K}_{\mathbb{Q}} \oplus j^* H^4_{pr}(A, \mathbb{Q}),$$

where

$$H^4_{pr}(A, \mathbb{Q}) := \mathrm{Ker}\left(H^4(A, \mathbb{Q}) \xrightarrow{\cup\theta^2} H^8(A, \mathbb{Q})\right).$$

Note that in the case of hypersurfaces in projective space the primal and primitive cohomology coincide.

Notation and conventions

(1) Unless otherwise specified, all singular cohomology groups are with $\mathbb{Q}$-coefficients.

(2) For a smooth curve C of genus g and integer $k > 0$, we choose a symplectic basis

$$\xi_i \in H^1(C,\mathbb{Z}) \cong H^1(\operatorname{Pic}^k C,\mathbb{Z}), \quad i = 1,\dots,2g.$$

We put $\xi_i' := \xi_{i+g}$, $\sigma_i = \xi_i\xi_i'$ for $i = 1,\dots,g$ and denote by $\theta = \sum_{i=1}^g \sigma_i$ the class of the theta divisor in $\operatorname{Pic}^k C$. We also denote by ξ_i, σ_i and θ the pull-backs to the k-th symmetric power $C^{(k)}$ under the natural map

$$C^{(k)} \to \operatorname{Pic}^k C.$$

Finally, we denote by $\eta \in H^2(C^{(k)},\mathbb{Z})$ the class of the cycle $p + C^{(k-1)} \subset C^{(k)}$ for some $p \in C$.

(3) We will interchangeably refer to elements of $\operatorname{Pic}^k C$ as invertible sheaves or complete linear systems. We use $\equiv$ to denote linear equivalence between divisors, and $D_1 \leq D_2$ means $D_2 - D_1$ is an effective divisor. As usual, we denote by $W_d^r \subset \operatorname{Pic}^d$ the scheme parametrizing complete linear systems of degree d and dimension r. By a g_d^r we will mean a linear system of degree d and dimension r.

(4) For products of symmetric powers of C, we denote $\omega_k := pr_k^*\omega \in H^\bullet(C^{(n_1)} \times \cdots \times C^{(n_k)} \times \dots)$, where $\omega \in H^\bullet(C^{(n_k)})$ and pr_k is the k-th projection.

(5) Via translation by an invertible sheaf of degree $g-1$, we identify $JC = \operatorname{Pic}^0 C$ with $\operatorname{Pic}^{g-1} C$ so that Θ_C is identified with Riemann's theta divisor $W_{g-1}^0 \subset \operatorname{Pic}^{g-1} C$.

(6) As usual, ω_C will denote the dualizing sheaf of C and K_C an arbitrary canonical divisor on C.

1. The family of curves in Θ: The general case

Let X be a smooth curve of genus 6 with an étale double cover $\widetilde{X}$ of genus 11. For a pencil M of degree 5 on X consider the curve B_M defined by the pull-back diagram

$$\begin{array}{ccc} B_M & \subset & \widetilde{X}^{(5)} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 = |M| & \subset & X^{(5)}. \end{array}$$

By [B2, p. 360] the curve B_M has two isomorphic connected components, say B_M^1 and B_M^2. Put $M' = |K_X - M|$. Then for any $D \in B_M \subset \widetilde{X}^{(5)}$ and any $D' \in B_{M'} \subset \widetilde{X}^{(5)}$, the push-forward to X of $D + D'$ is a canonical divisor on X. Hence the image of

$$\begin{array}{ccc} B_M \times B_{M'} & \longrightarrow & \operatorname{Pic}^{10} \widetilde{X} \\ (D, D') & \longmapsto & \mathcal{O}_{\widetilde{X}}(D + D') \end{array}$$

is contained in the preimage of ω_X by the Norm map $\mathrm{Nm}\colon \operatorname{Pic}^{10} \widetilde{X} \to \operatorname{Pic}^{10} X$. This preimage has two connected components, say A_1 and A_2, each a principal homogeneous space under the Prym variety (A, Θ) of the cover $\widetilde{X} \to X$ and parametrizing divisors whose spaces of global sections are even, respectively odd, dimensional. If we have labeled the connected components of B_M and $B_{M'}$ in such a way that $B^1_M \times B^1_{M'}$ maps into A_1, then $B^2_M \times B^2_{M'}$ also maps into A_1, while $B^1_M \times B^2_{M'}$ and $B^2_M \times B^1_{M'}$ map into A_2.

In order to obtain a family of curves in the theta divisor $\Theta = \Theta_{\widetilde{X} \to X} = \frac{1}{2}\Theta_{\widetilde{X}}|_{A_1}$ of the Prym variety A, we globalize the above construction.

The scheme $W^1_5(X)$ parametrizing complete linear systems of degree 5 and dimension at least 1 on X has a determinantal structure which is smooth for X sufficiently general. Let $G^1_5(X)$ denote the scheme over $W^1_5(X)$ parametrizing pencils of degree 5. Note that $W^1_5(X)$ is a surface unless X is hyperelliptic.

The universal family P^1_5 of divisors of the elements of G^1_5 is a $\mathbb{P}^1$ bundle over G^1_5 with natural maps

$$\begin{array}{ccc} P^1_5 & \longrightarrow & X^{(5)} \\ \downarrow & & \\ G^1_5 & & \\ \downarrow & & \\ W^1_5 & & \end{array}$$

whose pull-back via $\widetilde{X} \to X$ gives us the family of the curves B_M as M varies:

$$\begin{array}{ccc} B & \longrightarrow & \widetilde{X}^{(5)} \\ \downarrow & & \downarrow \\ P^1_5 & \longrightarrow & X^{(5)} \\ \downarrow & & \\ G^1_5. & & \end{array}$$

The parameter space of the connected components of the curves B_M is an étale double cover $\widetilde{G}^1_5$ of G^1_5.

The family of curves in the theta divisor of the Prym variety will be constructed as follows. Assuming that X is not a plane quintic, the natural map $G^1_5 \to W^1_5$ is an isomorphism. We have the involution $\iota\colon M \mapsto M' := |K_X - M|$ on W^1_5 and hence also on G^1_5. First define a family of surfaces ${}'F$

over G^1_5 as the fiber product

$$\begin{array}{ccc} {}'F & \longrightarrow & B \\ \downarrow & & \downarrow^{\iota\varrho} \\ B & \xrightarrow{\varrho} & G^1_5. \end{array}$$

As noted above, the image of ${}'F$ in $\mathrm{Pic}^{10}\,\widetilde{X}$ maps into $\mathrm{Nm}^{-1}(\omega_X) \subset \mathrm{Pic}^{10}\,\widetilde{X}$, which also shows that ${}'F$ has two connected components. One component, denoted ${}'F_1$, maps into A_1 and the other, denoted ${}'F_2$, maps into A_2. The fiber of ${}'F_1$ over a point $|M| \in G^1_5$ has two connected components, $B^1_M \times B^1_{M'}$ and $B^2_M \times B^2_{M'}$.

Therefore, if we make the base change

$$\begin{array}{ccc} {}''F_1 & \longrightarrow & {}'F_1 \\ \downarrow & & \downarrow \\ \widetilde{G}^1_5 & \longrightarrow & G^1_5, \end{array}$$

${}''F_1$ splits into two connected components (both isomorphic to ${}'F_1$ over $\mathbb{C}$, but their maps to $\widetilde{G}^1_5$ differ by the involution of $\widetilde{G}^1_5$). We denote by F the component which has fiber $B^1_M \times B^1_{M'}$ over the point parametrizing B^1_M.

Finally, we think of F as a correspondence

$$F \hookrightarrow \widetilde{G}^1_5 \times \widetilde{X}^{(5)} \times \widetilde{X}^{(5)}$$

and define our family of curves F_r by intersecting F with the pull-back of the divisor $r + \widetilde{X}^{(4)}$ in the first factor $\widetilde{X}^{(5)}$ for a general point $r \in \widetilde{X}$. The variety F'_r is the image of F_r in $\widetilde{G}^1_5 \times \Theta$:

$$\begin{array}{ccc} F_r & \xrightarrow{(\rho_1,\rho_2)} & F'_r \subset \widetilde{G}^1_5 \times \Theta \\ \downarrow{\scriptstyle \rho_1} & & \\ \widetilde{G}^1_5. & & \end{array}$$

Remark 1.1. It is easy to check that F_r maps generically injectively to $\widetilde{G}^1_5 \times \Theta$. So the push-forward of the cycle class $[F_r]$ is the cycle class $[F'_r]$.

2. The family of curves in Θ: The degeneration to a Wirtinger cover

Let $\widetilde{\mathcal{X}} \to \mathcal{X}$ be the family of étale double covers over T specializing to the Wirtinger cover $\widetilde{C}_{pq} \to C_{pq}$ at $0 \in T$ as explained in the introduction.

Also assume that $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are smooth.

Consider the smooth one-parameter family

$$\begin{array}{ccc} J^5 C_{pq} & \longrightarrow & \mathcal{J}^5 \\ \downarrow & & \downarrow \\ 0 & \in & T \end{array}$$

obtained as a compactification of the relative degree 5 Picard scheme of $\mathcal{X}$. The fiber of $\mathcal{J}^5 \to T$ is $\operatorname{Pic}^5 X_t$ for $t \neq 0$, and the fiber at $t = 0$ is the usual compactification $J^5 C_{pq}$ of $\operatorname{Pic}^5 C_{pq}$ obtained as follows.

2.1. The compactified Jacobian of C_{pq}. Let $\mathbb{P}\operatorname{Pic}^5 C_{pq}$ be the unique projective line bundle over $\operatorname{Pic}^5 C$ containing the $\mathbb{G}_m$-bundle $\operatorname{Pic}^5 C_{pq} \to \operatorname{Pic}^5 C$. Then $\mathbb{P}\operatorname{Pic}^5 C_{pq} \setminus \operatorname{Pic}^5 C_{pq}$ is the union of the zero section $\operatorname{Pic}^5_0 \cong \operatorname{Pic}^5 C$ and the infinity section $\operatorname{Pic}^5_\infty \cong \operatorname{Pic}^5 C$ of $\mathbb{P}\operatorname{Pic}^5 C_{pq} \to \operatorname{Pic}^5 C$. The compactification $J^5 C_{pq}$ is obtained from $\mathbb{P}\operatorname{Pic}^5 C_{pq}$ by identifying $x \in \operatorname{Pic}^5 C = \operatorname{Pic}^5_0$ with $x \otimes \mathcal{O}_C(p-q) \in \operatorname{Pic}^5 C = \operatorname{Pic}^5_\infty$. The points of $J^5 C_{pq} \setminus \operatorname{Pic}^5 C_{pq}$ are the push-forwards $\nu_* N$ where $\nu \colon C \to C_{pq}$ is the normalization map and $N \in \operatorname{Pic}^4 C$.

2.2. The support of $W^1_5(C_{pq})$ and of its compactification $\overline{W}^1_5(C_{pq})$. Let $\overline{W}^1_5(C_{pq})$ be the subvariety of $J^5 C_{pq}$ parametrizing torsion-free rank 1 sheaves M of degree 5 such that $h^0(M) \geq 2$. Let $W_{pq} \subset W^1_5(C) \subset \operatorname{Pic}^5 C$ be the surface consisting of those L such that $h^0(L - p - q) > 0$, and let X_p and X_q be the two curves $p + W^1_4(C)$ and $q + W^1_4(C)$ in W_{pq}. Pull-back via the normalization map gives a morphism

$$\nu^* \colon W^1_5(C_{pq}) \longrightarrow W_{pq}$$

whose image is $W_{pq} \setminus X_p \cup X_q$. We have

Lemma 2.1. *The morphism $\nu^* \colon W^1_5(C_{pq}) \to W_{pq}$ is injective. Its inverse extends to a birational morphism*

$$(\nu^*)^{-1} \colon W_{pq} \twoheadrightarrow \overline{W}^1_5(C_{pq})$$

that is bijective on $W_{pq} \setminus X_p \cup X_q$ and sends $p + g^1_4$ and $q + g^1_4$ to $\nu_* g^1_4$.

The involution ι extends to W_{pq} and sends L to $|K_C + p + q - L|$. It also descends to $\overline{W}^1_5(C_{pq})$ and sends $\nu_* g^1_4$ to $\nu_*(|K_C - g^1_4|)$.

Proof. If $M \in \overline{W}^1_5(C_{pq})$ is invertible, then the pull-back $\nu^* M$ is an invertible sheaf of degree 5 on C, and we have the usual exact sequence

$$0 \longrightarrow M \longrightarrow \nu_* \nu^* M \longrightarrow sk \longrightarrow 0$$

where sk is a skyscraper sheaf of length 1 supported at the singular point of C_{pq}. It follows that if $h^0(M) \geq 2$, then $h^0(\nu^* M) \geq 2$ also. Since C is a general curve of genus 5 and $\nu^* M$ has degree 5, we have $h^0(\nu^* M) \leq 2$. So the map $H^0(\nu^* M) = H^0(\nu_* \nu^* M) \to H^0(sk)$ obtained from the above sequence is zero. Since this map factors through the evaluation map $H^0(\nu^*M) \to (\nu^*M)_p \oplus (\nu^*M)_q$, a moment of reflection will show that a map $\nu_*\nu^*M \twoheadrightarrow sk$ that is zero on global sections and has locally free kernel exists if and only if neither p nor q is a base point of $|\nu^*M|$ and the unique nonzero section of ν^*M that vanishes at p also vanishes at q.

Conversely, given an invertible sheaf L of degree 5 on C such that neither p nor q is a base point of $|L|$ and the unique nonzero section of L vanishing at p also vanishes at q, one sees immediately that there is a unique quotient map

$$\nu_*L \twoheadrightarrow sk$$

onto a skyscraper sheaf of rank 1 supported at the singular point of C_{pq} such that the resulting map on global sections

$$H^0(\nu_*L) \longrightarrow H^0(sk)$$

is zero. The kernel of such a map is also immediately seen to be an invertible sheaf of degree 5 on C_{pq}. Thus $W^1_5(C_{pq})$ maps injectively into W_{pq} under ν^*.

If M is not locally free, then it is the direct image of a $g^1_4 \in W^1_4(C)$. We have two exact sequences,

$$\begin{aligned} 0 \longrightarrow \nu_*g^1_4 \longrightarrow \nu_*(p+g^1_4) \longrightarrow sk \longrightarrow 0,\\ 0 \longrightarrow \nu_*g^1_4 \longrightarrow \nu_*(q+g^1_4) \longrightarrow sk \longrightarrow 0, \end{aligned}$$

that give us two representations of M as the kernel of a surjective map from the push-forward of an invertible sheaf to sk. Thus ν^* maps $p+g^1_4$ and $q+g^1_4$ to $\nu_*g^1_4$. The statements about ι are immediate. □

Note that $W_{pq} \subset \operatorname{Pic}^5 C$ naturally embeds in $C^{(3)}$ via two different maps: $q_1\colon L \mapsto \Gamma_3 := |K_C - L|$ and $q_2\colon L \mapsto \Gamma'_3 := |L-p-q|$. We have

Proposition 2.2. *The surface W_{pq} is smooth for C, p and q general.*

Proof. For $L \in W_{pq}$, via the two embeddings of W_{pq} in $C^{(3)}$, the tangent space to W_{pq} at L is contained in the tangent spaces to $C^{(3)}$ at Γ_3 and Γ'_3. Embedding $C^{(3)}$ in $\operatorname{Pic}^0 C$ via subtraction of a fixed divisor of degree 3, the projectivizations of these two tangent spaces can be identified (after a translation) with the respective spans $\langle \Gamma_3 \rangle$ and $\langle \Gamma'_3 \rangle$ of Γ_3 and Γ'_3 in the canonical space $|K_C|^* \cong \mathbb{P}T_0 \operatorname{Pic}^0 C$. To prove W_{pq} is smooth at L, i.e., $T_L W_{pq}$ has dimension 2, it suffices to show that $\langle \Gamma_3 \rangle \neq \langle \Gamma'_3 \rangle$, since the intersection $\langle \Gamma_3 \rangle \cap \langle \Gamma'_3 \rangle$ is then a projective line which contains $\mathbb{P}T_L W_{pq}$.

Using Riemann Roch and Serre Duality it is immediately seen that a divisor of degree ≥ 5 on C cannot span a space of dimension ≤ 2 in $|K_C|^*$. So, if $\langle \Gamma_3 \rangle = \langle \Gamma'_3 \rangle$, then Γ_3 and Γ'_3 have a divisor of degree at least 2 in common: $\Gamma_3 = \Gamma_2 + t$ and $\Gamma'_3 = \Gamma_2 + t'$ for some $\Gamma_2 \in C^{(2)}$ and $t, t' \in C$. Note that by our assumptions $\Gamma_3 + \Gamma'_3 + p + q \in |K_C|$ is a canonical divisor.

If $t = t'$, then the span $\langle \Gamma_3 + \Gamma_3' + p + q \rangle$ is a hyperplane in $|K_C|^*$ which is tangent to the canonical image of C at three distinct points or has even higher tangency to the canonical curve. Such hyperplanes form a family of dimension 1 for C general; hence choosing p and q sufficiently general, this can be avoided.

If $t \neq t'$, then the span $\langle \Gamma_2 + t + t' \rangle$ is a plane, and by Riemann Roch and Serre Duality, $|\Gamma_2 + t + t'| \in W^1_4(C)$, hence $|K_C - \Gamma_2 - t - t'| \in W^1_4(C)$. However $|K_C - \Gamma_2 - t - t'| = |p + q + \Gamma_2|$. The divisors of $g^1_4 := |\Gamma_2 + t + t'|$ and $h^1_4 := |K_C - \Gamma_2 - t - t'| = |p + q + \Gamma_2|$ are cut on C by the two rulings of a quadric of rank 4 (since C is general, it is not contained in any quadrics of rank 3). Since Γ_2 appears in both g^1_4 and h^1_4, the line $\langle \Gamma_2 \rangle$ contains the singular point of this quadric of rank 4. There is a one-parameter family of such secants to C, and for each such secant $\langle \Gamma_2 \rangle$, there are exactly 5 (counted with multiplicities) divisors $p + q$ such that $h^0(\Gamma_2 + p + q) \geq 2$. Therefore there is a one-parameter family of divisors $p + q$ such that $h^0(\Gamma_2 + p + q) \geq 2$ for some Γ_2 such that $\langle \Gamma_2 \rangle$ contains the singular point of some quadric of rank 4 containing C. Taking $p + q$ outside this one-parameter family this case is also eliminated. □

The computation of the Hilbert polynomial of $\overline{W}^1_5(C_{pq})$ in Lemma 12.5 shows that $\overline{W}^1_5(C_{pq})$, with its reduced scheme structure, is the flat limit of $W^1_5(X_t)$. Therefore, if $\mathcal{W}^1_5 \subset \mathcal{J}^5$ is the family of sheaves with at least two independent global sections on each fiber X_t, then $\mathcal{W}^1_5 \to T$ is flat. Moreover, we have

Proposition 2.3. *The total space $\mathcal{W}^1_5$ is smooth.*

Proof. This is clear at any invertible sheaf $M \in W^1_5(X_t)$ or $M \in W^1_5(C_{pq})$. Suppose therefore that $M \in \overline{W}^1_5(C_{pq}) \setminus W^1_5(C_{pq})$. We need to prove that the Zariski tangent space to $\mathcal{W}^1_5$ at M is 3-dimensional; i.e., it is equal to the Zariski tangent space to $\overline{W}^1_5(C_{pq})$ at M. By Lemma 12.5, the morphism $\mathcal{W}^1_5 \to T$ is flat and its scheme-theoretical fiber at 0 is $\overline{W}^1_5(C_{pq})$ with its reduced structure. So the tangent space to $\overline{W}^1_5(C_{pq})$ at M is the kernel of the differential of the map $\mathcal{W}^1_5 \to T$ at M, and it is equal to the tangent space to $\mathcal{W}^1_5$ if and only if this differential is zero. Now the fact that this differential is zero follows from the fact that the differential of the map $\mathcal{J}^5 \to T$ is zero at M because $\mathcal{J}^5$ is smooth and $\mathcal{J}^5 \to T$ is flat. □

2.3. The surface $\widetilde{G}^1_5$ in the Wirtinger double cover case. Let P^1_5 be the universal $\mathbb{P}^1$ bundle over $\overline{W}^1_5(C_{pq})$ whose fiber over $M \in \overline{W}^1_5(C_{pq})$ is $\mathbb{P}H^0(C_{pq}, M)$. The following fibered diagram is the limit of the analogous diagram in the smooth case:

(2.1)
$$\begin{array}{ccc} B & \longrightarrow & \widetilde{C}_{pq}^{(5)} \\ \downarrow & & \downarrow \\ P_5^1 & \longrightarrow & C_{pq}^{(5)} \\ \downarrow & & \\ \overline{W}_5^1(C_{pq}). & & \end{array}$$

The horizontal map in the second row is as follows. If $M \in W_5^1(C_{pq})$ and $0 \neq s \in H^0(C_{pq}, M)$, then the image of s in $C_{pq}^{(5)}$ is $\mathrm{div}(s) = \nu_*(\mathrm{div}(\nu^* s))$. If $M \in \overline{W}_5^1(C_{pq}) \setminus W_5^1(C_{pq})$, then $M = \nu_* g_4^1$ and the image of $s \in H^0(C_{pq}, M) = H^0(C, g_4^1)$ is $\nu_*(\mathrm{div}(s) + p) = \nu_*(\mathrm{div}(s) + q) \in C_{pq}^{(5)}$.

Lemma 2.4. *The surface $\widetilde{G}_5^1(C_{pq})$ is the union of two copies of W_{pq}, denoted W_1 and W_2, where $X_{kp} = W_4^1(C)+p \subset W_k$ is identified with $X_{3-k,q} = W_4^1(C) + q \subset W_{3-k}$ for $k = 1, 2$.*

Proof. First note that $\widetilde{C}_{pq}^{(5)}$ has the following irreducible components:

$$C_1^{(5)} \cup (C_1^{(4)} \times C_2) \cup (C_1^{(3)} \times C_2^{(2)}) \cup (C_1^{(2)} \times C_2^{(3)}) \cup (C_1 \times C_2^{(4)}) \cup C_2^{(5)}.$$

Accordingly, for a given $M \in W_5^1(C_{pq})$, the two connected components B_M^1 and B_M^2 of the curve B_M embed, respectively, into

$$(C_1^{(4)} \times C_2) \cup (C_1^{(2)} \times C_2^{(3)}) \cup C_2^{(5)}$$

and

$$C_1^{(5)} \cup (C_1^{(3)} \times C_2^{(2)}) \cup (C_1 \times C_2^{(4)}).$$

This first shows that $\widetilde{G}_5^1(C_{pq})$ has two irreducible components and that the double cover $\widetilde{G}_5^1(C_{pq}) \to G_5^1(C_{pq})$ is split away from $\nu_* W_4^1(C)$. The claim of the lemma over $\nu_* W_4^1(C)$ follows from the fact that, for a fixed M, the components B_M^1 and B_M^2 are exchanged by the involution induced by that exchanging of C_1 and C_2 in $\widetilde{C}_{pq}$. □

An immediate consequence of Lemma 2.4 is

Corollary 2.5. *For a general double cover $\widetilde{X} \to X$, the double cover $\widetilde{G}_5^1 \to G_5^1$ is nontrivial.*

2.4. The pair (A_0, Θ_0). Denote by $(\mathcal{A}, \Theta) \to T$ the family of principally polarized Prym varieties of the above family of double covers of curves.

Following Beauville [B1, pp. 175-176], the Prym variety A_0 associated to the Wirtinger cover is given by the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & A_0 & \xrightarrow{\cong} & J(C) & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & J(\widetilde{C}_{pq}) & \xrightarrow{\nu^*} & J(\widetilde{C}) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \\
1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & J(C_{pq}) & \longrightarrow & J(C) & \longrightarrow & 0
\end{array}
$$

where $J(\widetilde{C}_{pq})$ and $J(C_{pq})$ are the generalized (noncompact) Jacobians, Nm is the Norm map, and $\nu\colon \widetilde{C} = C_1 \amalg C_2 \to \widetilde{C}_{pq}$ is the normalization map.

To obtain a canonical theta divisor in A_0, we fix a bidegree (d_1, d_2) such that $d_1 + d_2 = 10$ and the following holds.

- $\diamond$ There exists a line bundle N on $\widetilde{C}_{pq}$ of multidegree (d_1, d_2), so that $h^0(N) = 0$.

By [B1, p. 153], the only bidegrees satisfying $\diamond$ are $(6,4)$, $(4,6)$ and $(5,5)$. If there is a one-parameter family of line bundles $\mathcal{N}$ on $\widetilde{\mathcal{X}}$ with $N_0 = \mathcal{N}|_{\widetilde{C}_{pq}}$ of bidegree (d_1, d_2), we can modify $\mathcal{N}$ by twisting with a component of $\widetilde{C}_{pq}$ so that N_0 has either bidegree $(6,4)$ or $(5,5)$.

Proposition 2.6. *We have a canonical identification*

$$(A_0, \Theta_0) \cong (\mathrm{Pic}^4 C, \Theta_C).$$

Proof. Denote by $\mathrm{Pic}^{6,4}(\widetilde{C}_{pq})$ the principal homogeneous space over $J(\widetilde{C}_{pq})$ parametrizing line bundles of bidegree $(6,4)$ on $\widetilde{C}_{pq}$. We identify A_0 with the subvariety of $\mathrm{Pic}^{6,4}(\widetilde{C}_{pq})$ consisting of line bundles N such that $\mathrm{Nm}(N) = \omega_{C_{pq}}$. Note that $\mathrm{Nm}^{-1}(\omega_{C_{pq}})$ only has one connected component by the diagram above ($\mathrm{Pic}^{6,4}(\widetilde{C}_{pq})$ is not compact). We then define the theta divisor Θ_0 of A_0 as the locus of line bundles $N \in A_0 \subset \mathrm{Pic}^{6,4}(\widetilde{C}_{pq})$ such that $h^0(N) > 0$.

We have an isomorphism $A_0 \xrightarrow{\cong} \mathrm{Pic}^4 C$ which sends N to $N|_{C_2}$. This is an isomorphism because N is determined by $\nu^* N$ by the diagram above and $N|_{C_2}$ determines $\nu^* N$ since $\mathrm{Nm}(\nu^* N) = N|_{C_1} \otimes N|_{C_2} \cong \omega_C(p+q)$. We claim that if $h^0(\widetilde{C}_{pq}, N) \neq 0$, then $h^0(C_2, N|_{C_2}) \neq 0$. If not, let $0 \neq s \in H^0(\widetilde{C}_{pq}, N)$ be such that $s|_{C_2} = 0$; then $s|_{C_1}$ vanishes at p and q. Thus $0 \neq s|_{C_1} \in H^0(C_1, N|_{C_1}(-p-q))$. However, since $N|_{C_1}(-p-q) \otimes N|_{C_2} \cong \omega_C$, we have $h^0(C_1, N|_{C_1}(-p-q)) = h^1(C_2, N|_{C_2}) = h^0(C_2, N|_{C_2}) = 0$, a contradiction. Thus the canonical identification sends Θ_0 isomorphically to Θ_C. $\square$

2.5. The family of curves in the limit. Denote the total space of the family of the double covers $G_t := \widetilde{G}^1_5(X_t)$ by $\mathcal{G}$. The space $\mathcal{G}$ is an étale double cover of $\mathcal{W}^1_5$ and therefore smooth by Proposition 2.3. The central fiber G_0 of $\mathcal{G}$ is described in Lemma 2.4.

Assume we are also given a section $r\colon t \mapsto r_t$ of $\widetilde{\mathcal{X}} \to T$. Let $\mathcal{F}$ (resp. $\mathcal{F}_r$) be the closure in $\mathcal{G} \times_T \widetilde{\mathcal{X}}^{(5)} \times_T \widetilde{\mathcal{X}}^{(5)}$ of the family of fourfolds (resp. threefolds) F_t (resp. F_{r_t}) constructed in Section 1 for the fibers over $t \neq 0$.

By construction, the central fiber F_0 of $\mathcal{F}$ fibers over $G_0 = W_1 \cup W_2$. The fiber over $M \in W_k$ is the surface $B^k_M \times B^k_{M'}$, where B^1_M and $B^1_{M'}$ live in

$$(C_1^{(4)} \times C_2) \cup (C_1^{(2)} \times C_2^{(3)}) \cup C_2^{(5)}$$

and B^2_M and $B^2_{M'}$ live in

$$C_1^{(5)} \cup (C_1^{(3)} \times C_2^{(2)}) \cup (C_1 \times C_2^{(4)}).$$

3. The degeneration of theta divisors

Let $(\mathcal{A}, \Theta) = \{\,(A_t, \Theta_t)\,\}_{t \in T}$ be a one-parameter family of principally polarized abelian varieties of dimension 5 with smooth total space $\mathcal{A}$. Assume that for $t \neq 0$, the fiber Θ_t of Θ is smooth and that the fiber of $(\mathcal{A}, \Theta)$ at 0 is the polarized Jacobian $(A_0 = JC, \Theta_0 = \Theta_C)$ of a smooth curve C of genus 5.

We will obtain information about the cohomology of Θ_t from the cohomology of Θ_0 using limit mixed Hodge structures. We shall see below that the total space Θ is singular. We first need to modify the family $(\mathcal{A}, \Theta)$ using base change and blow-ups to obtain a family of theta divisors with smooth total space whose central fiber is a divisor with simple normal crossings.

3.1. The singularities of Θ. Denote by $\mathcal{H}$ the Siegel upper half space and consider the Riemann theta function $\theta(z, \tau)$ on $\mathbb{C}^5 \times \mathcal{H}$. After possibly replacing T with a finite cover we can assume that there is a map $\tau\colon T \to \mathcal{H}$ such that the family $(\mathcal{A}, \Theta)$ is the inverse image, via τ, of the universal family of polarized abelian varieties over $\mathcal{H}$. In particular, we can assume that the family Θ is defined by $\{\,(z,t) \in \mathbb{C}^5 \times T\colon \theta(z, \tau(t)) = 0\,\}$ (modulo the action of the lattice of A_t). Denote $F(z,t) := \theta(z, \tau(t))$.

Since Θ_t is smooth for $t \neq 0$, the total space Θ is smooth away from the special fiber Θ_0. In the case we are interested in, the singularities of the special fiber Θ_0 are all double points, and hence the singularities of the total family Θ are at worst double points.

We compute the singularities of Θ locally, using the heat equation: $\partial\theta/\partial\tau_{ij} = \partial^2\theta/\partial z_i \partial z_j$ modulo multiplication by a constant. Here the τ_{ij} denote coordinates on $\mathcal{H}$ and the z_i coordinates on an abelian variety A. We write

$\tau(t) = (\tau_{ij}(t))_{1\le i,j\le 5}$ and let $\dot\tau_{ij}(t)$ denote the derivative of $\tau_{ij}(t)$ with respect to t. We also put $\dot\tau := \dot\tau(0)$ and $\dot\tau_{ij} := \dot\tau_{ij}(0)$.

Throughout the rest of this section, we use the notation $\partial_i := \partial/\partial z_i$, $\partial_{ij} := \partial^2/\partial z_i \partial z_j$ and $\partial_{it} := \partial^2/\partial z_i \partial t$, etc. We also use the summation convention: when an index appears twice in a single term, it implies summation of that term as the index goes from 1 to 5.

Proposition 3.1. *A point $(z,0)$ is a singular point of Θ exactly when $(z,0)$ is a singular point of Θ_0 such that the equation $q_z \in S^2H^1(\mathcal{O}_{A_0})^*$ of the quadric tangent cone to Θ_0 at z vanishes on the infinitesimal deformation direction $\dot\tau = (\dot\tau_{ij})_{1\le i,j\le 5} \in S^2H^1(\mathcal{O}_{A_0})$ under duality.*

Proof. By the heat equation, at a point $(z,t) \in \Theta$ the equation of the tangent hyperplane to Θ in $\mathcal{A}$ is the pull-back from the Siegel space of the equation

$$Z_i\partial_i\theta + T_{ij}\partial_{ij}\theta = 0,$$

where the Z_i are the coordinates on the tangent space to a fiber A_t and the T_{ij} coordinates on the tangent space to $\mathcal{H}$ at $\tau(t)$.

This gives the equation

$$\partial_i F(z,t) Z_i + (\dot\tau_{ij}(t)\partial_{ij}F(z,t))\,\Omega = 0,$$

where Ω is the coordinate on the tangent space to T at t.

So the point $(z,0)$ is singular on Θ if and only if it is singular on Θ_0 and

$$\dot\tau_{ij}(t)\partial_{ij}F(z,0) = 0.$$

Since

$$q_z = Z_iZ_j\partial_{ij}F(z,0),$$

the proposition follows. □

The partial derivatives of F are:

$$\begin{aligned}
\partial_t F(z,t) &= \partial_{\tau_{ij}}\theta(z,\tau(t))\dot\tau_{ij}(t) = \partial_{ij}\theta(z,\tau(t))\dot\tau_{ij}(t) = \partial_{ij}F(z,t)\dot\tau_{ij}(t),\\
\partial_{it}F(z,t) &= \partial_t(\partial_i F(z,t)) = \partial_{ijk}F(z,t)\dot\tau_{jk}(t),\\
\partial_{tt}F(z,t) &= \partial_{ijkl}F(z,t)\dot\tau_{ij}(t)\dot\tau_{kl}(t) + \partial_{ij}F(z,t)\ddot\tau_{ij}(t).
\end{aligned}$$

3.2. The case $A_0 \simeq J(C) \simeq \mathrm{Pic}^4 C$. In this case the theta divisor $\Theta_0 = W_4^0(C)$ of the special fiber is smooth outside the curve $W_4^1 := W_4^1(C)$, and W_4^1 is an ordinary double curve on it. Therefore we have

$$\begin{aligned}
&F(p,0) = 0, && \forall p \in W_4^1,\\
&\partial_i F(p,0) = 0, && \forall i, \forall p \in W_4^1,\\
&\mathrm{rank}\Big(\partial_{ij}F(p,0)\Big)_{1\le i,j\le 5} = 4, && \forall p \in W_4^1.
\end{aligned}$$

Theorem 3.2. *For τ sufficiently general, the singularities of Θ consist of ten ordinary double points. In the case where $(\mathcal{A}, \Theta)$ is the family of Prym varieties of a family of double covers $(\widetilde{\mathcal{X}}, \mathcal{X})$ as in Section 2.4, the ten distinct singular points $g_1, \dots, g_5, h_1, \dots, h_5$ of Θ are the g^1_4's cut on C by quadrics of rank 4 containing C and its secant $\langle p+q \rangle$. In other words, $h^0(g_i - p - q) > 0$ and $h_i = |K - g_i|$ up to relabeling.*

Proof. We use the calculations in Section 3.1. The annihilator of the deformation direction

$$\dot{\tau} = (\dot{\tau}_{ij})_{1 \le i,j \le 5} \in S^2 H^1(\mathcal{O}_{A_0})$$

is a hyperplane in $S^2 H^0(\omega_C)$, which, for τ sufficiently general, gives a hyperplane in the space $I_2(C)$ of quadrics containing the canonical image of C and hence a line l in $\mathbb{P}I_2(C) \cong \mathbb{P}^2$. The quadrics of rank 4 containing the canonical model of C are the elements of Q, a plane quintic in $\mathbb{P}I_2(C)$. Those whose equations vanish on τ are the elements of the intersection $l \cap Q$, which, for τ sufficiently general, consists of five distinct points, say $q_1, \dots, q_5$. There are ten distinct points in the singular locus W^1_4 of Θ_0 above these five points: the g^1_4's cut on C by the rulings of $q_1, \dots, q_5$. Hence we see that Θ has exactly ten distinct singular points.

In the case where our family of abelian varieties is a family of Prym varieties of double covers with central fiber a Wirtinger cover, the deformation direction $\dot{\tau}$ is the image, via the differential of the Prym map, of the infinitesimal deformation direction, say η, of double covers induced by the family $(\widetilde{\mathcal{X}}, \mathcal{X})$. As the Prym map sends the locus $\mathcal{W}_6$ of Wirtinger covers in $\mathcal{R}_6$ into the Jacobian locus $\mathcal{J}_5$, its differential induces a linear map from the 1-dimensional normal space $N_{C_{pq}}$ to $\mathcal{W}_6$ to the 3-dimensional normal space N_{JC} to $\mathcal{J}_5$. It is well known (see e.g. [DS, p. 45]) that the normal space to $\mathcal{J}_5$ at JC can be canonically identified with the dual $I_2(C)^*$ to $I_2(C)$. By [DS, p. 86], the image of $\mathbb{P}N_{C_{pq}}$ in $\mathbb{P}I_2(C)^* = \mathbb{P}N_{JC}$ is the pencil of quadrics containing the canonical image of C together with its secant $\langle p+q \rangle$. This is also the line that we denoted by l above. Therefore the points $q_1, \dots, q_5$ are the quadrics of rank 4 containing C and $\langle p+q \rangle$. The line $\langle p+q \rangle$ is contained in exactly one ruling of q_i, and we denote by g_i the g^1_4 cut on C by that ruling. We then have $h^0(g_i - p - q) > 0$. The second ruling of q_i cuts $h_i := |K_C - g_i|$ on C.

It remains to prove that the ten singular points are ordinary double points. The degree 2 term of the Taylor expansion of F near a singular point (z, t) is (using the heat equation up to a scalar)

$$Z_i Z_j \partial_{ij} F + (Z_i \dot{\tau}_{jk} \partial_{ijk} F)\,\Omega + (\dot{\tau}_{ij}\dot{\tau}_{kl}\partial_{ijkl} F + \ddot{\tau}_{ij}\partial_{ij} F)\,\Omega^2.$$

The first part of the above is the equation of the quadric q_z which has rank 4. In a basis adapted to q_z we have the matrix of second partials of F:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dot\tau_{jk}\partial_{1jk}F \\ 1 & 0 & 0 & 0 & 0 & \dot\tau_{jk}\partial_{2jk}F \\ 0 & 0 & 0 & 1 & 0 & \dot\tau_{jk}\partial_{3jk}F \\ 0 & 0 & 1 & 0 & 0 & \dot\tau_{jk}\partial_{4jk}F \\ 0 & 0 & 0 & 0 & 0 & \dot\tau_{jk}\partial_{5jk}F \\ \dot\tau_{jk}\partial_{1jk}F & \dot\tau_{jk}\partial_{2jk}F & \dot\tau_{jk}\partial_{3jk}F & \dot\tau_{jk}\partial_{4jk}F & \dot\tau_{jk}\partial_{5jk}F & \begin{matrix}\dot\tau_{ij}\dot\tau_{kl}\partial_{ijkl}F \\ +\ddot\tau_{ij}\partial_{ij}F\end{matrix} \end{pmatrix}.$$

So we need to see that this matrix has rank 6 at the points of W^1_4. In other words, for τ sufficiently general the coefficient $\dot\tau_{jk}\partial_{5jk}F(z,0)$ is not zero. Taking $\dot\tau_{ij} = \lambda_i\lambda_j$ such that the point $(\lambda_i) \in \mathbb{C}^g = H^1(\mathcal{O}_{A_0}) = T_0A_0$ is on the osculating cone to Θ_0 and is otherwise general, this means that the vertex of the quadric q_z is not contained in the tangent space to the osculating cone to Θ_0 at the point (λ_i). For a general choice of the λ_i this is a consequence of [KS], page 353. □

4. The semistable reduction of the family of theta divisors

As before denote by $(\mathcal{A}, \Theta) \to T$ the family of principally polarized Prym varieties associated to the étale cover $\widetilde{\mathcal{X}} \to \mathcal{X}$. The central fiber is the Jacobian $A_0 \cong \operatorname{Pic}^4 C$ of a general curve of genus 5. By Theorem 3.2, the total space Θ has ten ordinary double points on W^1_4: $g_1, \dots, g_5$, which satisfy $h^0(g_i - p - q) > 0$, and $h_i := |K_C - g_i|$. We will construct a semistable reduction of Θ and, in Section 6, use the Clemens-Schmid exact sequence to compute the cohomology of Θ_t.

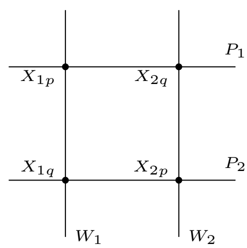
4.1. The base change and first blow-ups. To construct our semistable reduction, we first make a base change of degree 2, then resolve singularities. Let $T^b \to T$ be a degree 2 cover. After possibly shrinking T, we assume that the cover $T^b \to T$ has a unique branch point at $0 \in T$. Pulling back, we obtain the family $\Theta^b \subset \mathcal{A}^b \to T^b$ and Θ^b is singular along $W^1_4 \subset \Theta_0$. We define $\widetilde{\Theta}$ as the blow-up of Θ^b along its singular locus W^1_4. We will see that $\widetilde{\Theta}$ is a resolution of Θ^b whose special fiber $\widetilde{\Theta}_0$ is a simple normal crossings divisor ($\mathcal{A}^b \to T^b$ is still a smooth family):

$$\begin{array}{ccccc} \widetilde{\Theta} & \longrightarrow & \Theta^b & \longrightarrow & \Theta \\ & & \downarrow & & \downarrow \\ & & T^b & \longrightarrow & T. \end{array}$$

To make our family of curves compatible with the base change, we also need to take the base change of $\mathcal{G}$ to T^b and then blow up along the singular locus of $\mathcal{G}^b$ to obtain a semistable family. The resulting space is $\widetilde{\mathcal{G}}$:

$$\begin{array}{ccccc} \widetilde{\mathcal{G}} & \longrightarrow & \mathcal{G}^b & \longrightarrow & \mathcal{G} \\ & & \downarrow & & \downarrow \\ & & T^b & \longrightarrow & T. \end{array}$$

Recall that, by Lemma 2.4, the fiber of $\mathcal{G}$ at $t = 0$ is the union of two copies of W_{pq}, denoted W_1 and W_2, where $X_{kp} = W_4^1(C) + p \subset W_k$ is identified with $X_{3-k,q} = W_4^1(C) + q \subset W_{3-k}$. After blowing up along the singular locus $X_{1p} \amalg X_{1q} \subset \mathcal{G}^b$, the central fiber $\widetilde{G}_0$ of $\widetilde{\mathcal{G}}$ has four components: W_1, W_2, P_1 and P_2, where P_1, resp. P_2, is a $\mathbb{P}^1$ bundle over $X_{1p} = X_{2q}$, resp. $X_{1q} = X_{2p}$. The four components meet as below:



Notation 4.1. From now on we will replace T by T^b and the families $\Theta \subset \mathcal{A} \to T$, $\mathcal{G} \to T$, $\mathcal{F} \to T$ and $\widetilde{\mathcal{X}} \to \mathcal{X} \to T$ by their base changes to T^b.

4.2. The central fiber $\widetilde{\Theta}_0$.

Proposition 4.2. *The total space $\widetilde{\Theta}$ is smooth. Its special fiber $\widetilde{\Theta}_0$ is a divisor with simple normal crossings with the following two irreducible components.*

(1) *The component M_1 which is the blow-up of Θ_0 along W_4^1.*
(2) *The component M_2 which is the exceptional divisor, i.e. the projectivized normal cone to Θ along W_4^1. Therefore M_2 is a fibration over W_4^1. At the points g_i and h_i the fibers Q_{3i}^{sing} of M_2 are isomorphic to the singular quadric Q_3^{sing} of rank 4 in $\mathbb{P}^4$. At all the other points of W_4^1, the fibers of M_2 are isomorphic to the smooth quadric hypersurface $Q_3 \subset \mathbb{P}^4$.*

The intersection $M_{12} = M_1 \cap M_2$ is a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over W_4^1. In particular, it is smooth.

Proof. It immediately follows from the definition of $\widetilde{\Theta}$ that $\widetilde{\Theta}_0$ has two components, one of which is the blow-up M_1 of Θ_0 and the other the projectivized normal cone M_2 of W_4^1 in Θ. To prove the assertions about the smoothness of M_1, M_2 and M_{12} and the fibers of M_2 over W^1_4 we work in local coordinates near each of the ten points g_i and h_i of Theorem 3.2.

By Theorem 3.2, before our base change of degree 2 the local equation of Θ near one of the points g_i or h_i can be written as $xy + zw + st = 0$ in $\mathbb{A}^6$ where t is a local analytic coordinate on T centered at 0. Hence, after base change, the local equation is

$$xy + zw + st^2 = 0.$$

In the above coordinates, the equations of W^1_4 are

$$x = y = z = w = t = 0.$$

Hence, locally, $\widetilde{\Theta}$ is obtained from Θ by blowing up the ideal $\mathcal{I} = (x, y, z, w, t)$. Furthermore, s gives a local coordinate on W^1_4. For any given nonzero value of s, the local equation of Θ defines a quadric of rank 5 in $\mathbb{P}^4$ and, for $s = 0$, the equation defines a quadric of rank 4 in $\mathbb{P}^4$. This proves the assertions about the fibers of $M_2 \to W^1_4$.

Now let X, Y, Z, W, T be the homogeneous coordinates on the blow-up. We have the relations

$$\operatorname{rank}\begin{pmatrix} x & y & z & w & t \\ X & Y & Z & W & T \end{pmatrix} \leq 1.$$

By symmetry, we only need to check the following cases.

(1) $\{ X \neq 0 \}$. $\widetilde{\Theta}$ is locally isomorphic to

$$\operatorname{Spec} \frac{\mathbb{C}[x, Y, Z, W, T, s]}{(Y + ZW + sT^2)},$$

which is clearly smooth. $\widetilde{\Theta}_0$ is given by the equation $t = 0$, i.e., $xT = 0$, hence has two smooth components meeting transversely, defined locally by the equations $T = 0$ and $x = 0$. The equation $T = 0$ locally defines the component M_1 while $x = 0$ locally defines M_2.

(2) $\{ T \neq 0 \}$. $\widetilde{\Theta}$ is locally isomorphic to

$$\operatorname{Spec} \frac{\mathbb{C}[X, Y, Z, W, t, s]}{(XY + ZW + s)}.$$

In this open subset, the total space and the central fiber are both smooth, and $t = 0$ locally defines the component M_1.

□

Proposition 4.3.

(1) *The divisor* M_1 *can be identified with the correspondence*

$$M_1 = \{ (D_4, B_4) \in C^{(4)} \times C^{(4)} |\ D_4 + B_4 \in |K_C| \}$$

with the two projections p_1 and p_2 to $C^{(4)}$. We have a fibered diagram

$$\begin{array}{ccccc} & & M_1 & & \\ & \overset{p_1}{\swarrow} & & \overset{p_2}{\searrow} & \\ C^{(4)} & & & & C^{(4)} \\ & \underset{\phi}{\searrow} & & \underset{\psi}{\swarrow} & \\ & & \Theta_0 & & \end{array}$$

where ϕ is the natural map sending D_4 to $\mathcal{O}_C(D_4)$ and ψ sends B_4 to $\omega_C(-B_4)$.

(2) *Both p_1 and p_2 are birational morphisms and can be realized as the blow-up of $C^{(4)}$ along the smooth surface*

$$C_4^1 = \{\, D \in C^{(4)} \mid h^0(\mathcal{O}_C(D)) \geq 2 \,\}.$$

(3) *The double locus M_{12} is the fiber product*

$$\begin{array}{ccccc} & & M_{12} & & \\ & \overset{p_1'}{\swarrow} & & \overset{p_2'}{\searrow} & \\ C_4^1 & & & & C_4^1 \\ & \underset{\phi'}{\searrow} & & \underset{\psi'}{\swarrow} & \\ & & W_4^1 & & \end{array}$$

Proof. Immediate. □

5. General facts about the Clemens-Schmid exact sequence

We briefly review some general facts about the Clemens-Schmid exact sequence in this section. In Section 6, we will apply this general theory to compute the cohomology of $\widetilde{\Theta}_0$ and $\widetilde{\Theta}_t$.

5.1. The Clemens-Schmid exact sequence. Given any semistable degeneration

$$\begin{array}{ccc} Y_0 & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \{\,0\,\} & \longrightarrow & T \end{array}$$

of relative dimension n, where $\mathcal{Y}$ deformation retracts to Y_0, denote

$$\begin{aligned} H_t^m &:= H^m(Y_t, \mathbb{Q}), \\ H^m &:= H^m(\mathcal{Y}, \mathbb{Q}) \cong H^m(Y_0, \mathbb{Q}), \\ H_m &:= H_m(\mathcal{Y}, \mathbb{Q}) \cong H_m(Y_0, \mathbb{Q}). \end{aligned}$$

It follows from the work of Clemens-Schmid [C], [Sc] and Steenbrink [St] that one can define mixed Hodge structures on H_t^*, H^* and H_* such that we have an exact sequence of mixed Hodge structures

(5.1)

$$\longrightarrow H_{2n+2-m} \xrightarrow{\alpha} H^m \xrightarrow{i_t^*} H_t^m \xrightarrow{N} H_t^m \xrightarrow{\beta} H_{2n-m} \xrightarrow{\alpha} H^{m+2} \longrightarrow$$

where N is the logarithm of the monodromy operator, $i_t\colon Y_t \hookrightarrow \mathcal{Y}$ is the inclusion of the general fiber into the total space, α is the composition

$$H_{2n+2-m}(\mathcal{Y}) \xrightarrow{\text{PD}} H^m(\mathcal{Y}, \partial\mathcal{Y}) \longrightarrow H^m(\mathcal{Y}),$$

where PD denotes Poincaré Duality, and β is the composition

$$H^m(Y_t) \xrightarrow{\text{PD}} H_{2n-m}(Y_t) \xrightarrow{i_{t*}} H_{2n-m}(\mathcal{Y}).$$

5.2. The weight filtrations on H^m and H_m. Recall from [Mo, p. 103] that there is a Mayer-Vietoris type spectral sequence abutting to $H^\bullet(Y_0)$ with E_1 term

$$E_1^{p,q} = H^q(Y_0^{[p]}).$$

Here $Y_0^{[p]}$ is the disjoint union of the codimension p strata of Y_0, i.e.,

$$Y_0^{[p]} := \coprod_{i_0,\dots,i_p} Z_{i_0} \cap \ldots \cap Z_{i_p}$$

where the Z_{i_j} are distinct irreducible components of Y_0.

The differential d_1

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{d_1} & E_1^{p+1,q} \\ \downarrow \cong & & \downarrow \cong \\ H^q(Y_0^{[p]}) & \xrightarrow{d_1} & H^q(Y_0^{[p+1]}) \end{array}$$

is the alternating sum of the restriction maps on all the irreducible components. By [Mo, p. 103] this sequence degenerates at E_2.

The weight filtration is given by

$$W_k H^m := \bigoplus_{p+q=m,\ q\le k} E_\infty^{p,q} = \bigoplus_{p+q=m,\ q\le k} E_2^{p,q}.$$

Therefore the weights on H^m go from 0 to m and

$$\operatorname{Gr}_k H^m \cong E_2^{m-k,k} = \frac{\operatorname{Ker}(d_1\colon H^k(Y_0^{[m-k]}) \to H^k(Y_0^{[m-k+1]}))}{\operatorname{Im}(d_1\colon H^k(Y_0^{[m-k-1]}) \to H^k(Y_0^{[m-k]}))}.$$

We also put a weight filtration on H_m:

$$W_{-k}H_m := (W_{k-1}H^m)^{\perp}$$

under the perfect pairing between H^m and H_m. With this definition,

$$\operatorname{Gr}_{-k} H_m \cong (\operatorname{Gr}_k H^m)^{\vee}.$$

5.3. The monodromy weight filtration on H_t^m. Associated to the nilpotent operator N is an increasing filtration of $\mathbb{Q}$-vector spaces

$$0 \subset W_0 \subset W_1 \subset \cdots \subset W_{2m} = H_t^m.$$

Let $K_t^m := \operatorname{Ker} N \subset H_t^m$ be the monodromy invariant subspace. It inherits an induced weight filtration from H_t^m. We refer to [Mo, pp. 106-109] for the precise definition of the monodromy weight filtration and the fact that this filtration on H_t^m can be computed via its induced filtration on K_t^m:

$$\operatorname{Gr}_k H_t^m \cong \operatorname{Gr}_k K_t^m \oplus \operatorname{Gr}_{k-2} K_t^m \oplus \cdots \oplus \operatorname{Gr}_{k-2\lfloor \frac{k}{2} \rfloor} K_t^m \tag{5.2}$$

for $k \leq m$, and

$$\operatorname{Gr}_k H_t^m \cong \operatorname{Gr}_{2m-k} H_t^m, \tag{5.3}$$

for $k > m$.

The weight filtrations on H^m and K_t^m are related by the Clemens-Schmid exact sequence. Below are the basic facts we will use (see [Mo, pp. 107-109]).

(1) The pull-back map i_t^* induces an isomorphism

$$\operatorname{Gr}_k H^m \xrightarrow{\cong} \operatorname{Gr}_k K_t^m \quad \textit{for } k \leq m-1. \tag{5.4}$$

(2) There is an exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Gr}_{m-2} K_t^{m-2} \longrightarrow \operatorname{Gr}_{m-2n-2} H_{2n+2-m} &\xrightarrow{\alpha} \operatorname{Gr}_m H^m \\ &\longrightarrow \operatorname{Gr}_m K_t^m \longrightarrow 0. \end{aligned} \tag{5.5}$$

5.4. Mixed Hodge structures on $H_c^{\bullet}(\mathcal{Y})$. Now suppose furthermore that $\mathcal{Y}$ is an **analytic** open subset of a smooth projective variety $\overline{\mathcal{Y}}$ of dimension $n+1$. We have a sequence of isomorphisms

$$H_c^{2n+2-m}(\mathcal{Y}) \cong H^{2n+2-m}(\overline{\mathcal{Y}}, \overline{\mathcal{Y}} \setminus \mathcal{Y}) \cong H^{2n+2-m}(\overline{\mathcal{Y}}, \overline{\mathcal{Y}} \setminus Y_0),$$

where the last isomorphism follows from the fact that $\overline{\mathcal{Y}} \setminus Y_0$ deformation retracts to $\overline{\mathcal{Y}} \setminus \mathcal{Y}$.

Both $H^\bullet(\overline{\mathcal{Y}})$ and $H^\bullet(\overline{\mathcal{Y}}\setminus Y_0)$ admit canonical mixed Hodge structures ([De], [Du, pp. 1022-1024]). The relative singular cochain complex $S^\bullet(\overline{\mathcal{Y}}, \overline{\mathcal{Y}} \setminus Y_0)$ is quasi-isomorphic to the mapping cone of the chain map

$$S^\bullet(\overline{\mathcal{Y}}) \to S^\bullet(\overline{\mathcal{Y}} \setminus Y_0).$$

Using a standard mapping cone construction (see, for instance, [Du, pp. 1205-1207]), we can put a canonical mixed Hodge structure on $H^\bullet(\overline{\mathcal{Y}}, \overline{\mathcal{Y}} \setminus Y_0)$, and therefore on $H_c^\bullet(\mathcal{Y})$, such that the maps in the long exact sequence

$$\begin{aligned} \dots \longrightarrow H^{m-1}(\overline{\mathcal{Y}} \setminus Y_0) \longrightarrow H^m(\overline{\mathcal{Y}}, \overline{\mathcal{Y}} \setminus Y_0) \longrightarrow H^m(\overline{\mathcal{Y}}) \\ \longrightarrow H^m(\overline{\mathcal{Y}} \setminus Y_0) \longrightarrow \dots \end{aligned} \tag{5.6}$$

are morphisms of mixed Hodge structures.

There is also a spectral sequence [Du, pp. 1025-1027] for the mapping cone, dual to the Mayer-Vietoris type spectral sequence in Section 5.2, abutting to $H_c^\bullet(\mathcal{Y})$. This spectral sequence is in the second quadrant, degenerates at E_2, and has E_1 terms

$$E_{1,c}^{p,q} = H^{q+2p-2}(Y_0^{[-p]}),$$

for $p \leq 0$. The differential

$$\begin{array}{ccc} E_{1,c}^{p,q} & \longrightarrow & E_{1,c}^{p+1,q} \\ \| & & \| \\ H^{q+2p-2}(Y_0^{[-p]}) & \xrightarrow{d_1^c} & H^{q+2p}(Y_0^{[-p-1]}) \end{array}$$

is the alternating sum of Gysin morphisms. We have the duality

$$(E_1^{p,q})^\vee \cong E_{1,c}^{-p,2n+2-q}.$$

The increasing weight filtration is given by

$$W_k H_c^m(\mathcal{Y}) = \bigoplus_{p+q=m, q\leq k} E_{2,c}^{p,q}.$$

The weights on $H_c^m(\mathcal{Y})$ go from m to $2m-2$ and, for $m \leq k \leq 2m-2$,

$$\operatorname{Gr}_k H_c^m(\mathcal{Y}) \cong E_{2,c}^{m-k,k} = \frac{\operatorname{Ker}(H^{2m-k-2}(Y_0^{[k-m]}) \to H^{2m-k}(Y_0^{[k-m-1]}))}{\operatorname{Im}(H^{2m-k-4}(Y_0^{[k-m+1]}) \to H^{2m-k-2}(Y_0^{[k-m]}))}$$

with the convention that $Y_0^{[-1]} = \varnothing$.

The mixed Hodge structures on $H^m(\mathcal{Y})$ and

$$H_c^{2n+2-m}(\mathcal{Y}) \cong H^{2n+2-m}(\overline{\mathcal{Y}}, \overline{\mathcal{Y}} \setminus Y_0)$$

are dual to each other. We have

$$\operatorname{Gr}_k H^m(\mathcal{Y})^\vee \cong \operatorname{Gr}_{2n+2-k} H_c^{2n+2-m}(\mathcal{Y}).$$

6. The monodromy weight filtration on the cohomology of Θ_t

We apply the general theory in Section 5 to the case $\mathcal{Y} = \widetilde{\Theta}$ to compute the cohomology of Θ_t in this section. By the Hard Lefschetz Theorem,

$$H^m(\Theta_t) \cong H^{8-m}(\Theta_t) \tag{6.1}$$

and by the Lefschetz Hyperplane Theorem,

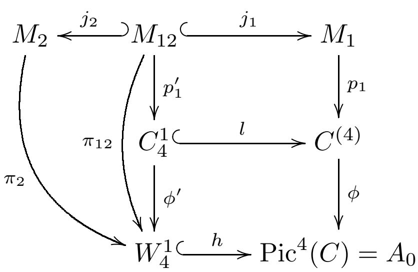
$$H^m(\Theta_t) \cong H^m(A_t) \cong \mathbb{Q}^{\binom{10}{m}} \tag{6.2}$$

for $m \leq 3$. The only remaining case is the middle cohomology $H^4(\Theta_t)$. We will describe the monodromy weight filtration on it.

According to the general theory explained in Section 5, in order to compute the monodromy weight filtration on the cohomology of Θ_t we first need to compute the cohomology of the central fiber $\widetilde{\Theta}_0 = M_1 \cup M_2$.

6.1. The cohomology of the strata of $\widetilde{\Theta}_0$. In this subsection we compute the cohomology of M_1, M_2 and M_{12} and describe their generators. The various spaces fit into the commutative diagram with Cartesian squares

(6.3)



where we denote by p_1' (resp. ϕ') the restriction of p_1 (resp. ϕ) to M_{12} (resp. C_4^1) and $j_k \colon M_{12} \to M_k$ the inclusion map.

Lemma 6.1. *We have the following table of Betti numbers.*

TABLE 1

	h^0	h^1	h^2	h^3	h^4	h^5	h^6	h^7	h^8
C_4^1	1	22	2	22	1	0	0	0	0
$C^{(4)}$	1	10	46	130	256	130	46	10	1
M_1	1	10	47	152	258	152	47	10	1
Q_3	1	0	1	0	1	0	1	0	0
Q_3^{sing}	1	0	1	0	2	0	1	0	0
M_{12}	1	22	3	44	3	22	1	0	0
M_2	1	22	2	22	12	22	2	22	1

Proof. This is a straightforward computation so we only sketch the idea.

(1) By Proposition 4.3, M_1 is the blow-up of $C^{(4)}$ along C^1_4. So we have $H^\bullet(M_1) = p_1^* H^\bullet(C^{(4)}) \oplus\ j_{1*}{p'_1}^* H^{\bullet-2}(C^1_4)$. The cohomology of $C^{(4)}$ was computed by Macdonald [Ma]:

$$H^k(C^{(4)}) = \bigoplus_{\beta=0}^{\lceil \frac{k}{2} \rceil} \eta^\beta \cdot H^{k-2\beta}(\operatorname{Pic}^4 C).$$

(2) Since M_{12} (resp. C^1_4) is a smooth fibration over W^1_4 with fibers $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{P}^1$), we can apply the Leray spectral sequence to $\pi_{12} \colon M_{12} \to W^1_4$ (resp. ϕ') to compute the cohomology of M_{12} and C^1_4.

(3) The variety M_2 is a fibration over W^1_4 with general fiber isomorphic to the smooth quadric threefold Q_3 and ten special fibers isomorphic to the singular quadric Q_3^{sing} of rank 4. Since the base is a curve, the Leray spectral sequence for π_2 degenerates at E_2.

We present the Leray spectral sequence computation for $H^4(M_2, \mathbb{Q})$; the other cohomology groups are similar and somewhat easier to compute. The E_2 terms are

$$E_2^{p,q} = H^p(W^1_4, R^q\pi_{2*}\mathbb{Q}).$$

Let $U \subset W^1_4$ be a small analytic disc, open neighborhood of a critical value of π_2. Then $\pi_2^{-1}(U)$ is homotopic to a smooth fiber $\pi_2^{-1}(t) = Q_3$ with a real 4-cell B^4 attached to $\pi_2^{-1}(t)$ along a vanishing sphere S^3. Since $h^3(Q_3) = 0$, this amounts to increasing h^4 by 1. Thus

$$R^q\pi_{2*}\mathbb{Q} \cong \begin{cases} \mathbb{Q} \oplus (\bigoplus_{i=1}^{10} \mathbb{Q}_i) & q = 4, \\ \mathbb{Q} & q = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{Q}_i$ is the skyscraper sheaf with stalk $\mathbb{Q}$ supported at the i-th critical point. Therefore

$$\dim_{\mathbb{Q}} E_2^{p,4-p} = \dim_{\mathbb{Q}} E_\infty^{p,4-p} = h^p(W^1_4, R^{4-p}\pi_{2*}\mathbb{Q}) = \begin{cases} 11 & p = 0, \\ 1 & p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

This gives $h^4(M_2) = 12$. □

Notation 6.2. Denote by $e_k \in H^2(M_k)$ the class of M_{12} in M_k, by $f \in H^2(M_{12})$ the class of a fiber $\pi_{12}^{-1}(t)$ and $\tau_1 := {p'_1}^* l^* \eta \in H^2(M_{12})$ (see Diagram (6.3)). Here $l^*\eta$ is represented by the curve $C^1_4 \cap (x + C^{(3)}) \subset C^1_4$ for $x \in C$ general, and τ_1 is represented by a $\mathbb{P}^1$ bundle over this curve. The product $\tau_1 \cdot f \in H^4(M_{12})$ is the class of the ruling of $\pi_{12}^{-1}(t) \cong \mathbb{P}^1 \times \mathbb{P}^1$ which projects to a point under p_1, and the product $j_2^* e_2 \cdot f$ is the hyperplane class in $\pi_{12}^{-1}(t)$. We also have the relation

$$-j_1^* e_1 = j_2^* e_2.$$

Furthermore, denote by $[\mathbb{P}^2_i]$, $i = 1, \ldots, 5$ (resp. $i = 6, \ldots, 10$) the class of the projective plane spanned by a line of the ruling corresponding to $\tau_1 \cdot f$ and the vertex of the singular quadric $Q^{sing}_{3i} = \pi_2^{-1}(g_i)$ (resp. $Q^{sing}_{3i} = \pi_2^{-1}(h_{i-5})$).

Lemma 6.3. *We have the following generators for the cohomology:*

$$\begin{aligned}
H^2(M_{12}) &= \langle f,\ \tau_1,\ j_2^* e_2 \rangle && \cong \mathbb{Q}^3,\\
H^3(M_{12}) &= \tau_1 \cdot \pi_{12}^* H^1(W^1_4) \oplus j_2^* e_2 \cdot \pi_{12}^* H^1(W^1_4) && \cong \mathbb{Q}^{44},\\
H^4(M_{12}) &= \langle f \cdot \tau_1,\ f \cdot j_2^* e_2,\ \tau_1 \cdot j_2^* e_2 \rangle && \cong \mathbb{Q}^3,\\
H^3(M_1) &= p_1^* H^3(C^{(4)}) \oplus j_{1*} \pi_{12}^* H^1(W^1_4) && \cong \mathbb{Q}^{152},\\
H^4(M_1) &= p_1^* H^4(C^{(4)}) \oplus \langle j_{1*} f, j_{1*} \tau_1 \rangle && \cong \mathbb{Q}^{258},\\
H^3(M_2) &= e_2 \cdot \pi_2^* H^1(W^1_4) && \cong \mathbb{Q}^{22},\\
H^4(M_2) &= \langle [\mathbb{P}^2_i],\ j_{2*} f,\ j_{2*} \tau_1 \mid i = 1, \ldots, 10 \rangle && \cong \mathbb{Q}^{12}.
\end{aligned}$$

Proof. The statements about M_1 follow from the formula for the cohomology of a blow-up, and the statements about M_2 and M_{12} follow from the Leray spectral sequence. □

6.2. The cohomology of $\widetilde{\Theta}_0$. Recall that $Q \subset \mathbb{P}I_2(C)$ is the plane quintic parametrizing quadrics of rank 4 and also the quotient of W^1_4 by the involution exchanging g^1_4 with $|K_C - g^1_4|$ (see the proof of Theorem 3.2). We have

Proposition 6.4. *The weight filtration on $H^4 := H^4(\widetilde{\Theta}_0) = H^4(M_1 \cup M_2)$ is as follows:*

$$\begin{aligned}
\operatorname{Gr}_k H^4 &= 0 && \textit{for } k \le 2,\\
\operatorname{Gr}_3 H^4 &\cong \frac{H^1(W^1_4)}{h^* H^1(\operatorname{Pic}^4 C)} \cong H^1(Q) && \cong \mathbb{Q}^{12},\\
\operatorname{Gr}_4 H^4 &= \operatorname{Ker}(H^4(M_1) \oplus H^4(M_2) \stackrel{j_1^* - j_2^*}{\longrightarrow} H^4(M_{12})) && \cong \mathbb{Q}^{267}.
\end{aligned}$$

Proof. We apply the spectral sequence of Section 5.2, which degenerates at E_2, to the case $\mathcal{Y} = \widetilde{\Theta}$. Since $\widetilde{\Theta}_0 = M_1 \cup M_2$, the E_1 term of the spectral sequence has only two nonzero columns corresponding to $p = 0$ and $p = 1$. Thus, from the definition of the weight filtration, we obtain

$$\begin{aligned}
\operatorname{Gr}_k H^4 &\cong E_2^{4-k,k} && = 0 \quad \text{for } k \le 2,\\
\operatorname{Gr}_3 H^4 &\cong E_2^{1,3} && = \operatorname{Coker}(H^3(M_1) \oplus H^3(M_2) \stackrel{j_1^* - j_2^*}{\longrightarrow} H^3(M_{12})),\\
\operatorname{Gr}_4 H^4 &\cong E_2^{0,4} && = \operatorname{Ker}(H^4(M_1) \oplus H^4(M_2) \stackrel{j_1^* - j_2^*}{\longrightarrow} H^4(M_{12})).
\end{aligned}$$

We compute $E_2^{1,3}$ in Lemma 6.5. By Lemma 6.6, the image of $j_1^* - j_2^*$ is equal to $H^4(M_{12})$; therefore $E_2^{0,4} \cong \mathbb{Q}^{267}$ by a dimension count. □

Lemma 6.5. *We have isomorphisms*

$$\mathrm{Coker}(H^3(M_1)\oplus H^3(M_2)\overset{j_1^*-j_2^*}{\longrightarrow} H^3(M_{12}))\cong \frac{H^1(W_4^1)}{h^*H^1(\mathrm{Pic}^4 C)}\cong H^1(Q)\cong \mathbb{Q}^{12}.$$

Proof. By Lemma 6.3,

$$H^3(M_1){=}p_1^*H^3(C^{(4)})\oplus j_{1*}\pi_{12}^*H^1(W_4^1)\ ,$$

and by [Ma, p. 325],

$$H^3(C^{(4)}) = H^3(\mathrm{Pic}^4(C))\oplus \eta\cdot H^1(\mathrm{Pic}^4 C).$$

Note that

$$H^3(\mathrm{Pic}^4 C)\xrightarrow{\ p_1^*\circ\ \phi^*\ } H^3(M_1)\xrightarrow{\ j_1^*\ } H^3(M_{12})$$

is zero since $\phi\circ p_1\circ j_1 = h\circ\phi'\circ p_1'$ (see diagram 6.3) and $H^3(W_4^1)=0$.

Furthermore, we see from Lemma 6.3 that the image of

$$H^3(M_2) = e_2\cdot\pi_2^*H^1(W_4^1)\xrightarrow{\ j_2^*\ } H^3(M_{12})$$

is equal to $j_1^*(j_{1*}\pi_{12}^*H^1(W_4^1)) = j_1^*e_1\cdot\pi_{12}^*H^1(W_4^1)$. This is because

$$j_1^*\circ j_{1*} = j_1^*e_1\cup\bullet = -j_2^*e_2\cup\bullet.$$

Therefore we have

$$\mathrm{Coker}(j_1^*-j_2^*)\cong\frac{\tau_1\cdot\pi_{12}^*H^1(W_4^1)}{j_1^*p_1^*(\eta\cdot H^1(\mathrm{Pic}^4 C))}\cong\frac{H^1(W_4^1)}{h^*H^1(\mathrm{Pic}^4 C)}\cong H^1(Q)\cong\mathbb{Q}^{12}.$$

□

Lemma 6.6. *The map $j_2^*\colon H^4(M_2)\to H^4(M_{12})$ acts as follows:*

$$\begin{aligned} j_2^*\colon j_{2*}f &\longmapsto f\cdot j_2^*e_2,\\ j_{2*}\tau_1 &\longmapsto \tau_1\cdot j_2^*e_2,\\ [\mathbb{P}_i^2] &\longmapsto f\cdot\tau_1 \text{ for } i=1,\dots,10.\end{aligned}$$

The map $j_1^*\colon H^4(M_1) = p_1^*H^4(C^{(4)})\oplus\langle\, j_{1*}f, j_{1*}\tau_1\,\rangle\to H^4(M_{12})$ acts as follows:

$$\begin{aligned} j_1^*\colon -j_{1*}f &\longmapsto f\cdot j_2^*e_2,\\ -j_{1*}\tau_1 &\longmapsto \tau_1\cdot j_2^*e_2,\\ p_1^*\omega &\longmapsto {p_1'}^*l^*\omega\in\mathbb{Q}f\cdot\tau_1,\ \forall\ \omega\in H^4(C^{(4)}).\end{aligned}\tag{6.4}$$

As a consequence, $j_k^*\colon H^4(M_k)\to H^4(M_{12})$ is surjective for $k=1,2$.

Proof. The lemma follows from the formula

$$j_k^*\circ j_{k*} = -\bigcup j_k^*e_k$$

for $k=1,2$ and the definition of $[\mathbb{P}_i^2]$ (see Notation 6.2). □

6.3. The monodromy weight filtration on H_t^4.

Proposition 6.7. *The weight filtration on $H^4(\Theta_t)$ is as follows:*

(1) $\mathrm{Gr}_k H_t^4 = 0$, *for* $k \leq 2$, *or* $k \geq 6$.

(2) $\mathrm{Gr}_5 H_t^4 \cong \mathrm{Gr}_3 H_t^4 = i_t^* \mathrm{Gr}_3 H^4 \cong \frac{H^1(W_4^1)}{h^* H^1(\mathrm{Pic}^4 C)} \cong H^1(Q) \cong \mathbb{Q}^{12}$.

(3) *There is an exact sequence*

$$0 \longrightarrow H^2(M_{12}) \xrightarrow{(-j_{1*}, j_{2*})} \mathrm{Gr}_4 H^4 \xrightarrow{i_t^*} \mathrm{Gr}_4 H_t^4 \longrightarrow 0.$$

Consequently, $\mathrm{Gr}_4 H_t^4 \cong \mathbb{Q}^{264}$ *and* $H^4(\Theta_t) \cong \mathbb{Q}^{288}$.

Proof. If $k \leq 3$, by (5.2) and (5.4),

$$\mathrm{Gr}_k H_t^4 \cong \mathrm{Gr}_k K_t^4 \oplus \cdots \oplus \mathrm{Gr}_{k-\lfloor \frac{k}{2} \rfloor} K_t^4 \cong \mathrm{Gr}_k H^4 \oplus \cdots \oplus \mathrm{Gr}_{k-\lfloor \frac{k}{2} \rfloor} H^4.$$

Therefore, the statements about $\mathrm{Gr}_k H_t^4$ for $k \leq 3$ follow immediately from the computation of the weight filtration on H^4 in Proposition 6.4.

For $k = 4$,

$$\mathrm{Gr}_4 H_t^4 \cong \mathrm{Gr}_4 K_t^4 \oplus \mathrm{Gr}_2 K_t^4 \oplus \mathrm{Gr}_0 K_t^4 \cong \mathrm{Gr}_4 K_t^4.$$

The exact sequence (5.5) becomes

$$0 \longrightarrow \mathrm{Gr}_2 K_t^2 \longrightarrow \mathrm{Gr}_{-6} H_6 \xrightarrow{\alpha} \mathrm{Gr}_4 H^4 \longrightarrow \mathrm{Gr}_4 K_t^4 \longrightarrow 0\ .$$

By Lemma 6.9 below, the image of α is equal to $(-j_{1*}, j_{2*})H^2(M_{12})$, and $(-j_{1*}, j_{2*})$ is clearly injective. Therefore (3) holds. The statements for $k \geq 5$ follow by symmetry (see Section 5.3). □

Proposition 6.8. *The induced monodromy filtration on the primal cohomology $\mathbb{K}_t \subset H_t^4$ and $\mathbb{H}_t = \mathbb{K}_t \oplus \theta H^2(A_t)$ satisfies the following:*

(1) $\mathrm{Gr}_k \mathbb{K}_t \cong \mathrm{Gr}_k H_t^4$ *for* $k = 3, 5$.

(2) *We have an exact sequence*

$$(I \oplus H^4(M_2)) \cap \mathrm{Gr}_4 H^4 \xrightarrow{i_t^*} \mathrm{Gr}_4 \mathbb{H}_t \longrightarrow 0,$$

where

$$I := p_1^*(\theta H^2(\mathrm{Pic}^4 C) \oplus \eta H^2(\mathrm{Pic}^4 C) \oplus \eta^2) \oplus \langle j_{1*} f, j_{1*}\tau_1 \rangle \subset H^4(M_1).$$

Proof. Since the family of Prym varieties A_t does not degenerate, we have $\mathrm{Gr}_6 H^6(A_t) \cong H^6(A_t)$ and $\mathrm{Gr}_5 H^6(A_t) = \mathrm{Gr}_7 H^6(A_t) = 0$. Therefore $\mathrm{Gr}_3 H^4(\Theta_4)$ and $\mathrm{Gr}_5 H^4(\Theta_t)$ map to zero under Gysin push-forward. This implies the first statement of the proposition. Now consider the commutative diagram

$$\begin{array}{ccccc} H^4(M_1 \cup M_2) & \xrightarrow{\cong} & H^4(\widetilde{\Theta}) & \xrightarrow{i_t^*} & H^4(\Theta_t) \\ \downarrow{\scriptstyle j_{0*}} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_{t*}} \\ H^6(\mathrm{Pic}^4 C) & \xrightarrow{\cong} & H^6(\mathcal{A}) & \xrightarrow{\cong} & H^6(A_t). \end{array}$$

Since the induced Gysin map on the graded piece

$$\mathrm{Gr}_4 H^4(M_1 \cup M_2) \xrightarrow{j_{0*}} \mathrm{Gr}_6 H^6(\mathrm{Pic}^4 C) \cong H^6(\mathrm{Pic}^4 C)$$

sends $(I \oplus H^4(M_2)) \cap \mathrm{Gr}_4 H^4$ to the subspace $\theta^2 H^2(\mathrm{Pic}^4 C)$ and the bottom horizontal maps are isomorphisms, we see that i_t^* sends $(I \oplus H^4(M_2)) \cap \mathrm{Gr}_4 H^4$ into $\mathrm{Gr}_4 \mathbb{H}_t$. By Proposition 6.7 (3), the kernel of i_t^* is 3-dimensional; therefore i_t^* sends $(I \oplus H^4(M_2)) \cap \mathrm{Gr}_4 H^4$ onto $\mathrm{Gr}_4 \mathbb{H}_t$ by a simple dimension count. □

It remains to describe the 3-dimensional image of α in (5.5). Recall from the definition of the spectral sequence in Section 5.2 that $\mathrm{Gr}_4 H^4$ fits in the exact sequence

$$0 \longrightarrow \mathrm{Gr}_4 H^4 \longrightarrow H^4(M_1) \oplus H^4(M_2) \xrightarrow{d_1^{0,4} = j_1^* - j_2^*} H^4(M_{12}) \longrightarrow 0.$$

The composition of the natural map

$$H^2(M_{12}) \xrightarrow{(-j_{1*}, j_{2*})} H^4(M_1) \oplus H^4(M_2)$$

with $d_1^{0,4}$ is zero; therefore $(-j_{1*}, j_{2*})$ factors through $\mathrm{Gr}_4 H^4$:

$$\begin{array}{ccccccccc} & & & & H^2(M_{12}) & & & & \\ & & & \swarrow{\scriptstyle (-j_{1*},j_{2*})} & \downarrow{\scriptstyle (-j_{1*},j_{2*})} & & & & \\ 0 & \longrightarrow & \mathrm{Gr}_4 H^4 & \longrightarrow & H^4(M_1) \oplus H^4(M_2) & \xrightarrow{d_1^{0,4}} & H^4(M_{12}) & \longrightarrow & 0. \end{array}$$

Lemma 6.9. *The image of* $\alpha\colon \mathrm{Gr}_{-6} H_6 \longrightarrow \mathrm{Gr}_4 H^4$ *is equal to the image of*

$$H^2(M_{12}) \xrightarrow{(-j_{1*}, j_{2*})} \mathrm{Gr}_4 H^4 \subset H^4(M_1) \oplus H^4(M_2).$$

Proof. We have the isomorphism $\mathrm{Gr}_{-6} H_6^\vee \cong \mathrm{Gr}_6 H^6$, and the latter fits into the exact sequence

$$0 \longrightarrow \mathrm{Gr}_6 H^6 \longrightarrow H^6(M_1) \oplus H^6(M_2) \xrightarrow{j_1^* - j_2^*} H^6(M_{12}) \longrightarrow 0$$

whose Poincaré dual is

$$0 \longrightarrow H^0(M_{12}) \xrightarrow{(j_{1*},-j_{2*})} H^2(M_1)\oplus H^2(M_2) \longrightarrow (\mathrm{Gr}_6\, H^6)^\vee \longrightarrow 0.$$

The map α is induced by

$$H_6(\widetilde{\Theta}) \xrightarrow{\mathrm{PD}} H^4(\widetilde{\Theta},\partial\widetilde{\Theta}) \longrightarrow H^4(\widetilde{\Theta}) \cong H^4(M_1\cup M_2)\ .$$

On the graded level,

$$\alpha\colon\ \mathrm{Gr}_{-6}\, H_6 = (\mathrm{Gr}_5\, H^6)^* = \frac{H^2(M_1)\oplus H^2(M_2)}{H^0(M_{12})} \longrightarrow \mathrm{Gr}_4\, H^4$$

is induced by the map

$$\begin{array}{rcl} H^2(M_1)\oplus H^2(M_2) & \longrightarrow & \mathrm{Gr}_4\, H^4 \subset H^4(M_1)\oplus H^4(M_2) \\ (\gamma_1,\gamma_2) & \longmapsto & (-j_{1*}(j_1^*\gamma_1 - j_2^*\gamma_2),\ j_{2*}(j_1^*\gamma_1 - j_2^*\gamma_2)). \end{array}$$

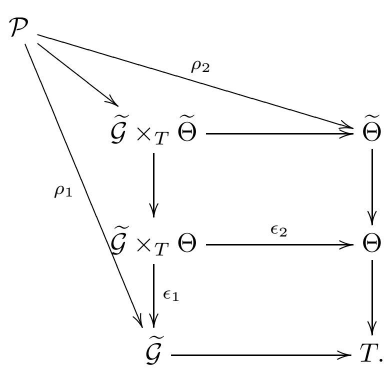
Since the map

$$j_1^* - j_2^*\colon\ H^2(M_1)\oplus H^2(M_2) \longrightarrow H^2(M_{12})$$

is surjective, the image of α is equal to the 3-dimensional image of $H^2(M_{12})$ via $(-j_{1*}, j_{2*})$. □

7. The semistable reduction of the fiber product

7.1. We need to construct a semistable reduction for the fiber product $\widetilde{\mathcal{G}}\times_T\widetilde{\Theta}$. The central fibers of $\widetilde{\mathcal{G}}$ and $\widetilde{\Theta}$ are described in Section 4.1 and Proposition 4.2 respectively. We follow the notation there. The total space of $\widetilde{\mathcal{G}}\times_T\widetilde{\Theta}$ is singular along $X_{kp}\times M_{12}$ and $X_{kq}\times M_{12}$ for $k=1,2$. The semistable reduction is simply the blow-up $\mathcal{P}$ of $\widetilde{\mathcal{G}}\times_T\widetilde{\Theta}$ along the union of $W_1\times M_1$ and $W_2\times M_1$, and it sits in the commutative diagram with Cartesian squares



Proposition 7.1. *The blow-up $\mathcal{P}$ of $\widetilde{\mathcal{G}} \times_T \widetilde{\Theta}$ along the union of $W_1 \times M_1$ and $W_2 \times M_1$ is a semistable family whose central fiber $\mathcal{P}_0$ has eight components:*

(1) *For $k = 1, 2$, the total transform $\widetilde{W_k \times M_1}$ of $W_k \times M_1$, which is isomorphic to the blow-up of $W_k \times M_1$ along $X_{kp} \times M_{12} \cup X_{kq} \times M_{12}$.*
(2) *The proper transforms $\widetilde{P_1 \times M_2}$ and $\widetilde{P_2 \times M_2}$ of $P_1 \times M_2$ and $P_2 \times M_2$ respectively, which are isomorphic to the blow-ups of $P_1 \times M_2$ and $P_2 \times M_2$ along $X_{1p} \times M_{12} \cup X_{2q} \times M_{12}$ and $X_{1q} \times M_{12} \cup X_{2p} \times M_{12}$ respectively.*
(3) *The proper transforms of $P_1 \times M_1$, $P_2 \times M_1$, $W_1 \times M_2$ and $W_2 \times M_2$, which are unchanged under the blow-up.*

Proof. We check locally that this is indeed a semistable reduction. Locally, the total space of the fiber product near, say, $X_{1p} \times M_{12}$ is isomorphic to the product of an affine space and

$$\text{Spec } \frac{\mathbb{C}[x, y, z, w, t]}{(xy - t, zw - t)} \cong \text{Spec } \frac{\mathbb{C}[x, y, z, w]}{xy - zw}. \tag{7.1}$$

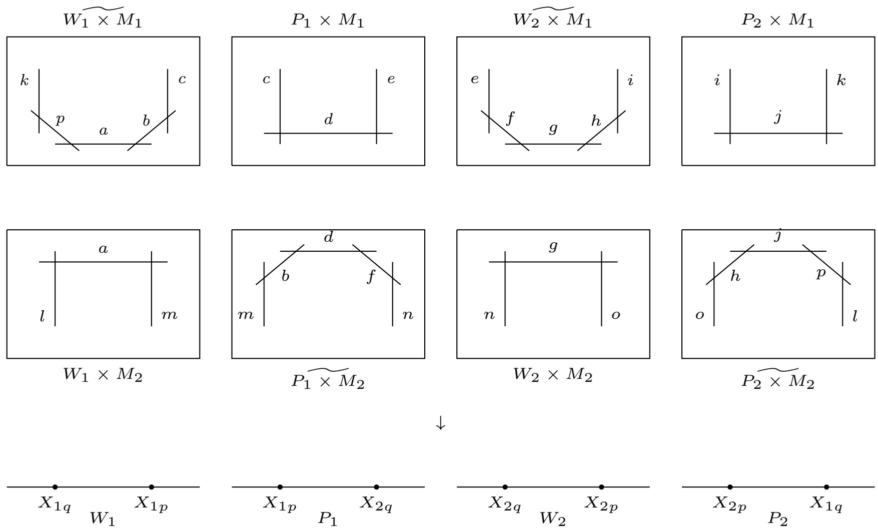
In the above local coordinates, $X_{1p} \times M_{12}$ is defined by the ideal (x, y, z, w), and blowing up $\widetilde{\mathcal{G}} \times_T \widetilde{\Theta}$ along $W_1 \times M_1$ amounts to blowing up (7.1) along the ideal (x, z). Let X, Z be the corresponding homogeneous coordinates in the blow-up. By symmetry, it is sufficient to check the result on the chart $\{ X \neq 0 \}$. Here $\mathcal{P}$ is isomorphic to the product of an affine space and

$$\text{Spec } \frac{\mathbb{C}[x, y, Z, w]}{y - Zw} \cong \text{Spec } \mathbb{C}[x, Z, w],$$

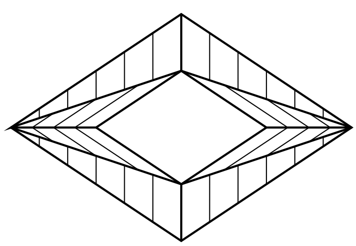
which is smooth. The central fiber in this chart is given by $t = xy = xZw$, which is a simple normal crossing divisor.

The other assertions about the components of the central fiber are immediate. □

7.2. The eight components of the central fiber $\mathcal{P}_0$ meet as follows:

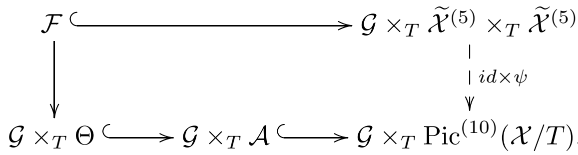


The lines with the same label indicate the subvarieties that are glued together to form the double loci of the central fiber. The horizontal lines represent the loci that project onto M_{12} via ρ_2, and the vertical lines the loci that project onto either X_{kp} or X_{kq} by ρ_1. The slanted lines represent exceptional loci: these are $\mathbb{P}^1$-bundles over the products $X_{kp} \times M_{12}$ and $X_{kq} \times M_{12}$, hence are contracted by ρ_1 and ρ_2. The dual graph of the central fiber is

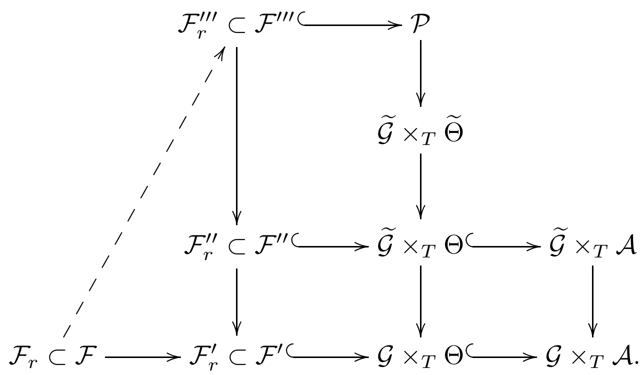


The four vertices of the inside square correspond to the four components in the top row of the previous picture and the four vertices of the outside square to the bottom row. The striped triangles correspond to triple intersections in the central fiber.

7.3. Let $\mathrm{Pic}^{(10)}(\mathcal{X}/T)$ be the (noncompact) relative Picard scheme whose central fiber is $\mathrm{Pic}^{6,4}(\widetilde{C}_{pq})$. There is a rational map $\psi \colon \widetilde{\mathcal{X}}^{(5)} \times_T \widetilde{\mathcal{X}}^{(5)} \dashrightarrow \mathrm{Pic}^{(10)}(\mathcal{X}/T)$ which is regular on the fibers over $t \neq 0$. We will show in Proposition 9.1 that the rational map $id \times \psi \colon \mathcal{G} \times_T \widetilde{\mathcal{X}}^{(5)} \times_T \widetilde{\mathcal{X}}^{(5)} \dashrightarrow \mathcal{G} \times_T \mathrm{Pic}^{(10)}(\mathcal{X}/T)$ restricted to $\mathcal{F} \subset \mathcal{G} \times_T \widetilde{\mathcal{X}}^{(5)} \times_T \widetilde{\mathcal{X}}^{(5)}$ is regular. In other words, we have the following commutative diagram:



Notation 7.2. Denote by $\mathcal{F}', \mathcal{F}'_r$ the images of $\mathcal{F}, \mathcal{F}_r$ in $\mathcal{G} \times_T \Theta$, and $\mathcal{F}'', \mathcal{F}''_r$ and $\mathcal{F}''', \mathcal{F}'''_r$ the proper transforms of $\mathcal{F}'$ and $\mathcal{F}'_r$ in $\widetilde{\mathcal{G}} \times_T \Theta$, $\mathcal{P}$ respectively. We summarize the relations between the various spaces in the diagram below:



8. Abel-Jacobi maps on the generic and special fibers: Outline of the proof of Theorem 3

The Abel-Jacobi map AJ on the total space is the composition

$$H^2(\widetilde{\mathcal{G}}) \xrightarrow{\rho_1^*} H^2(\mathcal{P}) \xrightarrow{\cup[\mathcal{F}'''_r]} H^8(\mathcal{P}) \xrightarrow{\rho_{2*}} H^4(\widetilde{\Theta}),$$

where the Gysin map ρ_{2*} is defined as

$$H^8(\mathcal{P}) \overset{\mathrm{PD}}{\cong} H^6_c(\mathcal{P})^\vee \xrightarrow{(\rho_2^*)^\vee} H^6_c(\widetilde{\Theta})^\vee \overset{\mathrm{PD}}{\cong} H^4(\widetilde{\Theta}),$$

where PD denotes Poincaré duality. As explained in Section 5.4, there exist canonical mixed Hodge structures on $H^6_c(\mathcal{P})$ and $H^6_c(\widetilde{\Theta})$, such that ρ_2^* (and therefore $(\rho_2^*)^\vee$) is a morphism of mixed Hodge structures. Thus the Abel-Jacobi map AJ, as a composition of such, is also a morphism of mixed Hodge structures.

By functoriality of the morphisms involved, we have a commutative diagram

$$\begin{array}{ccc} H^2(\widetilde{\mathcal{G}}) & \longrightarrow & H^2(G_t) \\ \downarrow{\scriptstyle \mathrm{AJ}} & & \downarrow{\scriptstyle \mathrm{AJ}_t} \\ H^4(\widetilde{\Theta}) & \longrightarrow & H^4(\Theta_t), \end{array}$$

where the images of the horizontal maps are the monodromy invariant parts of the cohomology groups of G_t and Θ_t.

8.1. The map AJ **on the** E_1 **terms.** The maps ρ_1^*, $\cup[\mathcal{F}_r''']$ and ρ_2^* are defined on the E_1 terms of the spectral sequences in Section 5 and commute with the differentials d_1.

For $k = 0, 1$, the map ρ_1^* on the E_1 terms is

$$\begin{array}{ccc} {}_{\widetilde{\mathcal{G}}}E_1^{k,2-k} & \xrightarrow{\rho_1^*} & {}_{\mathcal{P}}E_1^{k,2-k} \\ \| & & \| \\ H^{2-k}(\widetilde{G}_0^{[k]}) & \xrightarrow{\rho_1^*} & H^{2-k}(\mathcal{P}_0^{[k]}). \end{array}$$

If, for a stratum S in $\mathcal{P}_0^{[k]}$, $\rho_1(S)$ is not contained in $\widetilde{G}_0^{[k]}$, then the projection of ρ_1^* onto the summand $H^{2-k}(S) \subset H^{2-k}(\mathcal{P}_0^{[k]})$ is zero (some components of $\mathcal{P}_0^{[1]}$ map onto components in $\widetilde{G}_0^{[0]}$; cf. Section 7).

Cup-product with $[\mathcal{F}_r''']$ induces the horizontal maps

$$\begin{array}{ccc} {}_{\mathcal{P}}E_1^{k,2-k} & \xrightarrow{\cup[\mathcal{F}_r''']} & {}_{\mathcal{P}}E_1^{k,8-k} \\ \| & & \| \\ H^{2-k}(\mathcal{P}_0^{[k]}) & \xrightarrow{\cup[\mathcal{F}_r''']} & H^{8-k}(\mathcal{P}_0^{[k]}), \end{array}$$

where the lower horizontal map is cup-product with the cycle class of the *scheme-theoretic* intersection of $\mathcal{F}_r'''$ with each component in $\mathcal{P}_0^{[k]}$.

The map ρ_2^* on cohomology with compact supports is

$$\begin{array}{ccc} {}_{\widetilde{\Theta}}E_{1,c}^{-k,k+6} & \xrightarrow{\rho_2^*} & {}_{\mathcal{P}}E_{1,c}^{-k,k+6} \\ \| & & \| \\ H^{4-k}(\widetilde{\Theta}_0^{[k]}) & \xrightarrow{\rho_2^*} & H^{4-k}(\mathcal{P}_0^{[k]}). \end{array}$$

Similarly to the case of ρ_1^* above, we only pull back to the strata of $\mathcal{P}_0^{[k]}$ which map to $\widetilde{\Theta}_0^{[k]}$. Thus the dual map $\rho_{2*} = (\rho_2^*)^\vee$ is induced by the usual Gysin maps between the relevant strata:

$$\begin{array}{ccc} {}_{\mathcal{P}}E_1^{k,8-k} & \xrightarrow{\rho_{2*}} & {}_{\widetilde{\Theta}}E_1^{k,4-k} \\ \| & & \| \\ H^{8-k}(\mathcal{P}_0^{[k]}) & \xrightarrow{\rho_{2*}} & H^{4-k}(\widetilde{\Theta}_0^{[k]}). \end{array}$$

To compute the graded parts of the Abel-Jacobi maps (see (0.1) and (0.2)), we first compute the Abel-Jacobi map AJ^k on the E_1 terms for $k = 0, 1$:

$$\mathrm{AJ}^k\colon\ H^{2-k}(\widetilde{G}_0^{[k]}) \xrightarrow{\rho_1^*} H^{2-k}(\mathcal{P}_0^{[k]}) \xrightarrow{\cup[\mathcal{F}_r''']} H^{8-k}(\mathcal{P}_0^{[k]}) \xrightarrow{\rho_{2*}} H^{4-k}(\widetilde{\Theta}_0^{[k]}),$$

then pass to the E_2 terms of the corresponding spectral sequences.

8.2. Proof of the main theorem. Notation as in Section 6.1. We divide the proof of Theorem 3 into four propositions.

For the Abel-Jacobi map on the E_1 terms, we write

$$\mathrm{AJ}^0 =: (\mathrm{AJ}_1^0, \mathrm{AJ}_2^0)\colon H^2(\widetilde{G}_0^{[0]}) \to H^4(\widetilde{\Theta}_0^{[0]}) = H^4(M_1) \oplus H^4(M_2).$$

We have

Proposition 8.1. *The image of the map* $\mathrm{AJ}_1^0\colon H^2(\widetilde{G}_0^{[0]}) \to H^4(M_1)$ *contains the subspace*

$$I := p_1^*(\theta H^2(\operatorname{Pic}^4 C) \oplus \eta H^2(\operatorname{Pic}^4 C) \oplus \eta^2) \oplus \langle\, j_{1*}f, j_{1*}\tau_1 \,\rangle$$

modulo $\langle\, j_{1*}f, j_{1*}\tau_1 \,\rangle \oplus p_1^*(\theta H^2(\operatorname{Pic}^4 C))$.

Proposition 8.2. *The map* $\mathrm{AJ}_2^0\colon H^2(\widetilde{G}_0^{[0]}) \to H^4(M_2)$ *is surjective modulo* $\langle\, j_{2*}f, j_{2*}\tau_1 \,\rangle$.

For AJ^1, we have

Proposition 8.3. *The image of* $\mathrm{AJ}^1\colon H^1(\widetilde{G}_0^{[1]}) \to H^3(\widetilde{\Theta}_0^{[1]}) = H^3(M_{12})$ *contains* $\tau_1 \cdot \pi_{12}^* H^1(W_4^1)$.

Next we pass to the Abel-Jacobi map on the E_2 terms.

Proposition 8.4. *The image of the restriction of* $\mathrm{AJ}^0 = (\mathrm{AJ}_1^0, \mathrm{AJ}_2^0)$ *to*

$$\operatorname{Gr}_2 H^2(\widetilde{G}) = \operatorname{Ker}(H^2(\widetilde{G}_0^{[0]}) \xrightarrow{d_1} H^2(\widetilde{G}_0^{[1]}))$$

contains

$$(I \oplus H^4(M_2)) \cap \operatorname{Gr}_4 H^4(\widetilde{\Theta}) \text{ modulo } (-j_{1*}, j_{2*})H^2(M_{12}) + (p_1^*(\theta H^2(\operatorname{Pic}^4 C)), 0).$$

Assuming the above four propositions, we can prove our main theorem.

Proof of Theorem 3. Identifying $H^4(A_t)$ with a subspace of $H^4(\Theta_t)$ via pull-back, we have $H^4(\Theta_t) = (\mathbb{K}_t \otimes \mathbb{Q}) \oplus H^4(A_t)$, and, since A_t does not degenerate,

$$\operatorname{Gr}_3 H^4(\Theta_t) = \operatorname{Gr}_3(\mathbb{K}_t \otimes \mathbb{Q})$$

and

$$\mathrm{Gr}_4 H^4(\Theta_t) = \mathrm{Gr}_4(\mathbb{K}_t \otimes \mathbb{Q}) \oplus H^4(A_t).$$

Consider the commutative diagram

$$\begin{array}{ccc} H^2(\widetilde{\mathcal{G}}) & \xrightarrow{i_t^*} & H^2(G_t) \\ \downarrow{\scriptstyle \mathrm{AJ}} & & \downarrow{\scriptstyle \mathrm{AJ}_t} \\ H^4(\widetilde{\Theta}) & \xrightarrow{i_t^*} & H^4(\Theta_t). \end{array}$$

Proposition 8.3 implies that the image of AJ_t sends $\mathrm{Gr}_1 H^2(G_t) = i_t^* \mathrm{Gr}_1 H^2(\widetilde{\mathcal{G}})$ surjectively to $\mathrm{Gr}_3 H^4(\Theta_t) = i_t^* \mathrm{Gr}_3 H^4(\widetilde{\Theta}) \cong \frac{H^1(W_4^1)}{H^1(\mathrm{Pic}^4 C)} = H^1(Q)$. Since the logarithm of the monodromy operator N induces an isomorphism from $\mathrm{Gr}_5 H^4(\Theta_t)$ to $\mathrm{Gr}_3 H^4(\Theta_t)$ and from $\mathrm{Gr}_3 H^2(G_t)$ to $\mathrm{Gr}_1 H^2(G_t)$, we conclude that AJ_t sends $\mathrm{Gr}_3 H^2(G_t)$ surjectively to $\mathrm{Gr}_5 H^4(\Theta_t)$.

Next, by Lemma 6.9, the ambiguity $(-j_{1*}, j_{2*})H^2(M_{12})$ restricts to zero under i_t^*. Therefore, by Propositions 6.8 and 8.4, the image of $\mathrm{Gr}_2 H^2(G_t)$ by AJ_t contains $\mathrm{Gr}_4(\mathbb{H}_t \otimes \mathbb{Q})$ modulo $\theta_t H^2(A_t)$.

Combining the above, we see that the image of AJ_t contains $\mathbb{H}_t \otimes \mathbb{Q}$ modulo $\theta_t H^2(A_t)$. Since, as we observed earlier, $\theta_t H^2(A_t)$ is always contained in the image of $H^2(\Theta_t \cap \Theta_{ta})$ for $a \in A_t$ general, the theorem follows (here Θ_{ta} is the translate of Θ_t by a). □

9. The cycles at time zero: Before resolving the family of theta divisors

9.1. The central fiber F_0. We list the intersections of the central fiber F_0 of $\mathcal{F}$ with each component $W_k \times (C_1^{(d_1)} \times C_2^{(d_2)}) \times (C_1^{(e_1)} \times C_2^{(e_2)})$ in Tables 2 and 3. The left column lists the ambient spaces of all possible bidegrees. The middle column gives the conditions defining the cycles F_0 in each ambient space.

For each pair of bidegrees (d_1, d_2) and (e_1, e_2), we define a morphism

$$\psi_{(d_1,d_2)(e_1,e_2)} \colon F_0 \cap \Big(W_k \times (C_1^{(d_1)} \times C_2^{(d_2)}) \times (C_1^{(e_1)} \times C_2^{(e_2)})\Big) \longrightarrow \Theta_0 \subset \mathrm{Pic}^4 C$$
$$(L, D_{d_1}, D_{d_2}, D'_{e_1}, D'_{e_2}) \longmapsto \mathcal{O}_C(D_{d_2} + D'_{e_2} - m(p+q)),$$

where m is the integer such that $d_2 + e_2 = 4 + 2m$. These morphisms are listed case by case in the rightmost column of Tables 2 and 3.

TABLE 2. Cycles in $W_1 \times C_{pq}^{(5)} \times C_{pq}^{(5)}$

Ambient Spaces	F_0	Image under ψ
$(L, D_4, a, D_4', a') \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(4)} \times C_2)$	$\begin{cases} D_4 + a \in \lvert L\rvert \\ D_4' + a' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(a + a' + p + q)$
$(L, D_4, a, D_2', D_3') \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(2)} \times C_2^{(3)})$	$\begin{cases} D_4 + a \in \lvert L\rvert \\ D_2' + D_3' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(a + D_3')$
$(L, D_4, a, D_5') \in W_1 \times (C_1^{(4)} \times C_2) \times C_2^{(5)}$	$\begin{cases} D_4 + a \in \lvert L\rvert \\ D_5 \in \lvert L'\rvert \end{cases}$	$K_C(-D_4)$
$(L, D_2, D_3, D_4', a') \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times (C_1^{(4)} \times C_2)$	$\begin{cases} D_2 + D_3 \in \lvert L\rvert \\ D_3' + a' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(D_3 + a')$
$(L, D_2, D_3, D_2', D_3') \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times (C_1^{(2)} \times C_2^{(3)})$	$\begin{cases} D_2 + D_3 \in \lvert L\rvert \\ D_2' + D_3' \in \lvert L'\rvert \end{cases}$	$K_C(-D_2 - D_2')$
$(L, D_2, D_3, D_5) \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times C_2^{(5)}$	$\begin{cases} D_2 + D_3 \in \lvert L\rvert \\ D_5 \in \lvert L'\rvert \end{cases}$	$K_C(-D_2 - p - q)$
$(L, D_5, D_4', a') \in W_1 \times C_2^{(5)} \times (C_1^{(4)} \times C_2)$	$\begin{cases} D_5 \in \lvert L\rvert \\ D_4' + a' \in \lvert L'\rvert \end{cases}$	$K_C(-D_4')$
$(L, D_5, D_2', D_3') \in W_1 \times C_2^{(5)} \times (C_1^{(2)} \times C_2^{(3)})$	$\begin{cases} D_5 \in \lvert L\rvert \\ D_2' + D_3' \in \lvert L'\rvert \end{cases}$	$K_C(-D_2' - p - q)$
$(L, D_5, D_5') \in W_1 \times C_2^{(5)} \times C_2^{(5)}$	$\begin{cases} D_5 \in \lvert L\rvert \\ D_5' \in \lvert L'\rvert \end{cases}$	$K_C(-2p - 2q)$

TABLE 3. Cycles in $W_2 \times C_{pq}^{(5)} \times C_{pq}^{(5)}$

Ambient Spaces	F_0	Image under ψ
$(L, D_5, D_5') \in W_2 \times C_1^{(5)} \times C_1^{(5)}$	$\begin{cases} D_5 \in \lvert L\rvert \\ D_5' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(2p + 2q)$
$(L, D_5, D_3', D_2') \in W_2 \times C_1^{(5)} \times (C_1^{(3)} \times C_2^{(2)})$	$\begin{cases} D_5 \in \lvert L\rvert \\ D_3' + D_2' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(D_2' + p + q)$
$(L, D_5, a', D_4') \in W_2 \times C_1^{(5)} \times (C_1 \times C_2^{(4)})$	$\begin{cases} D_5 \in \lvert L\rvert \\ a' + D_4' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(D_4')$
$(L, D_3, D_2, D_5') \in W_2 \times (C_1^{(3)} \times C_2^{(2)}) \times C_1^{(5)}$	$\begin{cases} D_3 + D_2 \in \lvert L\rvert \\ D_5' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(D_2 + p + q)$
$(L, D_3, D_2, D_3', D_2') \in W_2 \times (C_1^{(3)} \times C_2^{(2)}) \times (C_1^{(3)} \times C_2^{(2)})$	$\begin{cases} D_3 + D_2 \in \lvert L\rvert \\ D_3' + D_2' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(D_2 + D_2')$
$(L, D_3, D_2, a', D_4') \in W_2 \times (C_1^{(3)} \times C_2^{(2)}) \times (C_1 \times C_2^{(4)})$	$\begin{cases} D_3 + D_2 \in \lvert L\rvert \\ a' + D_4' \in \lvert L'\rvert \end{cases}$	$K_C(-D_3 - a')$
$(L, a, D_4, D_5') \in W_2 \times (C_1 \times C_2^{(4)}) \times C_1^{(5)}$	$\begin{cases} a + D_4 \in \lvert L\rvert \\ D_5' \in \lvert L'\rvert \end{cases}$	$\mathcal{O}_C(D_4)$
$(L, a, D_4, D_3', D_2') \in W_2 \times (C_1 \times C_2^{(4)}) \times (C_1^{(3)} \times C_2^{(2)})$	$\begin{cases} a + D_4 \in \lvert L\rvert \\ D_3' + D_2' \in \lvert L'\rvert \end{cases}$	$K_C(-a - D_3')$
$(L, a, D_4, a', D_4') \in W_2 \times (C_1 \times C_2^{(4)}) \times (C_1 \times C_2^{(4)})$	$\begin{cases} a + D_4 \in \lvert L\rvert \\ a' + D_4' \in \lvert L'\rvert \end{cases}$	$K_C(-a - a' - p - q)$

9.2. The morphism to Θ_0.

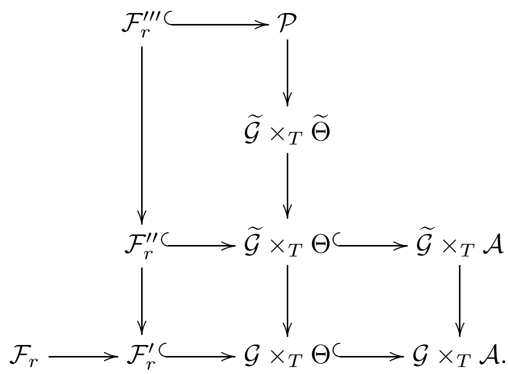
Proposition 9.1. *The rational map*

$$id \times \psi\colon \mathcal{G} \times_T \widetilde{\mathcal{X}}^{(5)} \times_T \widetilde{\mathcal{X}}^{(5)} \dashrightarrow \mathcal{G} \times_T \mathrm{Pic}^{(10)}(\mathcal{X}/T)$$

extends to a morphism when restricted to $\mathcal{F} \subset \mathcal{G} \times_T \widetilde{\mathcal{X}}^{(5)} \times_T \widetilde{\mathcal{X}}^{(5)}$ *(see Section 7.3 for the notation).*

Proof. We need to extend the rational map ψ to the central fiber F_0 of $\mathcal{F}$. As explained in Section 2.4, the natural extension of the map ψ to a general point of $W_k \times (C_1^{(d_1)} \times C_2^{(d_2)}) \times (C_1^{(e_1)} \times C_2^{(e_2)})$ is given by $\psi_{(d_1,d_2)(e_1,e_2)}$. Therefore we need to show that the morphisms $\psi_{(d_1,d_2)(e_1,e_2)}$ coincide on the intersection of F_0 with the overlaps of the different components of $G_0 \times \widetilde{C}_{pq}^{(5)} \times \widetilde{C}_{pq}^{(5)}$. For instance, a point $(p + g_4^1, D_2, D_3 = B_2 + p, D_4' = B_3' + q, a') \in F_0 \cap W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times (C_1^{(4)} \times C_2)$ is identified with $(q+g_4^1, D_2+q, B_2, B_3', a'+p) \in F_0 \cap W_2 \times (C_1^{(3)} \times C_2^{(2)}) \times (C_1^{(3)} \times C_2^{(2)})$. The images under $\psi_{(2,3)(4,1)}$ and $\psi_{(3,2)(3,2)}$ are both equal to $\mathcal{O}_C(B_2 + a' + p)$. Therefore all the $\psi_{(d_1,d_2)(e_1,e_2)}|_{F_0}$ glue together, and we obtain a morphism from F_0 to Θ_0. □

Recall that we have a tower of blow-ups and algebraic cycles in each blow-up:



Denote by $F_{(d_1,d_2)(e_1,e_2)}$ the intersection of F_{r_0} with $W_k \times (C_1^{(d_1)} \times C_2^{(d_2)}) \times (C_1^{(e_1)} \times C_2^{(e_2)})$, and by $\lambda := (\lambda_1, \lambda_2)\colon F_{r_0} \to G_0 \times \Theta_0$ the restriction of $id \times \psi$ to F_{r_0}.

9.3. The cycle F''_{r_0}. Recall that $\widetilde{G}_0$ has four components W_1, W_2, P_1, P_2, where P_k is a $\mathbb{P}^1$ bundle over X_{kp} for $k = 1, 2$ (see Section 4.1). We use the notation $F''_{r_0}|_{W_k \times \Theta_0}$, $F''_{r_0}|_{P_k \times \Theta_0}$ to denote the components of F''_{r_0} which lie in $W_k \times \Theta_0$, $P_k \times \Theta_0$ respectively.

Proposition 9.2.

(1) *The cycle $F''_{r_0}|_{W_1 \times \Theta_0}$ is the push-forward under λ of*

$$F_{r_0 1} := F_{(4,1)(4,1)} \amalg F_{(4,1)(2,3)} \amalg F_{(2,3)(4,1)} \amalg F_{(2,3)(2,3)}.$$

(2) *The cycle $F''_{r_0}|_{W_2 \times \Theta_0}$ is the push-forward of*

$$F_{r_0 2} := F_{(5,0)(3,2)} \amalg F_{(5,0)(1,4)} \amalg F_{(3,2)(3,2)} \amalg F_{(3,2)(1,4)} \amalg F_{(1,4)(3,2)} \amalg F_{(1,4)(1,4)}.$$

(3) *The cycle* $F''_{r_0}|_{P_k \times \Theta_0}$ *is the image of the fiber product*

$$\begin{array}{ccc} F_{r_0}|_{P_k} & \longrightarrow & F_{r_0}|_{X_{kp}} \\ \downarrow & & \downarrow \\ P_k & \longrightarrow & X_{kp}, \end{array}$$

where $F_{r_0}|_{P_k}$ *maps to* P_k *via projection and to* Θ_0 *via* λ_2*.*

Proof. Since any component of F_0 with bidegree $(0,5)+(e_1,e_2)$ does not intersect F_{r_0}, we see that $F_{(0,5)(e_1,e_2)}$ is empty. From Tables 2 and 3, we see that $F_{(4,1)(0,5)}$, $F_{(2,3)(0,5)}$ and $F_{(d_1,d_2)(5,0)}$ are contracted by λ and their image by λ is contained in the closure of the image of cycles of other bidegrees. For other bidegrees, λ is generically injective on any irreducible component. This proves the first two statements. The third statement follows immediately from the construction of $\mathcal{F}''_{r_0}$. □

10. The cycles at time zero: After resolving the family of theta divisors

10.1. The cycle $\mathcal{F}'''_r$ is the proper transform of $\mathcal{F}''_r$ under

$$\mathcal{P} \longrightarrow \widetilde{\mathcal{G}} \times_T \widetilde{\Theta} \longrightarrow \widetilde{\mathcal{G}} \times_T \Theta$$

where the arrow on the right is the blow-up of $\widetilde{\mathcal{G}} \times_T \Theta$ along $\widetilde{G}_0 \times W_4^1$ and the arrow on the left, which is a small resolution, is the blow-up of $\widetilde{\mathcal{G}} \times_T \widetilde{\Theta}$ along $\amalg_k(W_k \times M_1)$. The central fiber of $\mathcal{P}$ has eight components (see Section 7), where $\widetilde{W_k \times M_1}$ and $P_k \times M_1$ are the main components and $W_k \times M_2$, $\widetilde{P_k \times M_2}$ are the exceptional components.

Proposition 10.1. $F'''_{r_0}|_{\widetilde{W_k \times M_1}}$ *is the proper transform of* $F''_{r_0}|_{W_k \times \Theta_0}$ *under the birational morphism*

$$\widetilde{W_k \times M_1} \longrightarrow W_k \times M_1 \xrightarrow{(id,p_1)} W_k \times C^{(4)} \xrightarrow{(id,\phi)} W_k \times \Theta_0.$$

Proof. The inverse of the birational morphism

$$\widetilde{W_k \times M_1} \longrightarrow W_k \times M_1 \longrightarrow W_k \times \Theta_0$$

is defined on the open subset $(W_k \setminus (X_p \cup X_q)) \times (\Theta_0 \setminus W_4^1)$. This open subset contains an open dense subset of $F''_{r_0}|_{W_k \times \Theta_0}$. □

10.2. The center of the blow-up. Next we study $F'''_{r_0}|_{W_k\times M_2}$, which is the scheme-theoretic intersection of $\mathcal{F}'''_r$ with the exceptional divisor $W_k \times M_2$. So $F'''_{r_0}|_{W_k\times M_2}$ is the projectivized normal cone to the scheme-theoretic intersection $F''_{r_0} \cap (W_k \times W_4^1) \subset F''_{r_0}|_{W_k\times\Theta_0}$ in $\mathcal{F}''_r$. We first study the center of the blow-up.

By Proposition 9.2, $F''_{r_0}|_{W_k\times\Theta_0}$ is the image of

$$F_{r_0k} \xrightarrow{(\lambda_1,\lambda_2)} F''_{r_0}|_{W_k\times\Theta_0} \subset W_k \times \Theta_0 .$$

Denote by $Z_k \subset F_{r_0k}$ the inverse image scheme of $F''_{r_0} \cap (W_k \times W_4^1) \subset F''_{r_0}|_{W_k\times\Theta_0}$ and put $Z := Z_1 \cup Z_2$ and $Z_{(d_1,d_2)(e_1,e_2)} = Z \cap F_{(d_1,d_2)(e_1,e_2)}$. Then Z_k maps onto $W_4^1 \subset \Theta_0$ by λ_2, and we have the Cartesian diagram

(10.1)
$$\begin{array}{ccc} Z_k & \longrightarrow & W_4^1 \\ \downarrow & & \downarrow \\ F_{r_0k} & \xrightarrow{\lambda_2} & \Theta_0. \end{array}$$

Proposition 10.2. *The cycle Z_1 is 1-dimensional and, for $s_4^1 \neq g_i$ or h_i, the fiber $\lambda_2^{-1}(s_4^1) \cap F_{r_01}$ is finite. For $i = 1,\dots 5$, the fiber $\lambda_2^{-1}(g_i) \cap F_{r_01}$ is 1-dimensional (modulo finitely many points) and its support is listed in Table 4.*

TABLE 4

Ambient Spaces	*Support of* $\lambda_2^{-1}(g_i) \cap F_{r_01}$
$(L, D_4, a, D'_4, a') \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(4)} \times C_2)$	$\begin{cases} a + a' + p + q \equiv g_i \\ h^0(L - a - r_0) > 0 \\ D_4 \equiv L - a,\, D'_4 \equiv L' - a' \end{cases}$
$(L, D_4, a, D'_2, D'_3) \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(2)} \times C_2^{(3)})$	$\begin{cases} D'_3 + a \equiv g_i \\ L = a + g_j, j \neq i, a \in C \\ r_0 \leq D_4 \equiv g_j \\ D'_2 \equiv h_i - (g_i - p - q) \end{cases}$
$(L, D_4, a, D'_2, D'_3) \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(2)} \times C_2^{(3)})$	$\begin{cases} D'_3 + a \equiv g_i \\ L = h_i + p + q - c, c \in C \\ h^0(L - r_0 - a) > 0, D_4 \equiv L - a \\ D'_2 = a + c \end{cases}$
$(L, D_2, D_3, D'_4, a') \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times (C_1^{(4)} \times C_2)$	$\begin{cases} D_3 + r_0 \equiv g_i \\ L = c + g_i, c \in C \\ a' = r_0, D'_4 \equiv h_i + p + q - c - r_0 \\ D_2 = r_0 + c \end{cases}$
$(L, D_2, D_3, D'_4, a') \in W_1 \times (C_1^{(2)} \times C_2^{(3)}) \times (C_1^{(4)} \times C_2)$	$\begin{cases} D_3 + a' \equiv g_i \\ L = r_0 + g_i \\ D'_4 \equiv h_i + p + q - r_0 - a', a' \in C \\ D_2 = a' + r_0 \end{cases}$

The fiber $\lambda_2^{-1}(h_i) \cap F_{r_01}$ is also 1-dimensional with support described in Table 5.

TABLE 5

Ambient Spaces	*Support of* $\lambda_2^{-1}(h_i) \cap F_{r_0 1}$
$(L, D_4, a, D_2', D_3') \in W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(2)} \times C_2^{(3)})$	$\begin{cases} D_3' + a \equiv h_i \\ L = a + g_i, a \in C \\ r_0 \leq D_4 \equiv g_i \\ D_2' = p + q \end{cases}$

Proof. We study Z_1 case by case according to the bidegree. The proof is divided into three Lemmas: 10.4, 10.5 and 10.7. □

Proposition 10.3. *Z_2 is 1-dimensional and, for $s_4^1 \neq g_i$ or h_i, the fiber $\lambda_2^{-1}(s_4^1) \cap F_{r_0 2}$ is finite. For $i = 1, \dots 5$, the fiber $\lambda_2^{-1}(g_i) \cap F_{r_0 2}$ is 1-dimensional with support described in Table 6.*

TABLE 6

Ambient Spaces	*Support of* $\lambda_2^{-1}(g_i) \cap F_{r_0 2}$
$(L, D_5, D_3', D_2') \in W_2 \times C_1^{(5)} \times (C_1^{(3)} \times C_2^{(2)})$	$\begin{cases} D_2' + p + q \equiv g_i \\ h^0(L' - D_2') > 0, D_3' \equiv L' - D_2' \\ r_0 \leq D_5 \equiv L \end{cases}$
$(L, D_5, a', D_4') \in W_2 \times C_1^{(5)} \times (C_1 \times C_2^{(4)})$	$\begin{cases} c \leq D_4' \equiv g_i \\ L = h_i + p + q - c, c \in C \\ a' = c \\ r_0 \leq D_5 \equiv L \end{cases}$

The fiber $\lambda_2^{-1}(h_i) \cap F_{r_0 2}$ is 1-dimensional with support described in Table 7.

TABLE 7

Ambient Spaces	*Support of* $\lambda_2^{-1}(h_i) \cap F_{r_0 2}$
$(L, D_3, D_2, a', D_4') \in W_2 \times (C_1^{(3)} \times C_2^{(2)}) \times (C_1 \times C_2^{(4)})$	$\begin{cases} D_3 + a' \equiv g_i \\ r_0 \leq D_3 \\ L = c + g_i, c \in C \\ D_2 = a' + c, D_4' \equiv h_i + p + q - c - a' \end{cases}$
$(L, a, D_4, D_3', D_2') \in W_2 \times (C_1 \times C_2^{(4)}) \times (C_1^{(3)} \times C_2^{(2)})$	$\begin{cases} D_3' + r_0 \equiv g_i \\ L = h_i + p + q - c, c \in C \\ a = r_0, D_4 = h_i + p + q - c - r_0 \\ D_2' = r_0 + c \end{cases}$

Proof. The proof is entirely analogous to that of Proposition 10.2, so we omit the details. □

Lemma 10.4. *For any $s_4^1 \in W_4^1$, the intersection $\lambda_2^{-1}(s_4^1) \cap F_{(4,1)(4,1)}$ is empty except when $s_4^1 = g_i$. The support of the intersection $\lambda_2^{-1}(g_i) \cap F_{(4,1)(4,1)}$*

is of pure dimension 1 *and equal to*

$$\{ (L, D_4, a, D_4', a') \mid \; a + a' + p + q \equiv g_i, \; h^0(L - r_0 - a) > 0, \\ D_4 \equiv L - a, D_4' \equiv L' - a' \}.$$

Proof. The map $\lambda\colon F_{(4,1)(4,1)} \to W_1 \times \Theta_0$ factors through the projection of $F_{(4,1)(4,1)} \subset W_1 \times (C_1^{(4)} \times C_2) \times (C_1^{(4)} \times C_2)$ to $W_1 \times C_2 \times C_2$, which is generically injective. The image of $F_{(4,1)(4,1)}$ in $W_1 \times C_2 \times C_2$ consists of (L, a, a') such that

$$h^0(L - r_0 - a) > 0.$$

The image of $Z_{(4,1)(4,1)} = Z \cap F_{(4,1)(4,1)}$ under the projection is defined *scheme-theoretically* by imposing an extra condition

$$h^0(a + a' + p + q) > 1.$$

If $a + a' + p + q \equiv s_4^1 \in W_4^1$, then s_4^1 is equal to one of the g_i. This first shows that $\lambda_2^{-1}(s_4^1)$ is empty unless $s_4^1 = g_i$ for some i. Then it shows that there are only finitely many choices for a; hence $\lambda_2^{-1}(g_i) \cap F_{(4,1)(4,1)}$ is of pure dimension 1 and is as described. $\square$

Lemma 10.5. *For $s_4^1 \neq g_i$ or h_i, the intersection $\lambda_2^{-1}(s_4^1) \cap F_{(4,1)(2,3)}$ is finite. The intersection $\lambda_2^{-1}(h_i) \cap F_{(4,1)(2,3)}$ (up to finitely many points) has support*

$$\{ (L, D_4, a, D_2', D_3') \mid L = a + g_i, a \in C, r_0 \leq D_4 \equiv g_i, D_3' \equiv h_i - a, D_2' = p + q \},$$

and the intersection $\lambda_2^{-1}(g_i) \cap F_{(4,1)(2,3)}$ (again, up to finitely many points) has support

$$\{ (L, D_4, a, D_2', D_3') \mid L = a + g_j, j \neq i, a \in C, r_0 \leq D_4 \equiv g_j, D_3' \equiv g_i - a, \\ D_2' \equiv h_i - (g_i - p - q) \}$$

and

$$\{ (L, D_4, a, D_2', D_3') \mid L = h_i + p + q - c, c \in C, h^0(L - r_0 - a) > 0, D_3' \equiv g_i - a, \\ D_2' = (a + c) \}.$$

Proof. Consider the projection of $Z_{(4,1)(2,3)}$ to $W_1 \times C_2 \times C_2^{(3)}$ consisting of (L, a, D_3') satisfying the equations

$$h^0(L' - D_3') > 0 \tag{10.2}$$

$$h^0(L - r_0 - a) > 0 \tag{10.3}$$

$$h^0(a + D_3') > 1. \tag{10.4}$$

Fix any $s^1_4 \in W^1_4(C)$. Suppose $a + D'_3 \equiv s^1_4$. In the canonical space $|K_C|^*$, the span $\langle D'_3 \rangle$ is a plane (C is not trigonal). By Riemann-Roch, $a \in \langle D'_3 \rangle$. We have two cases:

(1) $a \not\leq \Gamma'_3 := K_C - L'$. In this case, $h^0(L' - D'_3) = h^0(K_C - \Gamma'_3 - D'_3) > 0$ implies that $h^0(K_C - \Gamma'_3 - s^1_4) > 0$, i.e. $h^0(L' - s^1_4) > 0$. If $s^1_4 \neq g_i$, then $L' = p + s^1_4$ or $L' = q + s^1_4$ because $h^0(L' - p - q) > 0$. In either case, there are finitely many choices of a satisfying condition (10.3), and therefore there are finitely many points in $Z_{(4,1)(2,3)}$ that map to s^1_4. If $s^1_4 = g_i$, then there exists $c \in C$ such that $L' = c + g_i$ and (10.3) becomes

$$h^0(K + p + q - (c + g_i) - r_0 - a) = h^0(h_i + p + q - c - r_0 - a) > 0.$$

For each c, there are 4 choices of a satisfying the above condition, so (L, a, D'_3) belongs to

$$\{\, (L = h_i + p + q - c, a, D'_3 = g_i - a) \mid c \in C,\ h^0(h_i + p + q - c - r_0 - a) > 0 \,\}.$$

Finally, there is a unique lifting of such (L, a, D'_3) to a point (L, D_4, a, D'_2, D'_3) in $Z_{(4,1)(2,3)}$ as described in the statement.

(2) $a \leq \Gamma'_3$. Write $\Gamma'_3 = a + \Gamma'_2$. The conditions defining the fiber of $Z_{(4,1)(2,3)}$ over s^1_4 are

$$\begin{cases} h^0(K_C - a - \Gamma'_2 - D'_3) = h^0(K_C - s^1_4 - \Gamma'_2) > 0 \\ h^0(K_C - \Gamma_3 - a - r_0) = h^0(\Gamma'_3 + p + q - a - r_0) = h^0(\Gamma'_2 + p + q - r_0) > 0 \\ a + D'_3 \equiv s^1_4. \end{cases}$$

Put $h^1_4 := |K_C - s^1_4|$ so that, by the above, $h^0(h^1_4 - \Gamma'_2) > 0$. There are two subcases:

(a) $h^0(\Gamma'_2 + p + q) = 2$. So the second condition above is automatically satisfied.

Here $\Gamma'_2 + p + q \in g_i$ for some i.

Claim 10.6. The five g^1_4s containing Γ'_2 are g_i and h_j for $j \neq i$. To prove this, denote by l_{pq} the line in $\mathbb{P}^2 = \mathbb{P}(I_2(C))$ consisting of quadrics vanishing on the secant line $\langle p + q \rangle$ in $|K_C|^*$. There are five rank 4 quadrics Q_j, $j = 1, \ldots, 5$, in l_{pq}, corresponding to the intersection of l_{pq} with the quintic curve parametrizing rank 4 quadrics in $\mathbb{P}(I_2(C))$. For each j, g_j is cut on C by one ruling of Q_j. Let S be the base locus of the pencil l_{pq}. Then S is a Del Pezzo surface of degree 4. By construction $\langle p + q \rangle$ is contained in S. Since the span $\langle p + q + \Gamma'_2 \rangle$ is a plane in $|K_C|^*$, $S \cap \langle p + q + \Gamma'_2 \rangle$ is a conic containing $\langle p + q \rangle$, thus $S \cap \langle p+q+\Gamma'_2 \rangle = \langle p+q \rangle \cup \langle \Gamma'_2 \rangle$. Therefore the pencil of quadrics

containing $\langle \Gamma_2' \rangle$ is also l_{pq}. We know that $\langle p+q+\Gamma_2' \rangle \subset Q_i$. For all $j \neq i$, $Q_j \cap \langle p+q+\Gamma_2' \rangle = S \cap \langle p+q+\Gamma_2' \rangle = \langle p+q \rangle \cup \langle \Gamma_2' \rangle$. So Γ_2' and $\langle p+q \rangle$ belong to different rulings of Q_j; i.e., Γ_2' is contained in the ruling of Q_j corresponding to h_j for $j \neq i$. The claim is proved.
Thus $h_4^1 = g_i$ or $h_4^1 = h_j$ for some $j \neq i$.
So those (L, a, D_3') which map to $s_4^1 = h_i$ are

$$\{ (L = a + g_i, a, D_3' \equiv h_i - a) \mid a \in C \}.$$

Similarly, the (L, a, D_3') which map to $s_4^1 = g_j$ for $j \neq i$ are

$$\{ (L = a + g_i, a, D_3' \equiv g_j - a) \mid a \in C,\ j \neq i \}.$$

There are unique liftings to points in $Z_{(4,1)(2,3)}$ as described in the statement of the proposition.

(b) $h^0(\Gamma_2' + p + q) = 1$. Then the second condition implies that $h^0(\Gamma_2' - r_0) > 0$. For each s_4^1, there are finitely many choices of $\Gamma_2' = r_0 + b$ satisfying the first condition and for each choice of Γ_2', there are finitely many choices of a such that $a + \Gamma_2' \in W_{pq}$ (because this means $h^0(K_C - a - \Gamma_2' - p - q) > 0$, and, since $h^0(\Gamma_2' + p + q) = 1$, we have $h^0(K_C - \Gamma_2' - p - q) = 1$ as well). Therefore, there are no positive dimensional fibers.

□

Lemma 10.7.

(1) *The only positive dimensional fibers in $Z_{(2,3)(4,1)}$ are $\lambda_2^{-1}(g_i) \cap Z_{(2,3)(4,1)}$. For each i, the 1-dimensional components of $\lambda_2^{-1}(g_i) \cap Z_{(2,3)(4,1)}$ are supported on the curve*

$$\{ (L, D_2, D_3, D_4', r_0) \mid L = c + g_i, c \in C, D_2 = r_0 + c, D_3 \equiv g_i - r_0,\\ D_4' \equiv h_i + p + q - c - r_0 \}$$

and

$$\{ (L, D_2, D_3, D_4', a') \mid L = r_0 + g_i, D_2 = a' + r_0, D_3 \equiv g_i - a',\\ D_4' \equiv h_i + p + q - r_0 - a', a' \in C \}.$$

(The second curve is contracted by $\lambda = (\lambda_1, \lambda_2)$ and therefore does not contribute to the Abel-Jacobi map in Section 11.)

(2) *All fibers in $Z_{(2,3)(2,3)}$ are finite.*

Proof. (1) The projection of $Z_{(2,3)(4,1)}$ to $W_{pq} \times C_2^{(3)} \times C_2$ is the locus of (L, D_3, a') satisfying

$$\begin{cases} h^0(L - D_3 - r_0) > 0 \\ h^0(a' + D_3) > 1. \end{cases}$$

As in the previous lemma, only the inverse image of g_i is positive dimensional. It is equal to

$$\{ (L = c + g_i, a' = r_0, D_3 \equiv g_i - r_0) \mid c \in C \} \\ \cup \{ (L = r_0 + g_i, D_3 \equiv g_i - a', a') \mid a' \in C \}.$$

As before, we can uniquely lift these curves to $Z_{(2,3)(4,1)}$.

(2) The projection of $Z_{(2,3)(2,3)}$ to $W_{pq} \times C_1^{(2)} \times C_1^{(2)}$ is the locus of (L, D_2, D_2') satisfying

$$\begin{cases} h^0(L - D_2) > 0 \\ h^0(L' - D_2') > 0 \\ r_0 \le D_2 \\ h^0(K_C - D_2 - D_2') > 1. \end{cases}$$

These cycles are also 1-dimensional, but there are only finitely many points mapping to a fixed s_4^1 (we choose r_0 general such that r_0+p+q is not in any g_4^1).

□

By the previous three lemmas, Proposition 10.2 is proved.

We also need to describe the components of Z_1 which lie over X_{1p} under λ_1. This will be needed in the computation of the Abel-Jacobi map in Section 11.3.

Lemma 10.8. *The scheme Z_1 has the following components which map onto X_{1p} by λ_1. Each component maps onto W_4^1 by λ_2. They are supported on*

$$\{ (L, D_2, D_3, D_4', r_0) \mid L = p + g_4^1, D_3 \equiv g_4^1 - r_0, D_2 = p + r_0, D_4' \equiv p + g_4^1 - r_0 \} \\ \subset Z_{(2,3)(4,1)},$$

$$\{ (L, D_4, p, D_2', D_3') \mid L = p + g_4^1, r_0 \le D_4 \equiv g_4^1, D_3' \equiv K_C - g_4^1 - a \} \subset Z_{(4,1)(2,3)},$$

and

$$\{ (L, D_4, a, D_2', D_3') \mid L = p + g_4^1, h^0(g_4^1 - r_0 - a) > 0, D_4 \equiv p + g_4^1 - a, \\ D_3' \equiv K_C - g_4^1 - a, D_2' = a + q \} \subset Z_{(4,1)(2,3)}.$$

Proof. Fix a general $L = p + g^1_4 \in X_{1p}$. One easily sees from Table 2 that only $Z_{(2,3)(4,1)}$ and $Z_{(4,1)(2,3)}$ have a point over L.

In $Z_{(2,3)(4,1)}$, the condition $h^0(p + g^1_4 - D_3 - r_0) > 0$ implies that either $D_3 \equiv g^1_4 - r_0$ or $D_3 = p + B_2$ with $h^0(g^1_4 - r_0 - B_2) > 0$. In the first case, $h^0(a' + D_3) > 1$ implies that $a' = r_0$. This is because D'_3 is contained in at most one pencil of degree 4. Thus we obtain the first curve in the statement of the lemma. The second case does not happen because $|a' + p + B_2|$ cannot be a pencil for p and g^1_4 general.

In $Z_{(4,1)(2,3)}$, there are four choices of a such that $h^0(p + g^1_4 - r_0 - a) > 0$. The condition $h^0(L' - D'_3) = h^0(q + K_C - g^1_4 - D'_3) > 0$ implies that either $h^0(K_C - g^1_4 - D'_3) > 0$ or $D'_3 = p + B'_2$ with $h^0(K_C - g^1_4 - B'_2) > 0$. In the first case, $h^0(a + D'_3) > 1$ implies that $D'_3 \equiv K_C - g^1_4 - a$. This is because D'_3 is contained in at most one pencil of degree 4. Thus we obtain the curves in the statement of the lemma. In the second case, $|a + q + D'_2|$ is not a pencil for q and g^1_4 general. Note that the last component is a degree 3 cover of X_{1p} under λ_1. □

10.3. Infinitesimal study of F_{r_0} and Z. In this subsection, we prove that each irreducible component of the center of the blow-up $F''_{r_0} \cap (W_k \times W^1_4)$ is generically smooth or, equivalently, generically reduced. We also prove that F''_{r_0} is smooth at a general point in each irreducible component of $F''_{r_0} \cap (W_k \times W^1_4)$.

The infinitesimal study is similar for all components. So let us take one component, say the image in $W_1 \times \Theta_0$ of the curve in $Z_{(4,1)(2,3)}$:

$$\{ (L, D_4, a, D'_2, D'_3) \mid L = a + g_i, a \in C, r_0 \leq D_4 \equiv g_i, D'_3 \equiv h_i - a, D'_2 = p + q \}.$$

This curve projects isomorphically to (with identification $C_1 = C_2 = C$)

(10.5)

$$Z'_{(4,1)(2,3)} = \{ (L, a, D'_3) \mid L = a + g_i, a, D'_3 \equiv h_i - a, a \in C \} \subset W_1 \times C \times C^{(3)}.$$

It suffices to show that the curve $Z'_{(4,1)(2,3)}$ is generically reduced. To this end, recall that by [ACGH, p. 189], for any line bundle M of degree d on C and $v \in H^1(\mathcal{O}_C) = T_M \operatorname{Pic}^d C$ a tangent vector, all sections in $H^0(C, M)$ extend to first order along v if and only if

$$(v, \operatorname{Im} \mu_M)_S = 0$$

where $(,)_S$ is the pairing for Serre duality and

$$\mu_M : H^0(M) \otimes H^0(K_C - M) \to H^0(K_C)$$

is the multiplication map.

Note that $\operatorname{Im}\mu_{g_i}$ is of codimension 1 in $H^0(K_C)$ by the base point free pencil trick.

If we embed $W_1 \times C \times C^{(3)}$ in $\operatorname{Pic}^5 C \times \operatorname{Pic}^1 C \times \operatorname{Pic}^3 C$, by the previous paragraph, the tangent space to $W_1 \times C \times C^{(3)}$ at the point (L, a, D_3') consists of $(v_1, v_2, v_3) \in H^1(\mathcal{O}_C)^{\oplus 3}$ such that

$$v_1 \in \operatorname{Im}\mu_L^\perp \cap \operatorname{Im}\mu_{L'}^\perp, \tag{10.6}$$

$$v_2 \in H^0(K_C - a)^\perp, \tag{10.7}$$

$$v_3 \in H^0(K_C - D_3')^\perp. \tag{10.8}$$

Lemma 10.9. *A tangent vector $(v_1, v_2, v_3) \in H^1(\mathcal{O}_C)^{\oplus 3}$ of $W_{pq} \times C \times C^{(3)}$ is tangent to $Z'_{(4,1)(2,3)} \subset W_{pq} \times C \times C^{(3)}$ at $(L = a + g_i, a, D_3' = h_i - a)$ if, in addition, the following hold:*

$$v_1 + v_3 \in H^0(K_C - p - q)^\perp, \tag{10.9}$$

$$v_1 - v_2 \in H^0(K_C - (g_i - r_0))^\perp, \tag{10.10}$$

$$v_2 + v_3 \in \operatorname{Im}\mu_{g_i}^\perp. \tag{10.11}$$

Proof. The cycle $Z'_{(4,1)(2,3)}$ is defined scheme-theoretically by (10.2), (10.3), and (10.4). These translate into the above conditions for infinitesimal deformations. □

Proposition 10.10. *Each irreducible component of $F''_{r_0} \cap (W_k \times W_4^1)$ is generically smooth.*

Proof. We only prove the proposition for the component which is the image in $W_1 \times \Theta_0$ of $Z'_{(4,1)(2,3)}$.

Fix a general point $(L = a + g_i, a, D_3' \equiv h_i - a)$. Consider the linear map from the tangent space of $Z_{(4,1)(2,3)}$ to $H^1(\mathcal{O}_C)$ which sends (v_1, v_2, v_3) to $v_2 + v_3$. Its image is 1-dimensional by (10.11). To show that the tangent space of $Z_{(4,1)(2,3)}$ is 1-dimensional, it suffices to show that the kernel of this linear map is trivial; i.e., if $v_2 + v_3 = 0$, then $v_1 = v_3 = 0$.

So assume $v_2 + v_3 = 0$. Then

$$v_1 + v_3 = v_1 - v_2 \in H^0(K_C - p - q)^\perp \cap H^0(K_C - (g_i - r_0))^\perp.$$

Since the pencil $K_C - (g_i - r_0) = h_i + r_0$ does not have base points at p or q and can separate p and q, we conclude that

$$H^0(K_C - p - q)^\perp \cap H^0(K_C - (g_i - r_0))^\perp = (H^0(K_C - p - q) + H^0(K_C - (g_i - r_0)))^\perp = 0.$$

Therefore $v_1 = -v_3$. Now by (10.6) and (10.7), $v_1 = v_2 = -v_3 \in \operatorname{Im}\mu_L^\perp \cap \operatorname{Im}\mu_{L'}^\perp \cap H^0(K_C - a)^\perp = 0$ for $a \in C$ general. This implies that $v_1 = v_2 = v_3 = 0$. □

Proposition 10.11. *The scheme F_{r_0k} is smooth at a general point of each component of Z_k.*

Proof. Again we only check the proposition for a general point of the image in $W_1 \times \Theta_0$ of (10.5). The defining conditions for $F_{(41)(23)} \subset W_1 \times C \times C^{(3)}$ are (10.2) and (10.3). The tangent space of F_{r_0} at $(L = a + g_i, a, D'_3 = h_i - a)$ consists of (v_1, v_2, v_3) satisfying the conditions from (10.6) to (10.10). Projection to the v_1 summand of (v_1, v_2, v_3) is surjective, and the kernel of this projection is 1-dimensional. The proposition follows. $\square$

10.4. The structure of the projectivized normal cone. Note that $F'''_{r_0}|_{W_k \times M_2}$ is the projectivized normal cone of $F_{r_0} \cap (W_k \times \Theta_0)$ in $\mathcal{F}''_r$.

We have the commutative diagram

(10.12)
$$\begin{array}{ccccc}
\mathcal{C}_k & \longrightarrow & F'''_{r_0}|_{W_k \times M_2} & \xrightarrow{\rho_2} & M_2 \\
\downarrow & & \downarrow & & \downarrow \pi_2 \\
Z_k & \xrightarrow{(\lambda_1,\lambda_2)} & F''_{r_0} \cap (W_k \times W_4^1) & \xrightarrow{Pr_2} & W_4^1 \\
& \searrow^{\lambda_1} & \downarrow Pr_1 & & \\
& & W_k & &
\end{array}$$

where $\mathcal{C}_k$ is defined by the left square being a fiber product.

Proposition 10.12. *$\mathcal{C}_k$ is generically a $\mathbb{P}^2$ bundle over the curve $\lambda_2^{-1}(\bigcup_i \{ g_i, h_i \}) \cap Z_k$.*

Proof. Since $W_k \times M_2$ is a divisor in the total space $\mathcal{P}$, $F'''_{r_0}|_{W_k \times M_2} = \mathcal{F}'''_r \cap (W_k \times M_2)$ is purely 3-dimensional. Furthermore, by Propositions 10.10 and 10.11, at a generic point of any component of $\lambda_2^{-1}(\bigcup_i \{ g_i, h_i \}) \cap Z_k$, both Z_k and F_{r_0k} are smooth. Thus there is an open dense subset of $\lambda_2^{-1}(\bigcup_i \{ g_i, h_i \}) \cap Z_k$ where the dominant map $\mathcal{C}_k \to Z_k$ is a $\mathbb{P}^2$ bundle. So the general fiber of $\mathcal{C}_k$ is a 2-dimensional linear subspace of the singular quadric threefold Q_3^{sing} which is the fiber of M_2 over one of the g_i or h_i. Therefore the general fiber is a $\mathbb{P}^2$ passing through the vertex of Q_3^{sing}. $\square$

11. The Abel-Jacobi map

We are now ready to prove Propositions 8.1 to 8.4.

11.1. Proof of Proposition 8.1. The map $\mathrm{AJ}_1^0 \colon H^2(\widetilde{G}_0^{[0]}) \to H^4(M_1)$.

We will show that it is enough to compute the restriction of AJ_1^0 to the direct summand $H^2(W_1)$ of $H^2(\widetilde{G}_0^{[0]})$. This map is the correspondence induced by the cycle $[F'''_{r_0}|_{\widetilde{W_1\times M_1}}] \in H^6(\widetilde{W_1\times M_1})$. We use the notation introduced in Section 6.1.

There are two reduction steps:

(1) First, since we are computing AJ^0_1 modulo $\langle\, j_{1*}f, j_{1*}\tau_1\,\rangle$ in Proposition 8.1 (recall that $H^4(M_1)\cong p_1^*H^4(C^{(4)})\oplus\langle\, j_{1*}f, j_{1*}\tau_1\,\rangle$), it suffices to check that the image of the composition

$$H^2(W_1)\xrightarrow{\mathrm{AJ}^0_1} H^4(M_1)\xrightarrow{p_{1*}} H^4(C^{(4)})$$

contains $\eta H^2(\mathrm{Pic}^4 C)\oplus\eta^2$ modulo $\theta H^2(\mathrm{Pic}^4 C)$. Recall (see Proposition 10.1) that $F'''_{r_0}|_{\widetilde{W_1\times M_1}}$ is the proper transform of $F''_{r_0}|_{W_1\times\Theta_0}$ under

$$\widetilde{W_1\times M_1}\longrightarrow W_1\times M_1\longrightarrow W_1\times C^{(4)}\longrightarrow W_1\times\Theta_0.$$

By the projection formula, $p_{1*}\circ\mathrm{AJ}^0_1$ is induced as a correspondence map by the proper transform $F''_{r_0}|_{W_1\times C^{(4)}}$ of $F''_{r_0}|_{W_1\times\Theta_0}$ in the intermediate space $W_1\times C^{(4)}$:

$$H^2(W_1)\longrightarrow H^2(W_1\times C^{(4)})\xrightarrow{\cup[F''_{r_0}|_{W_1\times C^{(4)}}]} H^8(W_1\times C^{(4)})\longrightarrow H^4(C^{(4)}).$$

(2) Second, we will prove that in fact the image by $p_{1*}\circ\mathrm{AJ}^0_1$ of the subspace $(q_1,q_2)^*H^2(C^{(3)}\times C^{(3)})$ of $H^2(W_1)$ contains $\eta H^2(\mathrm{Pic}^4 C)\oplus\eta^2$ modulo $\theta H^2(\mathrm{Pic}^4 C)$. We therefore compute the composition $\overline{\mathrm{AJ}^0_1}$,

$$\overline{\mathrm{AJ}^0_1}\colon\ H^2(C^{(3)}\times C^{(3)})\xrightarrow{(q_1,q_2)^*} H^2(W_1)\xrightarrow{p_{1*}\circ\mathrm{AJ}^0_1} H^4(C^{(4)})\longrightarrow\frac{H^4(C^{(4)})}{\theta H^2(\mathrm{Pic}^4 C)},$$

where

$$\begin{array}{rcl}(q_1,q_2)\colon W_{pq} & \longrightarrow & C^{(3)}\times C^{(3)}\\ L & \longmapsto & (\Gamma_3,\Gamma'_3)\end{array}$$

is the embedding used in Section 2.2.

Lemma 11.1. *The Kunneth component of* $[F''_{r_0}|_{W_1\times C^{(4)}}]\in H^6(W_1\times C^{(4)})$ *in* $H^2(W_1)\otimes H^4(C^{(4)})$ *is the restriction to* $W_1\times C^{(4)}\subset C^{(3)}\times C^{(3)}\times C^{(4)}$ *of*

$$(-2\theta_1+4\eta_1+4\eta_2)\eta_3^2+4\delta_{13}^2\eta_3+(\theta_1-\eta_1)\theta_3\eta_3 \tag{11.1}$$

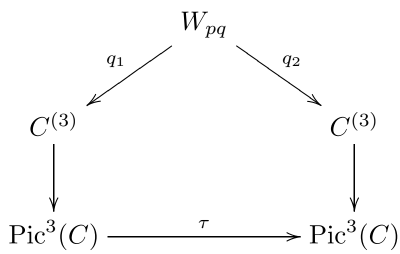
in $H^6(C^{(3)}\times C^{(3)}\times C^{(4)})$ *modulo* $\theta_3 H^2(\mathrm{Pic}^4 C)$, *where*

$$\delta_{kl}=\sum_{i=1}^{5}(\xi_{ki}\xi'_{li}+\xi_{li}\xi'_{ki}).$$

Proof. The Kunneth component of $[F''_{r_0}|_{W_1 \times C^{(4)}}]$ in $H^2(W_1) \otimes H^4(C^{(4)})$ is computed case by case for each bidegree in Section 12.4. It is the sum of the classes in (12.3), (12.4), (12.5), (12.6), (12.7), which is equal to the restriction to $W_1 \times C^{(4)} \subset C^{(3)} \times C^{(3)} \times C^{(4)}$ of

$$(11.2) \qquad [-2\theta_1 + 4\eta_1 + 4\eta_2]\,\eta_3^2 + \left[2\delta_{23}^2 + \delta_{13}^2 - \delta_{13}\delta_{23} + (\theta_1 - \eta_1)\theta_3\right]\eta_3.$$

Consider the commutative diagram



where τ is the involution sending M to $K_C - p - q - M$. Since $\tau^*(\xi_i) = -\xi_i$, we see immediately that

$$q_1^*(\xi_i) = -q_2^*(\xi_i),$$
$$\delta_{13}|_{W_1 \times C^{(4)}} = -\delta_{23}|_{W_1 \times C^{(4)}}.$$

Therefore (11.2) simplifies to (11.1). □

For any $\omega \in H^2(C^{(3)})$, denote by ω_1 its pull-back to $C^{(3)} \times C^{(3)}$ under the first projection (see "Notation and conventions" (4)). Now, using the class (11.1), we obtain

$$\overline{\mathrm{AJ}_1^0}(\omega_1) = pr_{C^{(4)}*}\left\{ \omega_1 \left[(-2\theta_1 + 4\eta_1 + 4\eta_2)\eta_3^2 + 4\delta_{13}^2 + (\theta_1 - \eta_1)\theta_3\eta_3\right] \right\}|_{W_1 \times C^{(4)}}.$$

Expanding $\delta_{13}^2 = \sum_{i,j=1}^5 \left[2\xi_{1i}\xi'_{1j}\xi'_{3i}\xi_{3j} - \xi_{1i}\xi_{1j}\xi'_{3i}\xi'_{3j} - \xi'_{1i}\xi'_{1j}\xi_{3i}\xi_{3j}\right]$, we obtain

$$\begin{aligned} \overline{\mathrm{AJ}_1^0}(\omega_1) = & \left[\int_{W_1} \omega_1(-2\theta_1 + 4\eta_1 + 4\eta_2)\right]\eta^2 + 8\sum_{i,j=1}^5 \left[\int_{W_1} \omega_1 \xi_{1i}\xi'_{1j}\right]\xi'_i\xi_j \\ & -4\sum_{i,j=1}^5 \left[\int_{W_1} \omega_1\xi_{1i}\xi_{1j}\right]\xi'_i\xi'_j - 4\sum_{i,j=1}^5 \left[\int_{W_1} \omega_1\xi'_{1i}\xi'_{1j}\right]\xi_i\xi_j \\ & + \left[\int_{W_1} \omega_1(\theta_1 - \eta_1)\right]\theta\eta. \end{aligned}$$

Noting that the class of the image of W_1 in $C^{(3)}$ by q_1 or q_2 is $\theta - \eta$ and $q_{1*}q_2^*\eta = \frac{1}{2}\theta^2 - \theta\eta + \eta^2 \in H^2(C^{(3)})$, the above formula becomes

(11.3)

$$\begin{aligned}\overline{\mathrm{AJ}_1^0}(\omega_1) = &\left[\int_{C^{(3)}} \omega(-2\theta + 4\eta)(\theta - \eta) + 4\int_{C^{(3)}} \omega(\frac{1}{2}\theta^2 - \theta\eta + \eta^2)\right]\eta^2 \\ &+ 8\sum_{i,j=1}^{5}\left[\int_{C^{(3)}} \omega\xi_i\xi_j'(\theta-\eta)\right]\xi_i'\xi_j - 4\sum_{i,j=1}^{5}\left[\int_{C^{(3)}} \omega\xi_i\xi_j(\theta-\eta)\right]\xi_i'\xi_j' \\ &- 4\sum_{i,j=1}^{5}\left[\int_{C^{(3)}} \omega\xi_i'\xi_j'(\theta-\eta)\right]\xi_i\xi_j + \left[\int_{C^{(3)}} \omega(\theta-\eta)^2\right]\theta\eta.\end{aligned}$$

Now a simple computation using the ring structure of $H^\bullet(C^{(3)})$ described in Macdonald [Ma] gives

$$\begin{aligned}\overline{\mathrm{AJ}_1^0}(\eta_1) &= 10\eta^2 - 11\theta\eta, \\ \overline{\mathrm{AJ}_1^0}(\xi_{1i}\xi_{1j}) &= c_{ij}\xi_i\xi_j\eta \text{ for } 0 \neq c_{ij} \in \mathbb{Z},\ j \neq i \pm 5, \\ \overline{\mathrm{AJ}_1^0}(\sigma_{1k}) &= 8\eta^2 - 11\theta\eta + 16\sigma_k\eta.\end{aligned}$$

Thus the image of $\overline{\mathrm{AJ}_1^0}$ contains $\eta H^2(\mathrm{Pic}^4 C) \oplus \eta^2$ modulo $\theta H^2(\mathrm{Pic}^4 C)$. □

11.2. Proof of Proposition 8.2. The map $\mathrm{AJ}_2^0 \colon H^2(\widetilde{G}_0^{[0]}) \longrightarrow H^4(M_2)$.

We will work with the restriction of AJ_2^0 to the direct summand $H^2(W_1) \oplus H^2(W_2)$ of $H^2(\widetilde{G}_0^{[0]})$:

$$H^2(W_k) \xrightarrow{\rho_1^*} H^2(W_k \times M_2) \xrightarrow{\cup[F_{r_0}'''|_{W_k\times M_2}]} H^8(W_k \times M_2) \xrightarrow{\rho_{2*}} H^4(M_2)\ .$$

The relations between the various spaces involved are summarized in diagram (10.12). The projection of $F_{r_0}'''|_{W_k\times M_2}$ to W_k is supported on curves. By Section 10.2, the image curve contains the following special curves in W_k:

$$\begin{aligned}C_i &:= \{\, c + g_i \mid c \in C \,\}, C_i' := \iota(C_i), i = 1, \dots 5, \\ X_{1p} &= \{\, p + g_4^1 \mid g_4^1 \in W_4^1(C) \,\}, \\ X_{1q} &= \{\, q + g_4^1 \mid g_4^1 \in W_4^1(C) \,\},\end{aligned}$$

where $\iota(L) = |K_C + p + q - L|$. By Lemma 6.3, $H^4(M_2)$ is generated by $j_{2*}f$, $j_{2*}\tau_1$, $[\mathbb{P}_i^2]$ and $[\mathbb{P}_{i+5}^2]$ (recall that f is the class of the fiber of $\pi_{12} \colon M_{12} \longrightarrow W_4^1$ and see Lemma 6.3 for the definition of $\mathbb{P}_i^2$).

Lemma 11.2. *Put* $[C]_{tot} := [C_1] + \cdots + [C_5]$. *For any* $(\alpha, \beta) \in H^2(W_1) \oplus H^2(W_2)$,

$$\mathrm{AJ}_2^0(\alpha) = \sum_{i=1}^{5} \left(\int_{W_1} \alpha \cdot [C_i] \right) [\mathbb{P}^2_{i+5}] + \sum_{i=1}^{5} \left(\int_{W_1} \alpha \cdot ([C]_{tot} + 4[C'_i] + 2q_1^*(\theta - \eta)) \right) [\mathbb{P}^2_i]$$

modulo $\langle j_{2*} f \rangle$, *and*

$$\mathrm{AJ}_2^0(\beta) = -\sum_{i=1}^{5} \left(\int_{W_2} \beta \cdot (3[C_i] + [C'_i]) \right) [\mathbb{P}^2_{i+5}] + \sum_{i=1}^{5} \left(\int_{W_2} \beta \cdot ([C'_i] + q_2^*(\theta - \eta)) \right) [\mathbb{P}^2_i]$$

modulo $\langle j_{2*} f \rangle$.

Proof. By Sections 10.2 and 10.3, the scheme $F''_{r_0} \cap (W_k \times W_4^1)$ is of pure dimension 1 and generically reduced on each of its components.

Represent α as the cohomology class of a real 2-chain in general position. By definition, $\mathrm{AJ}_2^0(\alpha)$ is the push-forward to M_2 of the pull-back of $\lambda_1^* \alpha \cup [Z_1]$ to $\mathcal{C}_k$. By Proposition 10.12, the fibers of $\mathcal{C}_k$ over $\lambda_2^{-1}(\bigcup_i \{ g_i, h_i \}) \cap Z_k$ are isomorphic to $\mathbb{P}^2$.

Since we are computing AJ_2^0 modulo $\langle j_{2*} f \rangle \in H^4(M_2)$, we only need to compute the intersection of $\lambda_1^* \alpha$ with $\lambda_2^{-1}(\bigcup_i \{ g_i, h_i \}) \cap Z_1$. The components of $\lambda_2^{-1}(\bigcup_i \{ g_i, h_i \}) \cap Z_1$ are described in Proposition 10.2. For instance, the curve supported on

$$\{ (L, D_4, a, D'_4, a') \mid a + a' + p + q \equiv g_i,\ h^0(L - r_0 - a) > 0, D_4 \equiv L - a, D'_4 \equiv L' - a' \}$$

has two components since we can switch a and a'. Each component projects to a curve in W_1 whose class is $(\theta_1 - \eta_1)|_{W_1}$ by the secant plane formula (Section 12.2). Thus the contribution of this curve is $\int_{W_1} \alpha \cdot 2(\theta_1 - \eta_1)[\mathbb{P}^2_i]$. The formula for $\mathrm{AJ}_2^0(\alpha)$ now easily follows.

The computation of $\mathrm{AJ}_2^0(\beta)$ is analogous. The minus sign in the formula for $\mathrm{AJ}_2^0(\beta)$ comes from the fact that the maps to Θ_0 on the curves

$$\{ (L, D_3, D_2, a', D'_4) \mid L = c + g_i, c \in C, r_0 \le D_3, a' + D_3 \equiv g_i, D_2 = a' + c, D'_4 \equiv h_i + p + q - c - a' \} \subset Z_{(3,2)(1,4)},$$

$$\{ (L, a, D_4, D_3', D_2') \mid L = h_i + p + q - c, a = r_0, r_0 + D_3' \equiv g_i, c \in C, D_2' = r_0 + c \} \subset Z_{(1,4)(3,2)}$$

are given by $\mathcal{O}_C(K_C - D_3 - a')$ and $\mathcal{O}_C(K_C - D_3' - a)$ respectively (instead of $\mathcal{O}_C(D_3 + a')$ and $\mathcal{O}_C(D_3' + a)$). Thus the $\mathbb{P}^2$ fibers over these curves are in the rulings opposite to those of $\mathbb{P}^2_{i+5}$. Since we work modulo $j_{2*}f$, the two rulings differ by a minus sign. □

We need the following lemma to study the rank of AJ_2^0.

Lemma 11.3. *We have the following intersection numbers in the smooth surface W_{pq}:*

$$C_i^2 = C_i'^2 = -2, \ C_iC_i' = C_iC_j = C_i'C_j' = 0, \ C_iC_j' = 2, \ \textit{for } i \neq j.$$

Proof. Clearly $C_iC_j = C_i'C_j' = 0$ for $i \neq j$. To compute C_i^2, consider the exact sequence

$$0 \longrightarrow N_{C_i|W_{pq}} \longrightarrow N_{C_i|C^{(3)}} \longrightarrow N_{q_2(W_{pq})|C^{(3)}}|_{C_i} \longrightarrow 0 .$$

Under the embedding $q_2 \colon W_{pq} \to C^{(3)}$ sending L to $|L-p-q|$, C_i is a complete intersection with cohomology class $\eta^2 \in H^4(C^{(3)})$. Therefore, $c_1(N_{C_i|C^{(3)}}) = 2$. We also have $c_1(N_{W_{pq}|C^{(3)}}|_{C_i}) = \int_{C^{(3)}} [W_{pq}] \cdot [C_i] = \int_{C^{(3)}} (\theta - \eta)\eta^2 = 4$. We conclude that $C_i^2 = -2$.

Now we compute C_iC_j'. Suppose $x + g_i \sim p + q + h_j - y$ for some $x, y \in C$. Then

$$D_{2i} := g_i - p - q = h_j - x - y.$$

By Claim 10.6, for a fixed i, the g^1_4s containing D_{2i} are g_i and h_l for $l \neq i$. This implies that $C_iC_i' = 0$ and $C_iC_j' = 2$ (embedding $C^{(3)}$ in $\mathrm{Pic}^3 C$, one easily sees that the intersection of C_i and C_j' is transverse for a general choice of $p + q$). □

Using the formula in Lemma 11.2 and the intersection numbers in Lemma 11.3 we compute

$$\begin{aligned} \mathrm{AJ}_2^0 : \ H^2(W_1) \oplus H^2(W_2) \ &\longrightarrow H^4(M_2)/\langle j_{2*}f \rangle \\ (12[C_i] - 3[C]_{tot} + 2q_2^*(\eta - \sigma_i), \ 3[C_i]) \\ &\longmapsto -58[\mathbb{P}^2_i] + 44 \sum_{j \neq i, j=1}^{5} [\mathbb{P}^2_j] \bmod \langle j_{2*}f \rangle. \end{aligned} \tag{11.4}$$

It immediately follows that the image of AJ_2^0 contains $\langle [\mathbb{P}^2_i] \mid i = 1, \dots, 5 \rangle$ modulo $j_{2*}f$. We then compute that

$$\mathrm{AJ}_2^0([C_i], [C_i']) = -6 \sum_{j \neq i, j=1}^{5} [\mathbb{P}^2_{j+5}] \tag{11.5}$$

modulo $\langle\, [\mathbb{P}^2_i], j_{2*}f \mid i = 1, \ldots, 5\,\rangle$. Proposition 8.2 follows immediately. $\square$

11.3. Proof of Proposition 8.3. The map $\mathrm{AJ}^1\colon H^1(\widetilde{G}_0^{[1]}) \longrightarrow H^3(M_{12})$. It follows from Sections 7.2 and 8.1 that the only double loci of the central fiber $\mathcal{P}_0$ inducing nontrivial Abel-Jacobi maps are those which map to X_{kp} or X_{kq} under ρ_1 and map to M_{12} under ρ_2. These are the slanted lines in the picture in Section 7.2. Recall (see Section 4.1) that $H^1(\widetilde{G}_0^{[1]}) = H^1(X_{1p}) \oplus H^1(X_{1q}) \oplus H^1(X_{2p}) \oplus H^1(X_{2p})$ and $H^3(M_{12}) = \tau_1 \cdot \pi_{12}^* H^1(W_4^1) \oplus j_2^* e_2 \cdot \pi_{12}^* H^1(W_4^1)$ (see Lemma 6.3). To prove Proposition 8.3, it is sufficient to prove that the image of the summand $H^1(X_{1q})$ by AJ^1 contains $\tau_1 \cdot \pi_{12}^* H^1(W_4^1)$. The map AJ^1 on this summand is given by

$$H^1(X_{1p}) \xrightarrow{\rho_1^*} H^1(E_{1p}) \xrightarrow{\cup [F'''_{r_0}|_{E_{1p}}]} H^7(E_{1p}) \xrightarrow{\rho_{2*}} H^3(M_{12})\ , \tag{11.6}$$

where E_{1p} corresponds to the slanted line labeled b in the picture in Section 7.2. Therefore E_{1p} is a $\mathbb{P}^1$-bundle over $X_{1p} \times M_{12}$ and fits into the diagram

$$\begin{array}{ccc}
E_{1p} & & \\
\downarrow & & \\
X_{1p} \times M_{12} & \longrightarrow & M_{12} \\
\downarrow & & \\
X_{1p}. & &
\end{array}$$

By the projection formula, to compute (11.6), it suffices to compute the correspondence induced by the push-forward cycle of $[F'''_{r_0}|_{E_{1p}}]$ to $X_{1p} \times M_{12}$. Denote by Y' the projectivized normal cone of $F''_{r_0} \cap (W_1 \times W_4^1)$ in $F''_{r_0}|_{W_1 \times \Theta_0}$. By construction, Y' has dimension 2 and $Y' = (W_1 \times M_{12}) \cap F'''_{r_0}|_{W_1 \times M_2}$.

The components of Z_1 which dominate X_{1p} are described in Lemma 10.8. Let Z_{1p} denote the union of these components and let Y be the fiber product $Z_{1p} \times_{F''_{r_0} \cap (W_k \times W_4^1)} Y'$, which is generically a $\mathbb{P}^1$-bundle over Z_{1p} (the $\mathbb{P}^1$ in the ruling corresponds to τ_1 because the map λ_2 from $F_{(4,1)(2,3)}$ and $F_{(2,3)(4,1)}$ factors through $C^{(4)} \xrightarrow{\phi} \Theta_0$):

$$\begin{array}{ccccc}
Y & \longrightarrow & Y' & \longrightarrow & M_{12} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_{12}} \\
Z_{1p} & \xrightarrow{(\lambda_1,\lambda_2)} & F''_{r_0} \cap (W_k \times W_4^1) & \xrightarrow{Pr_2} & W_4^1 \\
\downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle Pr_1} & & \\
X_{1p} & \longrightarrow & W_1. & &
\end{array}$$

For a real 1-cycle α in general position in X_{1p}, the inverse image of α in Y is a $\mathbb{P}^1$-bundle over α. The push-forward of the class of this $\mathbb{P}^1$-bundle to M_{12} is a class in $H^3(M_{12})$. As the class of α varies in $H^1(X_{1p}) \cong H^1(W_4^1)$, the class in $H^3(M_{12})$ spans $\tau_1 \cdot \pi_{12}^* H^1(W_4^1)$ because X_{1p} and W_4^1 are isomorphic to each other. □

11.4. Proof of Proposition 8.4. Passage to the E_2 terms.

Recall that $\widetilde{G}_0$ has four components and $E_2^{0,2} = \mathrm{Gr}_2\, H^2(\widetilde{G}_0)$ is the kernel of

$$\begin{array}{ccc} H^2(\widetilde{G}_0^{[0]}) & \xrightarrow{d_1} & H^2(\widetilde{G}_0^{[1]}) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{k=1}^2 H^2(W_k) \oplus H^2(P_k) & \longrightarrow & \bigoplus_{k=1}^2 H^2(X_{kp}) \oplus H^2(X_{kq}). \end{array}$$

Consider the subspace of $\mathrm{Gr}_2\, H^2(\widetilde{G}_0)$ consisting of $(x_1, x_2, \beta_1, \beta_2)$ with $x_k \in H^2(W_k)$ and $\beta_k \in H^2(P_k)$ such that β_k is a multiple of the class of fiber of the $\mathbb{P}^1$-bundle P_k. Since we always have

$$\int_{X_{kp}} \beta_k = \int_{X_{3-k,q}} \beta_k,$$

the compatibility condition defining $\mathrm{Ker}(d_1)$ becomes

$$\int_{X_{kp}} x_k = \int_{X_{3-k,q}} x_{3-k}. \tag{11.7}$$

Because the cycles $F_{r_0}'''|_{P_k \times M_1}$ and $F_{r_0}'''|_{P_k \times M_2}$ come from a base change (Proposition 9.2), the maps AJ_1^0 and AJ_2^0 are independent of $\beta_k \in H^2(P_k)$. We will therefore write $\mathrm{AJ}_i^0(x_1, x_2) := \mathrm{AJ}_i^0(x_1, x_2, \beta_1, \beta_2)$.

Now start with $(\gamma_1, \gamma_2) \in (I \oplus H^4(M_2)) \cap \mathrm{Gr}_4\, H^4(\widetilde{\Theta}_0)$. The condition $(\gamma_1, \gamma_2) \in \mathrm{Gr}_4\, H^4(\widetilde{\Theta}_0)$ means that $j_1^*\gamma_1 = j_2^*\gamma_2 \in H^4(M_{12})$ by Proposition 6.4. By Proposition 8.2, we can choose $(x_1, x_2) \in H^2(W_1) \oplus H^2(W_2)$ such that

$$\gamma_2 - \mathrm{AJ}_2^0(x_1, x_2) \in \langle\, j_{2*}f, j_{2*}\tau_1 \,\rangle.$$

Furthermore, note that in formulas (11.4) and (11.5), we have chosen x_1 and x_2 so that

$$\int_{X_{1p}} x_1 = \int_{X_{2q}} x_2, \qquad \int_{X_{1q}} x_1 = \int_{X_{2p}} x_2.$$

Subtracting $(\mathrm{AJ}_1^0(x_1, x_2), \mathrm{AJ}_2^0(x_1, x_2))$ from (γ_1, γ_2), we may assume that $\gamma_2 \in \langle\, j_{2*}f, j_{2*}\tau_1 \,\rangle \subset H^4(M_2)$. Now choose $\omega \in H^2(C^{(3)})$ such that for $i = 1, \dots, 5$ (see Section 11.2 for the notation),

$$\int_{X_{1p}} q_1^*\omega = \int_{X_{1q}} q_1^*\omega = 0 \tag{11.8}$$

and

$$\int_{C_i} q_1^*\omega = \int_{W_1} q_1^*\omega \cdot ([C]_{tot} + 4[C_i'] + 2q_1^*(\theta - \eta)) = 0. \tag{11.9}$$

The equations (11.8) imply that $(q_1^*\omega, 0) \in \mathrm{Gr}_2 H^2(\widetilde{G}_0)$. The equations (11.9) imply that

$$\mathrm{AJ}_2^0(q_1^*\omega, 0) \in \langle\, j_{2*}f, j_{2*}\tau_1 \,\rangle$$

by the formula for AJ_2^0 in Lemma 11.2.

By the secant plane formula,

$$\begin{aligned} q_{1*}[C_i] &= \frac{1}{2}\theta^2 - \theta\eta + \eta^2, \\ q_{1*}[C_i'] &= \eta^2, \\ q_{1*}[X_{1p}] &= q_{1*}[X_{1q}] = \frac{1}{2}\theta^2 - \theta\eta. \end{aligned}$$

Therefore the equations (11.8) and (11.9) together impose two conditions on ω since

$$\begin{aligned} \langle\, q_{1*}[C_i], q_{1*}(2[C_i'] + q_1^*(\theta - \eta)), q_{1*}[X_{1p}], q_{1*}[X_{1q}] \,\rangle &= \langle\, \theta^2, \theta\eta, \eta^2 \,\rangle \\ &= \langle\, \theta\eta, \eta^2 \,\rangle \subset H^4(C^{(3)}). \end{aligned}$$

So, if we choose

$$\omega \in \langle\, \xi_i\xi_j, \sigma_k - \sigma_1 |\; i \neq j \pm 5, k = 2, \ldots, 5 \,\rangle = \langle\, \theta\eta, \eta^2 \,\rangle^{\perp},$$

by the formula for AJ_2^0 in Lemma 11.2, $\mathrm{AJ}_2^0(q_1^*\omega, 0) \in \langle\, j_{2*}f, j_{2*}\tau_1 \,\rangle$. Similarly, we can choose $\omega' \in H^2(C^{(3)})$ such that $q_1^*(\omega')$ satisfies (11.8) and

$$\mathrm{AJ}_2^0(0, q_1^*\omega') \in \langle\, j_{2*}f, j_{2*}\tau_1 \,\rangle.$$

By formula (11.3), if we modify (γ_1,γ_2) by a linear combination of $(\mathrm{AJ}_1^0(q_1^*\omega, 0), \mathrm{AJ}_2^0(q_1^*\omega, 0))$, $(\mathrm{AJ}_1^0(0, q_1^*\omega'), \mathrm{AJ}_2^0(0, q_1^*\omega'))$ and $(p_1^*(\theta H^2(\mathrm{Pic}^4 C)), 0)$, we have $\gamma_1 = -j_{1*}y_1$ and $\gamma_2 = j_{2*}y_2$ for $y_1, y_2 \in H^2(M_{12})$. But since $j_1^*\gamma_1 = j_2^*\gamma_2 \in H^4(M_{12})$, we conclude immediately that $y_1 = y_2$, thus $(\gamma_1, \gamma_2) \in \mathrm{Im}(-j_{1*}, j_{2*})$. □

12. Appendix

12.1. The cohomology of $C^{(k)}$. For a smooth curve C of genus g, let m be the natural map from the Cartesian power C^k to $C^{(k)}$. We identify the cohomology $H^\bullet(C^{(k)})$ with its image under m^*, which is the invariant subring of $H^\bullet(C^k)$ under the action of the symmetric group $\mathfrak{S}_k$.

Macdonald [Ma] proved that the cohomology ring $H^\bullet(C^{(k)}, \mathbb{Z})$ is generated by (see "Notation and conventions" (2))

$$\xi_i \in H^1(C^{(k)}, \mathbb{Z}) \cong H^1(\mathrm{Pic}^k(C), \mathbb{Z}),\ i = 1, \dots, 2g$$

and the class $\eta \in H^2(C^{(k)}, \mathbb{Z})$ is subject to the following relations:

$$\xi_I \xi'_J (\sigma_K - \eta)\eta^d = 0 \tag{12.1}$$

where I,J,K are mutually disjoint subsets of $\{1, \dots, g\}$ and $|I| + |J| + 2|K| + d = k + 1$, $\xi_I = \Pi_{i\in I}\xi_i$, $(\sigma_K - \eta) = \Pi_{i\in K}(\sigma_i - \eta)$, etc.

12.2. The secant plane formula [ACGH, p. 342]**.** Let $|V| \subset |L|$ be a g^r_d. Fix $d \geq k \geq r$ and consider the following cycle:

$$\{ D \in C^{(k)} |\ E - D \geq 0 \text{ for some } E \in |V| \} \subset C^{(k)}.$$

The cohomology class of the above cycle is given by

$$\sum_{l=0}^{k-r} \binom{d-g-r}{l} \frac{\eta^l \theta^{k-r-l}}{(k-r-l)!} \tag{12.2}$$

12.3. The Gysin maps. If $\omega \in H^\bullet(C^k, \mathbb{Z})$, the Gysin push-forward for the sum map

$$m_* \colon H^\bullet(C^k, \mathbb{Z}) \to H^\bullet(C^{(k)}, \mathbb{Z})$$

is given by

$$m_*(\omega) = \sum_{\sigma \in \mathfrak{S}_k} \sigma^*(\omega).$$

If ω is $\mathfrak{S}_k$-invariant, then

$$m_*(\omega) = k!\ \omega,$$

reflecting the fact that m is generically $k!$ to 1.

Fix $k_1 + k_2 = k$, and let m_1 and m_2 be the symmetrization maps

$$C^k \xrightarrow{m_1} C^{(k_1)} \times C^{(k_2)} \xrightarrow{m_2} C^{(k)}.$$

For a cohomology class $\omega' \in H^\bullet(C^{(k_1)} \times C^{(k_2)})$ we have

$$m_{2*}(\omega') = \frac{1}{\deg(m_1)} m_*(m_1^*\omega') = \frac{1}{\deg(m_1)} \sum_{\sigma \in S_k} \sigma^*(m_1^*\omega').$$

In our case $g_C = 5$, and we have the following lemmas (whose proofs are straightforward computations).

Lemma 12.1. *The Gysin map* $m_*\colon H^2(C^{(2)}\times C^{(2)}) \longrightarrow H^2(C^{(4)})$ *acts as follows:*

$$\begin{array}{lcll} 1\otimes\theta & \longmapsto & \theta+10\eta, & \\ 1\otimes\eta & \longmapsto & 3\eta, & \\ \xi_i\otimes\xi_j & \longmapsto & 2\xi_i\xi_j & \text{for } j\neq i\pm 5, \\ \xi_i\otimes\xi_{i\pm5} & \longmapsto & 2\xi_i\xi_{i\pm5}\mp 2\eta, & \\ \xi_i\xi_j\otimes 1 & \longmapsto & \xi_i\xi_j & \text{for } j\neq i\pm 5. \end{array}$$

Lemma 12.2. *The Gysin map* $m_*\colon H^4(C^{(2)}\times C^{(2)}) \longrightarrow H^4(C^{(4)})$ *acts as follows:*

$$\begin{array}{lcll} \eta\otimes(\xi_i\cdot\xi_{i+5}) & \longmapsto & \eta\xi_i\xi_{i+5}+\eta^2, & \\ \eta\otimes\xi_i\xi_j & \longmapsto & \eta\xi_i\xi_j & \text{for } j\neq i\pm5, \\ \eta\otimes\eta & \longmapsto & 2\eta^2, & \\ \eta\xi_i\otimes\xi_j & \longmapsto & \eta\xi_i\xi_j & \text{for } j\neq i\pm5, \\ \eta\xi_i\otimes\xi_{i\pm5} & \longmapsto & \eta\xi_i\xi_{i\pm5}\mp\eta^2, & \\ \sigma_k\otimes\sigma_k & \longmapsto & 2\sigma_k\eta & \text{for } k=1,\dots,5, \\ \sigma_k\otimes\sigma_l & \longmapsto & \sigma_k\sigma_l+\eta^2\ \ k\neq l, & \\ \sigma_k\otimes\xi_k\xi_j & \longmapsto & \xi_k\xi_j\eta & \text{for } j\neq k+5, \\ \sigma_k\otimes\xi_i\xi_j & \longmapsto & \sigma_k\xi_i\xi_j & \text{for } i,j\notin\{k,k+5\}, \\ \eta^2\otimes 1 & \longmapsto & \eta^2. & \end{array}$$

Lemma 12.3. *The Gysin map* $m_*\colon H^4(C\times C^{(3)}) \longrightarrow H^4(C^{(4)})$ *acts as follows:*

$$\begin{array}{lcll} \eta\otimes\xi_i\xi_j & \longmapsto & \eta\cdot\xi_i\xi_j & \text{for } 1\leq i,j\leq 10, \\ 1\otimes\eta\xi_i\xi_j & \longmapsto & \eta\cdot\xi_i\xi_j & \text{for } j\neq i\pm5, \\ 1\otimes\eta\sigma_i & \longmapsto & \eta\cdot\sigma_i+\eta^2, & \\ \xi_i\otimes\xi_j\xi_k\xi_l & \longmapsto & \xi_i\xi_j\xi_k\xi_l & \text{for } j,k,l\neq i\pm5, \\ \xi_i\otimes\xi_{i\pm5}\xi_k\xi_l & \longmapsto & \xi_i\xi_{i\pm5}\xi_k\xi_l\mp\eta\cdot\xi_k\xi_l & \text{for } k,l\neq i\pm5, \\ \xi_i\otimes\eta\xi_j & \longmapsto & \eta\xi_i\xi_j & \text{for } j\neq i\pm5, \\ \xi_i\otimes\eta\cdot\xi_{i\pm5} & \longmapsto & \eta\cdot\xi_i\xi_{i\pm5}\mp\eta^2, & \\ \eta\otimes\eta & \longmapsto & \eta^2, & \\ 1\otimes\eta^2 & \longmapsto & 2\eta^2. & \end{array}$$

12.4. The cycle class of $F''_{r_0}|_{W_1\times C^{(4)}}$. We use the secant plane formula (Section 12.2) to compute the cycle class of $F''_{r_0}|_{W_1\times C^{(4)}}$ in each bidegree. For each bidegree $(d_1,d_2)+(e_1,e_2)$, the corresponding cycle $F_{(d_1,d_2)(e_1,e_2)}\subset W_k\times C_1^{(d_1)}\times C_1^{(e_1)}\times C_2^{(d_2)}\times C_2^{(e_2)}$ projects generically injectively to a product of some of the factors. Since the map $\lambda\colon F_{(d_1,d_2)(e_1,e_2)}\to W_k\times\Theta_0$ factors through these projections, we only need the cycle class of the projection of $F_{(d_1,d_2)(e_1,e_2)}$.

(1) (4,1)+(2,3). We first compute the class of the projection of $F_{(4,1)(2,3)}$ to $C^{(3)} \times C^{(3)} \times C \times C^{(3)}$ (with the identification $C_1 = C_2 = C$ and the embedding of W_1 into $C^{(3)} \times C^{(3)}$ via (q_1, q_2)). The cycles are given by the following conditions:

$$(\Gamma_3, \Gamma_3', a, D_3') \in C^{(3)} \times C^{(3)} \times C \times C^{(3)},$$

$$\begin{cases} h^0(K_C - p - q - \Gamma_3 - \Gamma_3') > 0 \\ h^0(K_C - \Gamma_3' - D_3') > 0 \\ h^0(K_C - \Gamma_3 - r_0 - a) > 0. \end{cases}$$

The map $\lambda_2|_{F_{(4,1)(2,3)}}$ factors through m, which sends $(\Gamma_3, \Gamma_3', a, D_3')$ to $(\Gamma_3, \Gamma_3', a + D_3') \in C^{(3)} \times C^{(3)} \times C^{(4)}$.

By the secant plane formula (12.2), the cycle class is given by the pull-back under the sum map from $C^{(3)} \times C^{(3)}$ (the first and fourth factor) to $C^{(6)}$ of the class

$$\frac{1}{2}\theta^2 - \eta\theta + \eta^2 \in H^4(C^{(6)})$$

cupped with the pull-back to $C^{(3)} \times C$ (second and third factors) of

$$\theta - \eta \in H^2(C^{(4)}),$$

then restriction to $W_1 \times C \times C^{(3)}$. Thus we obtain (cf. Notation (4))

$$\left[\frac{1}{2}\left(\theta_2 + \theta_4 + \delta_{24}\right)^2 - (\eta_2 + \eta_4)(\theta_2 + \theta_4 + \delta_{24}) + (\eta_2 + \eta_4)^2\right] \cdot \left[(\theta_1 + \theta_3 + \delta_{13}) - (\eta_1 + \eta_3)\right].$$

We only need the Kunneth component of this cycle class in $H^2(C^{(3)} \times C^{(3)}) \otimes H^4(C \times C^{(3)})$. We organize the terms according to the types in the Kunneth decomposition.

(a) Type $(2, 0, 0, 4)$.

$$\begin{aligned} &\left(\frac{1}{2}\theta_4^2 - \theta_4\eta_4 + \eta_4^2\right)(\theta_1 - \eta_1) \\ &\quad = \left(\sum_{i<j}[\sigma_{4i}\sigma_{4j}] - \theta_4\eta_4 + \eta_4^2\right)(\theta_1 - \eta_1) \\ &\quad = \left(\sum_{i<j}[(\sigma_{4i} + \sigma_{4j}) \cdot \eta_4 - \eta_4^2] - \theta_4\eta_4 + \eta_4^2\right)(\theta_1 - \eta_1) \\ &\quad = 3\left(\theta_4\eta_4 - 3\eta_4^2\right)(\theta_1 - \eta_1). \end{aligned}$$

(b) Type $(0,2,2,2)$.

$$\left(\frac{1}{2}\delta_{24}^2+\theta_2\theta_4-\theta_2\eta_4-\theta_4\eta_2+2\eta_2\eta_4\right)(\theta_3-\eta_3)$$
$$=\left(\frac{1}{2}\delta_{24}^2+\theta_2\theta_4-\theta_2\eta_4-\theta_4\eta_2+2\eta_2\eta_4\right)4\eta_3.$$

(c) Type $(1,1,1,3)$.

$$(\theta_4\delta_{24}-\eta_4\delta_{24})\,\delta_{13}.$$

By Lemma 12.3, the push-forwards of these classes to $C^{(3)}\times C^{(3)}\times C^{(4)}$ are

(a)

$$m_*\left(3\theta_4\eta_4-9\eta_4^2\right)(\theta_1-\eta_1) \quad = \quad 3\left(\theta_3\eta_3-\eta_3^2\right)(\theta_1-\eta_1). \tag{12.3}$$

(b)

$$\begin{aligned} & m_*\left(\frac{1}{2}\delta_{24}^2+\theta_2\theta_4-\theta_2\eta_4-\theta_4\eta_2+2\eta_2\eta_4\right)4\eta_3 \\ = \quad & m_*\left(2\eta_3\delta_{24}^2\right)+4m_*\left[(\theta_2\theta_4-\theta_2\eta_4-\theta_4\eta_2+2\eta_2\eta_4)\,\eta_3\right] \\ = \quad & m_*2\eta_3\sum_{i,j=1}^{5}\left[-\xi_{2i}\xi_{2j}\xi'_{4i}\xi'_{4j}+2\xi_{2i}\xi'_{2j}\xi'_{4i}\xi_{4j}-\xi'_{2i}\xi'_{2j}\xi_{4i}\xi_{4j}\right] \\ & +4m_*\left[(\theta_2\theta_4-\theta_2\eta_4-\theta_4\eta_2+2\eta_2\eta_4)\eta_3\right] \\ = \quad & 2\eta_3\sum_{i,j=1}^{5}\left[-\xi_{2i}\xi_{2j}\xi'_{3i}\xi'_{3j}+2\xi_{2i}\xi'_{2j}\xi'_{3i}\xi_{3j}-\xi'_{2i}\xi'_{2j}\xi_{3i}\xi_{3j}\right] \\ & +4\left[\theta_2\eta_3\theta_3-\theta_2\eta_3^2-\eta_2\eta_3\theta_3+2\eta_2\eta_3^2\right] \\ = \quad & 2\eta_3\delta_{23}^2+4\left[\theta_2\eta_3\theta_3-\theta_2\eta_3^2-\eta_2\eta_3\theta_3+2\eta_2\eta_3^2\right]. \end{aligned} \tag{12.4}$$

(c) For $i\neq j$, using the formula

$$m_*\xi'_{3i}\xi'_{4j}\sigma_{4k}=\begin{cases}0, & k=j\\ \xi'_{3i}\xi'_{3j}\eta_3, & k=i\\ \xi'_{3i}\xi'_{3j}\sigma_{3k}, & k\neq i,j\end{cases}$$

and

$$m_*\xi'_{3i}\xi'_{4j}\eta_4=\xi'_{3i}\xi'_{3j}\eta_3,$$

we compute that

(12.5)

$$
\begin{aligned}
&m_*(\theta_4-\eta_4)\delta_{13}\delta_{24}\\
&= m_*(\theta_4-\eta_4)\sum_{i,j=1}^{5}\left[-\xi_{1i}\xi_{2j}\xi'_{3i}\xi'_{4j}-\xi'_{1i}\xi'_{2j}\xi_{3i}\xi_{4j}+\xi_{1i}\xi'_{2j}\xi'_{3i}\xi_{4j}+\xi'_{1i}\xi_{2j}\xi_{3i}\xi'_{4j}\right]\\
&= m_*(\theta_4-\eta_4)\sum_{i=1}^{5}\left[-\xi_{1i}\xi_{2i}\xi'_{3i}\xi'_{4i}-\xi'_{1i}\xi'_{2i}\xi_{3i}\xi_{4i}+\xi_{1i}\xi'_{2i}\xi'_{3i}\xi_{4i}+\xi'_{1i}\xi_{2i}\xi_{3i}\xi'_{4i}\right]\\
&\quad + m_*(\theta_4-\eta_4)\sum_{i,j=1,i\neq j}^{5}\left[-\xi_{1i}\xi_{2j}\xi'_{3i}\xi'_{4j}-\xi'_{1i}\xi'_{2j}\xi_{3i}\xi_{4j}+\xi_{1i}\xi'_{2j}\xi'_{3i}\xi_{4j}\right.\\
&\qquad\qquad\left.+\xi'_{1i}\xi_{2j}\xi_{3i}\xi'_{4j}\right]\\
&= 0+\sum_{i=1}^{5}\left[\xi_{1i}\xi'_{2i}(-\sigma_{3i}\theta_3+\eta_3\theta_3-\eta_3^2)+\xi'_{1i}\xi_{2i}(\sigma_{3i}\theta_3-\eta_3\theta_3+\eta_3^2)\right]\\
&\quad + \sum_{i,j=1,i\neq j}^{5}\left[-\xi_{1i}\xi_{2j}\xi'_{3i}\xi'_{3j}-\xi'_{1i}\xi'_{2j}\xi_{3i}\xi_{3j}+\xi_{1i}\xi'_{2j}\xi'_{3i}\xi_{3j}+\xi'_{1i}\xi_{2j}\xi_{3i}\xi'_{3j}\right]\theta_3\\
&= \delta_{12}(\eta_3\theta_3-\eta_3^2)+\delta_{13}\delta_{23}\theta_3.
\end{aligned}
$$

(2) (2,3)+(4,1).
The cycle is

$$\{\,(\Gamma_3,\Gamma'_3,a',D_3)\in C^{(3)}\times C^{(3)}\times C\times C^{(3)}\mid h^0(\mathcal{O}_C(K_C-r_0-\Gamma_3-D_3)>0\,\}.$$

The map m sends $(\Gamma_3,\Gamma'_3,a',D_3)$ to $(\Gamma_3,\Gamma'_3,a'+D_3)\in C^{(3)}\times C^{(3)}\times C^{(4)}$.

Its class is the pull-back under the sum map to $H^6(C^{(3)}\times C^{(3)})$ of

$$\frac{\theta^3}{6}-\frac{\eta\theta^2}{2}+\eta^2\theta-\eta^3\in H^6(C^{(6)}),$$

which is equal to

$$\frac{1}{6}\left(\theta_1+\theta_4+\delta_{14}\right)^3-\frac{1}{2}(\eta_1+\eta_4)\left(\theta_1+\theta_4+\delta_{14}\right)^2+\left(\eta_1+\eta_4\right)^2\left(\theta_1+\theta_4+\delta_{14}\right)-\left(\eta_1+\eta_4\right)^3.$$

The contributing terms in the Kunneth decomposition have type $(2,0,0,4)$:

$$\frac{1}{2}\left(\theta_1\theta_4^2+\theta_4\delta_{14}^2\right)-\frac{1}{2}\left(\eta_1\theta_4^2+\eta_4\delta_{14}^2\right)-\theta_1\eta_4\theta_4+2\eta_1\eta_4\theta_4+\theta_1\eta_4^2-3\eta_1\eta_4^2$$
$$=\frac{1}{2}(\theta_1-\eta_1)\left(8\theta_4\eta_4-20\eta_4^2\right)+\frac{1}{2}(\theta_4-\eta_4)\delta_{14}^2+(2\eta_1-\theta_1)\eta_4\theta_4$$
$$+(\theta_1-3\eta_1)\eta_4^2$$
$$=2(\theta_1-\eta_1)\left(2\theta_4\eta_4-5\eta_4^2\right)+\left(\eta_4\delta_{14}^2+4\theta_1\eta_4^2-\theta_1\theta_4\eta_4\right)$$
$$+(2\eta_1-\theta_1)\eta_4\theta_4+(\theta_1-3\eta_1)\eta_4^2$$
$$=\eta_4\delta_{14}^2+2(\theta_1-\eta_1)\eta_4\theta_4+(-5\theta_1+7\eta_1)\eta_4^2.$$

Pushing forward to $C^{(3)}\times C^{(3)}\times C^{(4)}$:

$$\begin{aligned}(12.6)\qquad & m_*\left[\eta_4\delta_{14}^2+2(\theta_1-\eta_1)\eta_4\theta_4+(-5\theta_1+7\eta_1)\eta_4^2\right]\\ = \quad & \left(\eta_3\delta_{13}^2-2\theta_1\eta_3^2\right)+2(\theta_1-\eta_1)\left(\eta_3\theta_3+5\eta_3^2\right)+2(-5\theta_1+7\eta_1)\eta_3^2\\ = \quad & \eta_3\delta_{13}^2+2(\theta_1-\eta_1)\eta_3\theta_3+2(-\theta_1+2\eta_1)\eta_3^2.\end{aligned}$$

(3) (2,3)+(2,3).

The cycle consists of $(\Gamma_3,\Gamma_3',D_2,D_2')\in C^{(3)}\times C^{(3)}\times C^{(2)}\times C^{(2)}$ given by the conditions

$$\begin{cases} h^0(K_C-\Gamma_3-D_2)>0,\\ h^0(K_C-\Gamma_3'-D_2')>0,\\ r_0\in D_2.\end{cases}$$

The map m sends $(\Gamma_3,\Gamma_3',D_2,D_2')$ to $(\Gamma_3,\Gamma_3',D_2+D_2')\in C^{(3)}\times C^{(3)}\times C^{(4)}$. Note that in this bidegree, m is not a lifting of $\lambda_2|_{F_{(2,3)(2,3)}}$. As in the previous case, the cycle class is the restriction to $W_1\times C^{(2)}\times C^{(2)}$ of

$$[(\theta_1+\theta_3+\delta_{13})-(\eta_1+\eta_3)]\cdot[(\theta_2+\theta_4+\delta_{24})-(\eta_2+\eta_4)]\cdot\eta_3.$$

The contributing terms in the Kunneth decomposition are

(a) Type $(2,0,2,2)$.

$$[\theta_1\theta_4+\eta_1\eta_4-\eta_1\theta_4-\theta_1\eta_4]\cdot\eta_3.$$

(b) Type $(2,0,4,0)$.

$$(\theta_3-\eta_3)(\theta_2-\eta_2)\eta_3=4(\theta_2-\eta_2)\cdot\eta_3^2.$$

(c) Type $(1,1,3,1)$.

$$\delta_{13}\delta_{24}\cdot\eta_3.$$

Pushing these classes forward to $H^4(C^{(3)} \times C^{(3)} \times C^{(4)})$ by m_*, we obtain

(a)

$$\begin{aligned} m_*\left[(\theta_1-\eta_1)\eta_3\theta_4+(\eta_1-\theta_1)\eta_3\eta_4\right] &= (\theta_1-\eta_1)\left(\eta_3\theta_3+5\eta_3^2\right)+(\eta_1-\theta_1)\left(2\eta_3^2\right) \\ &= (\theta_1-\eta_1)\eta_3\theta_3+3(\theta_1-\eta_1)\eta_3^2. \end{aligned}$$

(b)

$$m_*\left[4(\theta_2-\eta_2)\cdot\eta_3^2\right] = 4(\theta_2-\eta_2)\cdot\eta_3^2.$$

(c)

$$\begin{aligned} & m_*\left[(\delta_{13}\delta_{24})\eta_3\right] \\ = \; & m_*\sum_{i,j=1}^{5}\left[-\xi_{1i}\xi_{2j}\xi'_{3i}\xi'_{4j}+\xi_{1i}\xi'_{2j}\xi'_{3i}\xi_{4j}+\xi'_{1i}\xi_{2j}\xi_{3i}\xi'_{4j}-\xi'_{1i}\xi'_{2j}\xi_{3i}\xi_{4j}\right]\eta_3 \\ = \; & \sum_{i,j=1}^{5}\left[-\xi_{1i}\xi_{2j}\xi'_{3i}\xi'_{3j}+\xi_{1i}\xi'_{2j}\xi'_{3i}\xi_{3j}+\xi'_{1i}\xi_{2j}\xi_{3i}\xi'_{3j}-\xi'_{1i}\xi'_{2j}\xi_{3i}\xi_{3j}\right]\eta_3 \\ & +\eta_3^2\sum_{i=1}^{5}\left[\xi_{1i}\xi'_{2i}-\xi'_{1i}\xi_{2i}\right] \\ = \; & \delta_{13}\delta_{23}\eta_3+\delta_{12}\eta_3^2. \end{aligned}$$

Finally, since λ_2 sends $(\Gamma_3, \Gamma'_3, D_2, D'_2)$ to $K_C(-D_2 - D'_2)$ (instead of $\mathcal{O}_C(D_2 + D'_2)$), we apply the involution $p_{2*}p_1^*$ to the sum of the classes in (a), (b), (c) as in Lemma 12.4 to obtain the cycle class

(12.7)

$$\begin{aligned} &(\theta_1-\eta_1)\theta_3(\theta_3-\eta_3)+3(\theta_1-\eta_1)\left(\frac{1}{2}\theta_3^2-\theta_3\eta_3+\eta_3^2\right) \\ &+4(\theta_2-\eta_2)\left(\frac{1}{2}\theta_3^2-\theta_3\eta_3+\eta_3^2\right)+\delta_{13}\delta_{23}(\theta_3-\eta_3)+\delta_{12}\left(\frac{1}{2}\theta_3^2-\theta_3\eta_3+\eta_3^2\right). \end{aligned}$$

Lemma 12.4. *The correspondence*

$$M_1 = \{\,(D_4, B_4) \in C^{(4)} \times C^{(4)} \mid D_4 + B_4 \equiv K_C\,\}$$

induces an involution $p_{2*}p_1^*\colon H^4(C^{(4)}) \to H^4(C^{(4)})$ where p_1 and p_2 are the two birational projections to $C^{(4)}$. Under the decomposition

$$H^4(C^{(4)}) \cong H^4(\operatorname{Pic}^4 C) \oplus \eta H^2(\operatorname{Pic}^4 C) \oplus \mathbb{C}\cdot\eta^2,$$

$p_{2*}p_1^*$ acts as an identity on $H^4(\operatorname{Pic}^4 C)$, sends $\eta\cdot\omega$ to $(\theta-\eta)\cdot\omega$ for any $\omega \in H^2(\operatorname{Pic}^4 C)$, and η^2 to $\frac{\theta^2}{2} - \eta\theta + \eta^2$.

Proof. First note that the proper transform of the algebraic cycle $r_0 + C^{(3)}$ under the birational map $p_2 p_1^{-1}$ is the cycle

$$\{ B_4 \in C^{(4)} | \; h^0(K_C - r_0 - B_4) > 0 \}$$

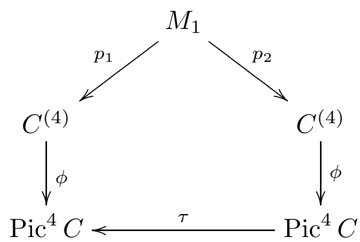
whose cohomology class is $\theta - \eta$ by the secant plane formula (12.2). Therefore $p_{2*}p_1^*$ sends η to $\theta - \eta$. Similarly, the proper transform of $2r_0 + C^{(2)}$ is

$$\{ B_4 \in C^{(4)} | \; h^0(K_C - 2r_0 - B_4) > 0 \}$$

whose cohomology class is $\frac{\theta^2}{2} - \eta\theta + \eta^2$, i.e.,

$$p_{2*}p_1^*\eta^2 = \frac{\theta^2}{2} - \eta\theta + \eta^2.$$

Now let us prove the statement on the summand $\eta H^2(\operatorname{Pic}^4 C)$. Consider the commutative diagram



where τ sends L to $K_C - L$. For any $\omega \in H^2(C^{(4)})$,

$$p_1^* (\eta \cdot \phi^*\omega) = p_1^*\eta \cdot \; p_1^*\phi^*\omega = p_1^*\eta \cdot (p_2^*\phi^*\tau^*\omega) \,.$$

By the projection formula,

$$p_{2*}p_1^*(\eta \cdot \phi^*\omega) = (p_{2*}p_1^*\eta) \cdot (\phi^*\tau^*\omega) = (\theta - \eta) \cdot \phi^*\omega.$$

Similarly, the statement about the $H^4(\operatorname{Pic}^4 C)$ summand is a consequence of the projection formula. □

12.5. The reducedness of $W_5^1(C_{pq})$ and of its compactification $\overline{W}_5^1(C_{pq})$.

Lemma 12.5. *The surface $\overline{W}_5^1(C_{pq})$ is reduced and is the flat limit of the family of $W_5^1(C_t)$ as t goes to* 0.

Proof. We will prove that $\overline{W}_5^1(C_{pq})$ with its reduced scheme structure is the flat limit of the family of $W_5^1(C_t)$ as t goes to 0. By [So] the family of theta divisors specializes to the ample Cartier divisor

$$\Theta_{pq} := \{ M \in J^5 C_{pq} \mid h^0(M) > 0 \}$$

on $J^5 C_{pq}$. We will prove that the Hilbert polynomial of $\overline{W}_5^1(C_{pq})$ with its reduced scheme structure and with respect to Θ_{pq} is equal to the Hilbert polynomial of $W_5^1(C_t)$ with respect to Θ_t for $t \neq 0$.

To compute the Hilbert polynomial of $\overline{W}^1_5(C_{pq})$, we use the normalization map (see Lemma 2.1)

$$\mu = (\nu^*)^{-1} \colon W_{pq} \longrightarrow \overline{W}^1_5(C_{pq}).$$

From this we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{W}^1_5(C_{pq})} \longrightarrow \mu_*\mathcal{O}_{W_{pq}} \longrightarrow \mathcal{M} \longrightarrow 0$$

where $\mathcal{M}$ is a sheaf supported on the image of $W^1_4(C)$ in $\overline{W}^1_5(C_{pq})$. It is immediately seen, by restricting the above sequence to $W^1_4(C)$, that

$$\mathcal{M} \cong \mathcal{O}_{W^1_4(C)}$$

so that we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{W}^1_5(C_{pq})} \longrightarrow \mu_*\mathcal{O}_{W_{pq}} \longrightarrow \mathcal{O}_{W^1_4(C)} \longrightarrow 0. \tag{12.8}$$

To compute $\chi(\mathcal{O}_{\overline{W}^1_5(C_{pq})}(n\Theta_{pq}))$, we therefore compute $\chi(\mathcal{O}_{W_{pq}}(n\Theta_{pq}))$ and $\chi(\mathcal{O}_{W^1_4(C)}(n\Theta_{pq}))$.

By [BC, p. 57], the inverse image of the divisor Θ_{pq} in $\mathbb{P}\operatorname{Pic}^5 C_{pq}$ is numerically equivalent to the sum of reduced divisors

$$\overline{(\nu^*)^{-1}\Theta_{C,x}} + \operatorname{Pic}^5_0$$

where we use the notation of Section 2.1, $\Theta_{C,x}$ is the image of $\Theta_C \subset \operatorname{Pic}^4 C$ in $\operatorname{Pic}^5 C$ by the addition of the general point $x \in C$ and $\overline{(\nu^*)^{-1}\Theta_{C,x}}$ is the closure of $(\nu^*)^{-1}\Theta_{C,x} \subset \operatorname{Pic}^5 C_{pq}$ in $\mathbb{P}\operatorname{Pic}^5 C_{pq}$.

Now, we have

$$(\nu^*)^{-1}\Theta_{C,x} = \{\, M \in \operatorname{Pic}^5 C_{pq} \mid h^0(\nu^* M(-x)) > 0 \,\}.$$

The trace of $\overline{(\nu^*)^{-1}\Theta_{C,x}}$ on the image of W_{pq} in $\mathbb{P}\operatorname{Pic}^5 C_{pq}$ is reduced for a general choice of x and is equal to

$$\Theta_{C,x}|_{W_{pq}} = \{\, L \in W_{pq} \mid h^0(L(-x)) > 0 \,\}.$$

Furthermore, it is immediate that

$$\operatorname{Pic}^5_0|_{W_{pq}} = X_q.$$

To compute the degree of Θ_{pq} on $W^1_4(C)$, we use the isomorphism $W^1_4(C) \cong X_p$. In this way we immediately see that the restriction of Pic^5_0 to $W^1_4(C)$ is zero while $\overline{(\nu^*)^{-1}\Theta_{C,x}}$ pulls back to $\Theta_C|_{W^1_4(C)}$ via the natural embedding $W^1_4(C) \subset \operatorname{Pic}^4 C$. Therefore, summarizing the above, we have

$$\chi\left(n\Theta_{pq}|_{W_{pq}}\right) = \chi\left(n\Theta_{C,x}|_{W_{pq}} + X_q\right)$$

and

$$\chi\left(n\Theta_{pq}|_{W^1_4(C)}\right) = \chi\left(n\Theta_C|_{W^1_4(C)}\right).$$

To compute $\chi(n\Theta_C|_{W_{pq}})$, we use the embedding q_1 of W_{pq} in $C^{(3)}$ given by $g^1_5 \mapsto |K - g^1_5|$. Via this embedding W_{pq} is identified with the reduced surface in $C^{(3)}$,

$$\{\, \Gamma_3 \mid h^0(K_C - p - q - \Gamma_3) > 0 \,\},$$

whose cohomology class by the secant plane formula (Section 12.2) is $\theta - \eta$. By Hirzebruch-Riemann-Roch

$$\begin{aligned}\chi\left(n\left(\Theta_C|_{W_{pq}} + X_q\right)\right) = \frac{1}{2} n\left(\Theta_C|_{W_{pq}} + X_q\right)\left(c_1\left(T_{W_{pq}}\right) + n\left(\Theta_C|_{W_{pq}} + X_q\right)\right) \\ + \frac{1}{12}\left(c_1^2\left(T_{W_{pq}}\right) + c_2\left(T_{W_{pq}}\right)\right).\end{aligned}$$

By [Ma, p. 332 (14.5)], the total Chern class of $C^{(3)}$ is

$$(1+\eta)^{-6}\prod_{i=1}^{5}(1+\eta+\sigma_i) = 1 - \eta - \theta - 9\eta^2 + 6\eta\theta - 56\eta^3.$$

So, using the tangent bundle sequence

$$0 \longrightarrow T_{W_{pq}} \longrightarrow T_{C^{(3)}}|_{W_{pq}} \longrightarrow \mathcal{O}_{W_{pq}}(W_{pq}) \longrightarrow 0,$$

we compute

$$c\left(T_{W_{pq}}\right) = \left(1 - 2\theta - 9\eta^2 + 4\eta\cdot\theta + 2\theta^2\right)|_{W_{pq}}.$$

Now, since X_q is the restriction of the zero section of a $\mathbb{P}^1$-bundle to W_{pq}, we have

$$X_q^2 = 0.$$

Furthermore, the degree of Θ_C on X_q is 10 since this is a Prym-embedded curve in $\operatorname{Pic}^4 C$. By the above,

$$c_1\left(T_{W_{pq}}\right) = 2\theta|_{W_{pq}};$$

hence the degree of $c_1(T_{W_{pq}})$ on X_q is 20. Putting all this together with the relations in [Ma, p. 325 (6.3)], we obtain

$$\chi\left(n\Theta_{pq}|_{W_{pq}}\right) = 30n^2 - 50n + 22.$$

To compute $\chi(n\Theta_C|_{W^1_4(C)})$, note that $W^1_4(C)$ has genus 11 and its cohomology class in $\operatorname{Pic}^4 C$ is twice the minimal class, i.e.,

$$\left[W^1_4(C)\right] = 2\frac{[\Theta_C]^4}{4!}.$$

Therefore, by Riemann-Roch for curves,

$$\chi\left(n\Theta_C|_{W^1_4(C)}\right) = 1 - 11 + \deg\left(n\Theta_C|_{W^1_4(C)}\right) = 10n - 10.$$

Finally, by (12.8),

$$\chi\left(\mathcal{O}_{\overline{W}^1_5(C_{pq})}\left(n\Theta_{pq}\right)\right) = \chi\left(n\Theta_{pq}|_{W_{pq}}\right) - \chi\left(n\Theta_{pq}|_{W^1_4(C)}\right) = 30n^2 - 60n + 32.$$

To compute the Hilbert polynomial of $W^1_5(C_t)$ for $t \neq 0$, we only need to do so for one smooth curve X of genus 6 such that $\dim_{\mathbb{C}} W^1_5(X) = 2$. If X is trigonal, $W^1_5(X)$ is the reduced union of two copies of $X^{(2)}$ (see [T]):

$$W^1_5(X) = X^{(2)} + g^1_3 \bigcup K_X - \left(X^{(2)} + g^1_3\right).$$

The intersection of these two components is the reduced curve

$$X_2\left(g^1_4\right) = \{\, D_2 \mid h^0(g^1_4 - D_2) > 0 \,\} \subset X^{(2)}$$

where $g^1_4 = |K_X - 2g^1_3|$. As in the previous case, we have the normalization exact sequence

$$0 \longrightarrow \mathcal{O}_{W^1_5(X)} \longrightarrow \mu_* \mathcal{O}_{X^{(2)} \coprod X^{(2)}} \longrightarrow \mathcal{O}_{X_2(g^1_4)} \longrightarrow 0.$$

So

$$\chi\left(n\Theta_X|_{W^1_5(X)}\right) = 2\chi\left(n\Theta_X|_{X^{(2)}}\right) - \chi\left(n\Theta_X|_{X_2(g^1_4)}\right).$$

This time, using similar methods, we compute

$$\chi\left(n\Theta_X|_{X^{(2)}}\right) = 15n^2 - 24n + 10,$$

$$\chi\left(n\Theta_X|_{X_2(g^1_4)}\right) = 12n - 12$$

and

$$\chi\left(n\Theta_X|_{W^1_5(X)}\right) = 30n^2 - 60n + 32.$$

□

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