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Half twists and the cohomology of hypersurfaces

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Abstract. A Hodge structure V of weight k on which a CM field acts defines, under certain conditions, a Hodge structure of weight $k-1$, its half twist. In this paper we consider hypersurfaces in projective space with a cyclic automorphism which defines an action of a cyclotomic field on a Hodge substructure in the cohomology. We determine when the half twist exists and relate it to the geometry and moduli of the hypersurfaces. We use our results to prove the existence of a Kuga-Satake correspondence for certain cubic 4-folds.

Introduction

Given a rational Hodge structure V of weight k with an action of a CM-field K such that no element of K has complex conjugate eigenvalues on $V^{k,0}$, one can associate to it a polarized Hodge structure $V_{1/2}$ (the half twist of V) of weight $k-1$ with the same underlying vector space. This half twist depends on the choice of a CM-type for K and was defined by one of us in [vG]. It has the property that there is an inclusion of Hodge structures $V_{1/2} \subset V \otimes H^1(A_K)$ (up to Tate twist) where A_K is an abelian variety with CM by K. The Hodge conjectures translate into interesting problems on

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the geometry of varieties having Hodge structures which allow a half twist, some of which we consider in this paper.

The half twist is also related to Kuga-Satake varieties. In case V has weight 2 with $V^{2,0}$ of dimension 1, Kuga and Satake [KS] define an abelian variety $KS(V)$ such that $H^1(KS(V))$ is the even part of the Clifford algebra of the polarization of V. They show that V is a direct summand of $H^1(KS(V))^{\otimes 2}$. If an imaginary quadratic field K acts on V then $V_{1/2}$ and $H^1(A_K)$ are direct summands of $H^1(KS(V))$ and $V \subset V_{1/2} \otimes H^1(A_K)$ (see [vG] where the other summands are also determined). The half twist is thus a partial (it exists only when a CM field acts on V) generalization (it works for any weight) of the Kuga-Satake construction.

The half twist is also implicit in the work of Kondo [K] on $K3$ surfaces which are degree 4 covers of $\mathbb{P}^2$ totally branched along plane quartics. These come with an action of $\mathbb{Z}/4\mathbb{Z}$ and the transcendental part V of $H^2(S)$ is a vector space over the field $\mathbf{Q}(i)$. Kondo shows that the moduli space for such V is isomorphic to the moduli space of certain weight one Hodge structures V' on which the field $\mathbf{Q}(i)$ acts. In fact V' is $V_{1/2}$, the half twist of V. The moduli space for V' is a quotient of a 6-ball and Kondo proves that the moduli space of curves of genus 3 is birationally isomorphic to this ball quotient. In the same paper, Kondo makes a similar construction for the moduli space of curves of genus 4. Kondo's work was motivated by results of Allcock, Carlson and Toledo [ACT1], [ACT2] who produce a complex hyperbolic structure on the moduli space of cubic surfaces by using cubic threefolds which are triple covers of $\mathbb{P}^3$ totally branched along a cubic surface.

The half twist also appears in the study of variations of Hodge structures. The families of weight two Hodge structures considered by Carlson and Simpson in [CS] are obtained as (negative) half twists of families of weight one Hodge structures with a $\mathbf{Q}(\sqrt{-1})$-action.

In this paper, we consider hypersurfaces Y_k of degree $d \geq 3$ and dimension $k \geq 1$ in $\mathbb{P}^{k+1}$ which are d-fold covers of $\mathbb{P}^k$ totally branched along hypersurfaces X_{k-1} of degree d in $\mathbb{P}^k$. Such hypersurfaces come with an action of $\mathbb{Z}/d\mathbb{Z}$. Letting σ be a generator of $\mathbb{Z}/d\mathbb{Z}$, we define V as the part of the cohomology of Y_k where the eigenvalues of σ are primitive d-th roots of 1. Then V is a vector space over the cyclotomic field K of d-th roots of unity. We will fix a CM-type Σ_0 of K in 2.1 which is optimal for the examples under consideration. That is, if V has a half twist for some CM-type, then it has a half twist for this CM-type.

We determine which V allow a half twist. If V does not have maximal level (i.e. if $V^{k,0} = 0$), V has a half twist. It is then also interesting to see if $V(q)$, the Tate twist of V whose level is equal to its weight, has a half twist. In theorem 2.6 we determine the values of d and k for which the half twist of $V(q)$ exists.

In case $V_{1/2}$ exists, we would like to have it as a Hodge substructure of the cohomology of a 'nice' algebraic variety. A basic result on half twists (cf. Proposition 1.6) implies that

$$V_{1/2}(-1) \subset H^k(Y_k) \otimes H^1(A_K) \subset H^{k+1}(Y_k \times A_K)$$

where A_K is an abelian variety with CM by K. We find two nicer varieties:

Theorem (cf. 3.4, 3.6). *Let Y_1 be the Fermat curve of degree d.*

1. *There is an embedding $H^1(A_K) \hookrightarrow H^1(Y_1)$. So if $V_{1/2}$ exists, then*
$$V_{1/2}(-1) \subset H^k(Y_k) \otimes H^1(Y_1) \subset H^{k+1}(Y_k \times Y_1).$$
2. *Let Z_{k+1} be the d-fold cover of $\mathbb{P}^{k+1}$ totally ramified along Y_k. If $V_{1/2}$ is exists, then*
$$V_{1/2}(-1) \subset H_0^{k+1}(Z_{k+1})$$
where the subscript 0 denotes primitive cohomology.

The second part of the above theorem follows from the first part and work of Shioda.

Since $V_{1/2}(-1)$ is not of maximal level in $H_0^{k+1}(Z_{k+1})$ and $H^{k+1}(Y_k \times Y_1)$, the general Hodge conjecture predicts that $V_{1/2}$ is a Hodge substructure in the cohomology of subvarieties of codimension one of Z_{k+1} and $Y_k \times Y_1$. We don't know how to find such a subvariety except in a few examples (cf. [vG] for the case of Calabi-Yau varieties).

A different and equally interesting type of question is a Torelli type problem: to what extent does the half twist $V_{1/2}$ determine X_{k-1}? In Sect. 4 we prove

Theorem (cf. 4.3). *Suppose $d = 3$, $k > 3$ and $V(q)_{1/2}$ is well-defined, i.e., by Theorem 2.6, $k = 3q + 1$. Then the differential of the period map which to X_{k-1} associates $V(q)_{1/2}$ is generically injective.*

An important application of our results is on Kuga-Satake correspondences. For a Hodge structure V of weight 2 with $V^{2,0}$ of dimension 1 there is the inclusion of Hodge structures $V \subset H^1(KS(V))^{\otimes 2}$. Supposing V is a Hodge substructure of the cohomology of a variety S, the Hodge conjecture asserts that this inclusion is induced by a correspondence on the product of S and $KS(V)^2$. The following theorem generalizes a result of Voisin [V2].

Theorem (cf. 5.6.) *There exists a Kuga-Satake correspondence for the weight two Hodge structure $V = H_0^4(Y_4)(1)$ where Y_4 is a triple cover of $\mathbf{P}^4$ branched along a (general) smooth cubic threefold.*

1. Polarized rational Hodge structures with automorphisms

1.1. We recall the basic results on half twists from [vG].

1.2. Definition. A (rational) Hodge structure of weight k $(\in \mathbf{Z}_{\geq 0})$ is a $\mathbf{Q}$-vector space V with a decomposition of its complexification $V_{\mathbf{C}} := V \otimes_{\mathbf{Q}} \mathbf{C}$ (where complex conjugation is given by $\overline{v \otimes z} := v \otimes \bar{z}$ for $v \in V$ and $z \in \mathbf{C}$):

$$V_{\mathbf{C}} = \oplus_{p+q=k} V^{p,q}, \qquad \text{such that} \quad \overline{V^{p,q}} = V^{q,p}, \qquad (p,\, q \in \mathbf{Z}_{\geq 0}).$$

Note that we insist on p and q being non-negative integers throughout this paper, so we only consider 'effective' Hodge structures.

1.3. CM-type. Recall that a CM-field K has $2r = [K : \mathbf{Q}]$ complex embeddings $K \hookrightarrow \mathbf{C}$ which are pairwise conjugate and that a CM-type is a subset $\{\sigma_1, \ldots, \sigma_r\}$ of distinct embeddings with the property that no two are complex conjugate. Hence if we define embeddings $\sigma_{r+i}(x) := \overline{\sigma_i(x)}$ then any embedding of K in $\mathbf{C}$ is a σ_j for some j, $1 \leq j \leq 2r$.

1.4. Half twists. Let V be a Hodge structure of weight k on which a CM-field K acts and let $\Sigma = \{\sigma_1, \ldots, \sigma_r\}$ be a CM-type. The eigenspaces of the K-action on $V^{p,q}$ are denoted by:

$$V_j^{p,q} := \{v \in V^{p,q} :\ xv = \sigma_j(x)v \quad \forall x \in K\}, \qquad 1 \leq j \leq 2r.$$

We define two subspaces of $V_{\mathbf{C}}$ whose direct sum is $V^{p,q}$:

$$V_+^{p,q} := \oplus_{i=1}^r V_i^{p,q}, \qquad V_-^{p,q} := \oplus_{i=1}^r V_{r+i}^{p,q}.$$

We define the Hodge decomposition of the negative half twist of V (w.r.t. Σ) by:

$$V_{-1/2}^{p,q} := V_+^{p-1,q} \oplus V_-^{p,q-1}.$$

It is not hard to see that this is a Hodge structure of weight $k+1$ on V. By successively performing the negative half twist one obtains $V_{-n/2}$, a Hodge structure on V of weight $k+n$. However, half twists do not give Tate twists: $V_{-2m/2} \neq V(-m)$. Half twists and Tate twists are compatible in the sense that $(V_a)(b) = (V(b))_a$ for all $a \in (1/2)\mathbf{Z}$ and $b \in \mathbf{Z}$.

A classical example is obtained by taking $V = K$, with the trivial Hodge structure $V^{0,0} = V_{\mathbf{C}}$. Then $K_{-1/2} \cong H^1(A_K, \mathbf{Q})$ for any abelian variety A_K of dimension r with CM by K and CM-type Σ.

1.5. Positive half twists. To define the (positive) half twist one would put:

$$V_{1/2}^{p,q} := V_+^{p+1,q} \oplus V_-^{p,q+1}.$$

This works if $V^{k,0}=0$. However, if $V^{k,0}\neq 0$, then the subspaces $V_-^{k,0}$ and $V_+^{0,k}$ of $V_{\mathbf{C}}$ do not appear in $(V_{1/2})_{\mathbf{C}}$ and therefore this definition does not define a Hodge structure on V (and in general not on any $\mathbf{Q}$-subspace of V). One can define the half twist of V only if $V_-^{k,0}=0$ (the complex conjugate of this space is $V_+^{0,k}$ which is then also 0). So one needs the eigenvalues of any $x\in K$ on $V^{k,0}$ to be in the set $\{\sigma(x)\}_{\sigma\in\Sigma}$. The following proposition shows that half twists appear naturally in certain tensor products.

1.6. Proposition. (See [vG].) *Let V be a Hodge structure with CM by K and fix a CM-type Σ of K. Then we have an inclusion of Hodge structures (both of weight $k+1$):*

$$V_{-1/2}\subset V\otimes_{\mathbf{Q}}K_{-1/2},$$

given by:

$$V_{-1/2}=\left\{w\in V\otimes_{\mathbf{Q}}K_{-1/2}:\ (x\otimes 1)w=(1\otimes x)w\quad\forall x\in K\right\}.$$

Similarly, if V admits a positive half twist, the Hodge structure $V_{1/2}(-1)$ of weight $k-1+2=k+1$ is a Hodge substructure of $V\otimes K_{-1/2}$:

$$V_{1/2}(-1)=\left\{w\in V\otimes_{\mathbf{Q}}K_{-1/2}:\ (x\otimes 1)w=(1\otimes\bar{x})w\quad\forall x\in K\right\}.$$

2. Generalities on Hodge structures of hypersurfaces

2.1. The Hodge substructure V. For a smooth $(k-1)$-dimensional hypersurface X_{k-1} of degree d

$$X_{k-1}:=Zeroes(F_d(x_0,\dots,x_k))\qquad(\subset\mathbf{P}^k),$$

the cyclic d-fold cover of $\mathbf{P}^k$ branched along X_{k-1} is the smooth k-fold Y_k defined by

$$Y_k:=Zeroes(x_{k+1}^d+F_d(x_0,\dots,x_k))\qquad(\subset\mathbf{P}^{k+1}).$$

We denote by α_k the automorphism of order d defined by

$$\begin{array}{lrcl}\alpha_k: & Y_k & \longrightarrow & Y_k\\ & (x_0:\ldots:x_k:x_{k+1}) & \longmapsto & (x_0:\ldots:x_k:\zeta x_{k+1})\ (\zeta=e^{2\pi i/d}).\end{array}$$

The action of α_k on $H_0^k(Y_k,\mathbf{Q})$ makes this space a $\mathbf{Q}[T]/(T^d-1)$-module. We will denote by

$$V\hookrightarrow H_0^k(Y_k,\mathbf{Q})$$

the largest subspace on which the eigenvalues of α_k^* are primitive d-roots of unity. If d is a prime number we have $V = H_0^k(Y_k, \mathbf{Q})$. The restriction of α_k^* to V is denoted by α:

$$\alpha := (\alpha_k^*)_{|V} : V \longrightarrow V.$$

The subring $K := \mathbf{Q}[\alpha]$ of $End(V)$ is isomorphic to the field of d-th roots of unity. The complex embeddings of K are:

$$\sigma_a : K \longrightarrow \mathbf{C}, \qquad \sum x_j \alpha^j \longmapsto \sum x_j \zeta^{aj} \qquad (x_j \in \mathbf{Q}, \zeta = e^{2\pi i/d})$$

with $a \in (\mathbf{Z}/d\mathbf{Z})^*$. We define the CM-type Σ_0 for K by:

$$\Sigma_0 := \{\sigma_a\}_{0<a<d/2}.$$

All half twists throughout this paper are taken w.r.t. this CM-type. Thus V has a half twist (w.r.t. Σ_0) if and only if all eigenvalues of α on $V^{k,0}$ are in the set $\{\sigma_a(\zeta)\}_{0<a<d/2}$.

We will often identify K with $\mathbf{Q}(\zeta)$ via σ_1. With this convention, Proposition 1.6 implies that the half twist, when it exists, may also be defined via:

$$V_{1/2}(-1) = \left\{ w \in V \otimes_{\mathbf{Q}} K_{-1/2} : \ (\alpha \otimes \zeta) w = w \right\}$$

since $1 \otimes \bar{\zeta} = (1 \otimes \zeta)^{-1}$. We observe that if the half twist exists, then the eigenvalues of $\alpha \otimes \zeta$ on $V^{k,0} \otimes (K_{-1/2})^{1,0}$ are contained in the set $\{\sigma_a(\zeta) \cdot \sigma_b(\zeta)\}_{0<a,b<d/2}$, so none of these is equal to 1 and hence $V_{1/2}$ is indeed an 'effective' Hodge structure of weight $k-1$.

We now review some basic results on Hodge structures of hypersurfaces and the action of automorphisms on their cohomology and apply them to V.

2.2. Hodge numbers. The primitive cohomology of a degree d hypersurface X_k in $\mathbf{P}^{k+1}$ can be calculated using Griffiths residues (see [G] page 44):

$$H_0^{k-q,q}(X_k) \cong H^0\left(\mathcal{O}_{\mathbb{P}^{k+1}}(d(q+1)-k-2)\right)/J_{d(q+1)-k-2}$$

where $J = \oplus J_r$ is the graded jacobian ideal of X_k, i.e.., the ideal generated by the partial derivatives of an equation for X_k.

2.3. Automorphisms. We compute the dimensions of the eigenspaces in the primitive cohomology

$$H_0^{k-q,q}(Y_k)(i) := \{x \in H_0^{k-q,q}(Y_k) : \ \alpha_k^* x = \zeta^i x\ \}$$

where α_k is the automorphism of Y_k defined in 2.1 above. The following lemma is an immediate consequence of a result of Shioda (cf. [S] Theorem I).

2.4. Lemma. *Let J' be the Jacobian ideal of $F_d \in \mathbf{C}[x_0, \ldots, x_k]$. Then we have:*

$$h_0^{k-q,q}(Y_k)(i) = \dim H^0\left(\mathcal{O}_{\mathbb{P}^k}(d(q+1)-k-1-i)\right)/J'_{d(q+1)-k-1-i}.$$

Proof. According to [S], Theorem I, the dimension of the eigenspace is given by the number of $(k+1)$-tuples $a_0, \ldots, a_k$ with $1 \le a_j \le d-1$ such that $(a_0 + \ldots + a_k + i)/d = q+1$. Moreover $h_0^{k-q,q}(Y_k)(i) = 0$ if $i = 0$ hence we take $1 \le i \le d-1$. Writing $b_j := a_j - 1$, so $0 \le b_j \le d-2$, we obtain $b_0 + \ldots + b_k = d(q+1) - k - 1 - i$. To each such $(k+1)$-tuple we associate the monomial $x_0^{b_0} \ldots x_k^{b_k}$, which is homogeneous of degree $d(q+1) - k - 1 - i$ and has no variable to power $\ge d-1$. Since these dimensions are constant in families of smooth hypersurfaces, we may assume that $F_d = \sum x_i^d$ and we obtain the desired equality. $\square$

2.5. The half twist for V. Since V is the subspace of $H_0^k(Y_k, \mathbf{Q})$ on which the eigenvalues of α_k^* are primititive d-th roots of unity, we have:

$$V^{k-q,q} = \oplus_{i \in (\mathbf{Z}/d\mathbf{Z})^*} H_0^{k-q,q}(Y_k)(i).$$

In case $V^{k,0} = 0$, the half twist of V always exists. It may happen that $V_{1/2}$ has higher level than V (for example, a cubic surface Y_2 has $V_{\mathbf{C}} = V^{1,1}$ and thus level 0, but $V_{1/2}$ has weight one and hence has level 1). In case $V^{k,0} = 0$, we will therefore also consider the Tate twist $V(q)$ of V with q choosen such that $V^{k-q,q} \neq 0$ but $V^{u,s} = 0$ if $u > k-q$. Then $V_{1/2}$ has lower level than V exactly when $V(q)$ has a half twist. Recall that all our half twists are for the CM-type Σ_0 fixed in 2.1.

2.6. Theorem. *Let Y_k be a hypersurface of degree d in $\mathbf{P}^{k+1}$ as in Sect. 2.1. Define $q \in \mathbf{Z}_{\ge 0}$ and t by:*

$$k = qd + t \qquad \text{with} \quad t \in \{-1, 0, \ldots d-2\}.$$

Then $H_0^{k-q,q}(Y_k)$ is the 'extremal' summand of $H_0^k(Y_k, \mathbf{C})$, i.e.,

$$H_0^{k-q,q}(Y_k) \neq 0, \qquad H_0^{u,s}(Y_k) = 0 \quad \text{if } u > k-q.$$

The Tate twist $V(q)$ of the Hodge substructure V of $H_0^k(Y_k, \mathbf{Q})$ has a half twist if and only if

1. *either $d \not\equiv 2 \bmod 4$ and:*

$$t > \frac{d-4}{2},$$

2. or $d \equiv 2 \bmod 4$ *and*

$$t > \frac{d-6}{2}.$$

Proof. The integer q is the smallest nonnegative integer for which $H_0^{k-q,q}(Y_k) \neq 0$. Therefore, by the formula for the Hodge numbers of Y_k (Sect. 2.2), we have that q is the nonnegative integer such that $d(q+1)-k-2 \geq 0$ and $dq-k-2 < 0$, i.e., q is the integer satisfying

$$q < \frac{k+2}{d} \leq q+1.$$

Now it is clear that $k = qd + t$ as in the statement of the proposition. We have

2.7. Lemma. *For all* i

$$h_0^{k-q,q}(Y_k)(i) > h_0^{k-q,q}(Y_k)(i+1)$$

except if $d = 3$ *and* $k = 2$ *in which case the two dimensions are equal.*

Proof. Put $a = d(q+1)-k-2 = d-t-2$. Recall that (Lemma 2.4)

$$h_0^{k-q,q}(Y_k)(i) = \dim H^0\left(\mathcal{O}_{\mathbb{P}^k}(a+1-i)\right)/J'_{a+1-i}.$$

We may assume that J' is generated by $x_0^{d-1}, \ldots, x_k^{d-1}$, and then $H^0(\mathcal{O}_{\mathbb{P}^k}(a+1-i))/J'_{a+1-i}$ is spanned by monomials $x_0^{a_0}\ldots x_k^{a_k}$ with $\sum a_j = a+1-i$ and $a_j \leq d-2$ for all j. Since $a = d-t-2$ and $-1 \leq t \leq d-2$, multiplication by x_k induces an injection

$$H_0^{k-q,q}(Y_k)(i+1) \hookrightarrow H_0^{k-q,q}(Y_k)(i)$$

except if $a = d-1$ and $i = 1$. In this last case we can define a different map which is also injective: we send x_k^{d-2} to $x_0^{d-2}x_1$ and we send all other monomials to their product by x_k. This inclusion is not surjective except when $d = 3$ and $k = 2$. $\square$

To finish the proof of the theorem note that when $d = 3$ and $k = 2$, we have $q = 1$ and $V(-1)$, of weight 0, does not have a half twist. Now $V(q)$ has a half twist, for our chosen CM type Σ_0, if and only if $H_0^{k-q,q}(Y_k)(i) = 0$ for $i \in (\mathbf{Z}/d\mathbf{Z})^*$ and $d/2 < i < d$. Using Lemma 2.7, we find that $V(q)$ has a half twist if and only if $h_0^{k-q,q}(Y_k)(j) = 0$ where $j \in (\mathbf{Z}/d\mathbf{Z})^*$ is the smallest unit with $j > d/2$.

If $d = 2e$ with e even or $d = 2e+1$, the smallest unit in $(\mathbf{Z}/d\mathbf{Z})^*$ larger than $d/2$ is $e+1 \in (\mathbf{Z}/d\mathbf{Z})^*$. Therefore V has a half twist if and only if

$h_0^{k-q,q}(Y_k)(e+1) = 0$. This is equivalent to $a - e < 0$, that is, if $d = 2e$ with e even:

$$(d-t-2) - d/2 < 0 \iff t > \frac{d-4}{2}$$

and the same result holds if $d = 2e+1$.

If $d = 2e$ with e odd, the smallest unit in $(\mathbf{Z}/d\mathbf{Z})^*$ larger than $d/2$ is $e+2$, hence V has a half twist if and only if $a - e - 1 < 0$, that is:

$$(d-t-2) - (d/2) - 1 < 0 \iff t > \frac{d-6}{2}.$$

□

2.8. Example. For a cubic hypersurface Y_k the Hodge structure $V(q)$ has a half twist if and only if $k = 3q+1$. In that case we have $h_0^{k-q,q}(Y_k) = 1$.

2.9. Corollary. *Let Y_k have degree d. Then $V \subset H^k(Y_k, \mathbf{Q})_0$ has a half twist if and only if*

$$\begin{array}{cccc} \textit{either} & d < 2k+4 & \textit{and} & k \equiv 0 \bmod 2, \\ \textit{or} & d \leq 2k+4 & \textit{and} & k \equiv 1 \bmod 2. \end{array}$$

Proof. If $d < k+2$, then $V^{k,0} = 0$ and, as we saw in 2.5, V has a half twist. Else $k \leq d-2$ and we have $q = 0$ and $t = k$. Therefore V has a half twist if and only if either $k > (d-4)/2$, that is $d < 2k+4$, or $d \equiv 2 \bmod 4$ in which case $k > (d-6)/2$ hence $d < 2k+6$. In the last case we must actually have $d \leq 2k+4$ with equality only if k is odd. □

3. The half twist of V for a cyclic cover

3.1. Cyclic covers. In Sect. 2.1 we defined a Hodge substructure $V \subset H_0^k(Y_k, \mathbf{Q})$ which has CM by the field K of d-th roots of unity. In case V has a positive half twist $V_{1/2}$, we saw in Proposition 1.6 that we have an embedding

$$V_{1/2}(-1) \subset V \otimes K_{-1/2} \subset H_0^k(Y_k, \mathbf{Q}) \otimes K_{-1/2}.$$

Recall that α_k is the automorphism of $Y_k = Zeroes(x_{k+1}^d + F_d(x_0, \ldots, x_k))$ defined by

$$\alpha_k : Y_k \longrightarrow Y_k, \qquad (x_0 : \ldots : x_k : x_{k+1}) \longmapsto (x_0 : \ldots : x_k : \zeta x_{k+1})$$

where $\zeta = e^{2\pi i/d}$. In particular, the Fermat curve Y_1 has the automorphism α_1 (taking $F_d := x_0^d + x_1^d$).

3.2. Lemma. *The Hodge structure $K_{-1/2}$ is a Hodge substructure of $H^1(Y_1, \mathbf{Q})$. In particular, we have the following geometric realization of the half twist:*

$$V_{1/2}(-1) \subset H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}) \subset H^{k+1}(Y_k \times Y_1).$$

Proof. On the Fermat curve Y_1 defined by $x_0^d + x_1^d + x_2^d$ consider the automorphism

$$\gamma : Y_1 \longrightarrow Y_1, \qquad (x_0 : x_1 : x_2) \longmapsto (x_0 : \zeta^{-1} x_1 : \zeta x_2).$$

The holomorphic one forms on Y_1 are the $y_1^{a-(d-1)} y_2^b \mathrm{d}y_2$ (where we put $y_i = \frac{x_i}{x_0}$), with $0 \leq a + b \leq d - 3$ and $a, b \geq 0$. For the γ-invariants we have $-a + (d-1) + b + 1 \equiv 0 \bmod d$, that is $a \equiv b \bmod d$. Hence the possible values for (a, b) are $(0,0)$, $(1,1), \dots, ([\frac{d-3}{2}], [\frac{d-3}{2}])$. In particular $h^1(Y_1, \mathbf{Q})^\gamma = 2[\frac{d-1}{2}]$ and the eigenvalues of α_1^* on $H^{1,0}(Y_1)^\gamma$ are $\zeta, \zeta^2, \dots, \zeta^{[(d-1)/2]}$. Since $\alpha_1^d = 1$ and α_1 commutes with γ, the space $H^1(Y_1, \mathbf{Q})^\gamma$ is a $\mathbf{Q}[T]/(T^d - 1)$-module (via $T \mapsto \alpha_1$). The eigenvalue ζ has multiplicity one, hence $H^1(Y_1, \mathbf{Q})^\gamma$ has a unique submodule M isomorphic to $K = \mathbf{Q}(\zeta)$. As a Hodge structure, $M \cong K_{-1/2}$ since both have the same CM-type. The geometric realization follows from

$$V_{1/2}(-1) \subset V \otimes K_{-1/2} \subset H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}).$$

This concludes the proof of the lemma. $\square$

3.3. We can improve on the above by embedding $V_{1/2}(-1)$ in $H^{k+1}(Z_{k+1}, \mathbf{Q})$ where

$$Z_{k+1} := Zeroes(x_{k+2}^d + x_{k+1}^d + F_d(x_0, \dots, x_k)) \qquad (\subset \mathbf{P}^{k+2}).$$

3.4. Theorem. *With the notation of 2.1 and 3.1, assume that the half twist $V_{1/2}$ of the Hodge structure V $(\subset H_0^k(Y_k, \mathbf{Q}))$ exists. Let W be the Hodge substructure of $H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q})$ on which the automorphism $\beta := (\alpha_k^*, \alpha_1^*)$ acts trivially:*

$$W := \left(H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}) \right)^{\langle \beta \rangle}.$$

Then there is an inclusion of Hodge structures of weight $k+1$:

$$V_{1/2}(-1) \hookrightarrow W = \left(H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}) \right)^{\langle \beta \rangle}.$$

Proof. By Lemma 3.2,

$$W = \left(V \otimes K_{-1/2}\right)^{\langle\beta\rangle} \oplus \dots .$$

Since $\beta(v) = (\alpha_k^* \otimes \alpha_1^*)v$ for $v \in H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q})$ we also have:

$$\beta(v) = v \qquad \text{if and only if} \qquad (\alpha_k^* \otimes 1)v = (1 \otimes \bar{\alpha}_1^*)v.$$

Hence, by Proposition 1.6, we have $\left(V \otimes K_{-1/2}\right)^{\langle\beta\rangle} \cong V_{1/2}(-1)$ and thus $V_{1/2}(-1) \subset W$. □

We have the following immediate consequence of results of Shioda.

3.5. Proposition. (Shioda.) *The primitive cohomology of Z_{k+1} has a direct sum decomposition*

$$H_0^{k+1}(Z_{k+1}, \mathbf{Q}) \cong H_0^{k-1}(X_{k-1}, \mathbf{Q})(-1)^{\oplus(d-1)} \oplus W.$$

Proof. In the case of Fermat varieties the direct sum decomposition of $H_0^{k+1}(Z_{k+1}, \mathbf{Q})$ is proved in [S], Theorem II and it applies in this case as well since the result is topological in nature. □

3.6 Corollary. *We have an inclusion of Hodge structures of weight $k-1$:*

$$H_0^{k-1}(X_{k-1}, \mathbf{Q})^{\oplus(d-1)} \oplus V_{1/2} \hookrightarrow H_0^{k+1}(Z_{k+1}, \mathbf{Q})(1).$$

Proof. Immediate consequence of 3.4 and 3.5. □

We could then ask whether we can have

$$V_{1/2}(-1) = W.$$

The following dimension computation shows that this is the case if and only if $d = 3$.

3.7. Lemma. *For all $k \geq 2$ we have the identity*

$$h_0^{k+1} = (d-1)h_0^{k-1} + (d-2)h_0^k.$$

where h_0^k is the dimension of the primitive cohomology of a smooth k-dimensional hypersurface of degree d in $\mathbf{P}^{k+1}$. Therefore, by Proposition 3.5

$$\dim W = (d-2)h_0^k.$$

Proof. It suffices to prove that $h_0^{k+1} = (d-1)h_0^{k-1} + (d-2)h_0^k$ for special hypersurfaces X_k which are cyclic degree d covers of $\mathbb{P}^k$ totally ramified along a degree d hypersurface X_{k-1}. Then the Zeuthen-Hurwitz formula gives

$$\chi_{top}(X_k) = d\chi_{top}(\mathbb{P}^k) - (d-1)\chi_{top}(X_{k-1})$$

or

$$k+1+(-1)^k h_0^k = d(k+1) - (d-1)(k+(-1)^{k-1}h_0^{k-1})$$

or

$$(-1)^k h_0^k = (d-1)(1-(-1)^{k-1}h_0^{k-1}).$$

From this we calculate

$$\begin{aligned}(-1)^{k+1}h_0^{k+1} + (-1)^k(d-2)h_0^k &= (d-1)(1-(-1)^k h_0^k) \\ &\quad +(-1)^k(d-2)h_0^k \\ &= (d-1)-(-1)^k h_0^k \\ &= (d-1)-(d-1)(1-(-1)^{k-1}h_0^{k-1}) \\ &= (d-1)(-1)^{k-1}h_0^{k-1},\end{aligned}$$

multiplying by $(-1)^{k+1}$ gives the desired result. $\square$

4. Torelli theorems

4.1. Let $X_{k-1} \subset \mathbf{P}^k$ be a hypersurface such that $V \subset H_0^k(Y_k, \mathbf{Z})$ allows a half twist. Then we can associate to X_{k-1} two polarized Hodge structures of weight $k-1$, one is its primitive cohomology $H_0^{k-1}(X_{k-1}, \mathbf{Q})$ and the other $V_{1/2}$. We have thus two period maps: one sending X_{k-1} to $H_0^{k-1}(X_{k-1}, \mathbf{Q})$ and the other sending X_{k-1} to $V_{1/2}$. The first map has been extensively studied and is known to be generically injective in many cases. The injectivity of the second map means that we can recover X_{k-1} from $V_{1/2}$. We know this is the case for example if $k=2$ and $d=4$ by the work of Kondo [K] and if $k=4$ and $d=3$ by recent work of Allcock, Carlson and Toledo (see Sect. 5 below).

In case we do not only have the polarized Hodge structure $V_{1/2}$ but we also know the action of $\zeta = e^{2\pi i/d}$ on its underlying $\mathbf{Z}$-module (this action comes from the action of α on V, which has the same underlying $\mathbf{Z}$-module), we show that V can be recovered from $V_{1/2}$. In case d is prime we thus recover $H_0^k(Y_k, \mathbf{Z})$ which in many cases determines Y_k and hence X_{k-1}. In case d is not prime we do not know to what extent V determines $H_0^k(Y_k, \mathbf{Z})$ and thus, via the Torelli theorems for hypersurfaces, the variety Y_k.

4.2 Lemma. *Let V be a Hodge structure of weight k which allows a half twist $V_{1/2}$. Then we recover V from $V_{1/2}$ in the following way*

$$V = (V_{1/2})_{-1/2} = \{w \in V_{1/2} \otimes K_{-1/2} : \ (\alpha \otimes \zeta^{-1})w = w \}.$$

Proof. This follows from 1.6 (cf. [vG]). □

Recall that $V_{1/2}(-1)$ is a Hodge substructure of W, defined in Theorem 3.4, with equality if and only if $d = 3$. In case we do not know the action of ζ on $(V_{1/2})_{\mathbf{Z}}$, we use the inclusion $W \subset H_0^{k+1}(Z_{k+1}, \mathbf{Q})$ to prove the following.

4.3. Lemma *Suppose $k \geq 2$. The differential of the period map which to X_{k-1} associates the polarized Hodge structure W is generically injective. In particular, this period map is generically finite.*

Proof. The tangent space to the space of periods for W is the space of compatible $(k+1)$-tuples of homomorphisms $W^{p,r} \to W^{p-1,r+1}$ which preserve the polarization. The space of infinitesimal deformations of X_{k-1} is isomorphic to

$$H^0\left(\mathcal{O}_{\mathbb{P}^k}(d)\right)/J'_d.$$

Given a degree d polynomial G_d (modulo J'), the image of the corresponding deformation in the tangent space to the space of periods for W is the $(k+1)$-tuple of homomorphisms given by multiplication by G_d in the description

$$W^{k+1-p,p} \cong \frac{H^0\left(\mathcal{O}_{\mathbb{P}^{k+2}}(d(p+1)-k-3)\right)/J_{d(p+1)-k-3}}{\left(\sum_{i=0}^{d-2} x_{k+1}^i x_{k+2}^{d-2-i} H^0(\mathcal{O}_{\mathbb{P}^k}(dp-k-1))\right)}$$

(see 2.2 and 3.5). To show that the differential is injective, it suffices to show that one such homomorphism is non-zero, i.e., there is a p and an element of the above space which is not sent to zero by multiplication by G_d. Again, as before, it is enough to check this for the Fermat hypersurface of degree d. In this case, a basis of the above vector space is given by the monomials of degree $d(p+1)-k-3$ in $x_0, \ldots, x_k$ containing no variable to power $\geq d-1$ and products $x_{k+1}^i x_{k+2}^j M$ where $i, j \leq d-2, i+j \neq d-2$ and M is a monomial of degree $d(p+1)-k-3-i-j$ in $x_0, \ldots, x_k$ containing no variable to power $\geq d-1$. An infinitesimal deformation of the Fermat hypersurface of degree d can be uniquely represented by a degree d polynomial G_d in the variables $x_0, \ldots, x_k$ containing no variable to power $\geq d-1$. After possibly dividing G_d by a non-zero constant, we have

$$G_d = \Pi_{j=0}^k x_j^{i_j} + H_d$$

where H_d does not contain the monomial $\Pi_{j=0}^k x_j^{i_j}$ and $\sum_{j=1}^k i_j = d$. Then, the image of $x_{k+1}^r x_{k+2}^s \Pi_{j=0}^k x_j^{m_j}$, where $m_j + l_j \leq d-2$ for all j, $\quad r, s \leq d-2, r+s \neq d-2$ and $\sum m_j = d(p+1)-k-3-r-s$, by multiplication by G_d is non-zero in $W^{k+1-p,p}$. $\square$

4.4. Cubics. In case Y_k has degree 3, we have $V = H_0^k(Y_k, \mathbf{Q})$. Since for $k > 1$ we have $V^{k,0} = 0$, its half twist exists and 3.4, 3.5 and 3.6 imply that $V_{1/2}(-1) = W$ and

$$H_0^{k+1}(Z_{k+1}, \mathbf{Q})(1) = H_0^{k-1}(X_{k-1}, \mathbf{Q})^{\oplus 2} \,\oplus\, H_0^k(Y_k, \mathbf{Q})_{1/2}.$$

So Lemma 4.3 is a Torelli Theorem for $V_{1/2}$ in this case.

4.5. Quartics. In case Y_k has degree 4, Theorem 2.6 with $d = 4$ shows that the Hodge substructure $V(q) \subset H^k(Y_k, \mathbf{Q})$ has a half twist if and only if $k = 4q+1$ or $4q+2$. In case $k = 4q+2$ we have $\dim V^{k-q,q} = h_0^{k-q,q}(Y_k) = 1$. From Corollary 2.9 it follows that V has a half twist for any k (in fact $V^{k,0} = 0$ for $k > 2$). We have:

4.6. Lemma. *The Hodge structure W splits as follows:*

$$W(1) = V_{1/2}^{\oplus 2} \oplus \left(V' \otimes K_{-1/2}\right),$$

where $V' \subset H_0^k(Y_k, \mathbf{Q})$ is the Hodge substructure on which α_k acts as -1. Therefore by 3.5

$$H_0^{k+1}(Z_{k+1}, \mathbf{Q})(1) \cong H_0^{k-1}(X_{k-1}, \mathbf{Q})^{\oplus 3} \oplus V_{1/2}^{\oplus 2} \oplus \left(V' \otimes K_{-1/2}\right).$$

Proof. We have $H_0^k(Y_k, \mathbf{Q}) = V \oplus V'$ and $H^1(Y_1, \mathbf{Q}) \cong K_{-1/2}^{\oplus 3}$ because the action of α_1^* on $H^{1,0}(Y_1)$ has eigenvalue i with multiplicity 2 and eigenvalue -1 with multiplicity 1 (the (-1)-eigenspace is a copy of $K_{-1/2}$, using permutations of the coordinates one finds the other two copies). Hence:

$$W(1) = \left(H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q})\right)^{\langle\beta\rangle} = V_{1/2}^{\oplus 2} \oplus \left(V' \otimes K_{-1/2}\right).$$

$\square$

4.7. In general we do not know to what extent $V_{1/2}$ determines V'. Note that V' is the primitive cohomology of the double cover of $\mathbf{P}^k$ ramified along X_{k-1}. In case $k = 2$, this double cover is a Del Pezzo surface and thus $V'(1)$ is a trival Hodge structure (of dimension 7). Therefore Lemma 4.6 is a Torelli Theorem for $V_{1/2}$ if $k = 2$ and $d = 4$.

We study the case of the quartic surface Y_2 in some detail since it is related to recent work of Kondo [K]. Start with a curve $C := X_1$ $(\subset \mathbf{P}^2)$ of degree 4. The surface Y_2 is a K3 surface, in particular $h^{2,0}(Y_2) = 1$. The primitive cohomology of a general Y_2 decomposes into the direct sum of its transcendental part V and its algebraic part $NS_0(Y_2)_{\mathbf{Q}}$:

$$H_0^2(Y_2, \mathbf{Q}) = V \oplus NS_0(Y_2)_{\mathbf{Q}}, \quad \dim_{\mathbf{Q}} V = 14, \quad \dim NS_0(Y_2)_{\mathbf{Q}} = 7.$$

Kondo [K] proves, (using the Torelli theorem for K3 surfaces) that the map $C \mapsto V_{\mathbf{Z}}$ defines an injective morphism from the moduli space of non-hyperelliptic curves of genus 3 to a 6-ball quotient and he studies its extension to hyperelliptic curves of genus 3 and to nodal plane quartics. This ball quotient is a moduli space of abelian varieties of dimension 7 with (1,6)-action by $K = \mathbf{Q}(i)$.

Kondo's construction can also be done with half twists: We can apply the half twist to the transcendental part V. The Hodge structure $V_{1/2}$ is of weight one, has dimension 14 and defines an (isogeny class of) abelian variety A_C of dimension 7 on which the action of the field K is of type (1,6):

$$V_{1/2} = H^1(A_C, \mathbf{Q}), \qquad \dim A_C = 7.$$

Since we can also carry out the half twist over $\mathbf{Z}$, that is we can define $(V_{\mathbf{Z}})_{1/2}$ by the same formula as for $V_{1/2}$, we can in fact define an abelian variety A_C (with $H^1(A_C, \mathbf{Z}) = (V_{\mathbf{Z}})_{1/2}$ and polarization induced by the polarization ψ on $V_{\mathbf{Z}}$ via $E(v, w) := \psi(v, iw)$). We can of course recover $V_{\mathbf{Z}}$ from $H^1(A_C, \mathbf{Z})$. This abelian variety is in fact Kondo's abelian variety.

Since $V_{1/2}(-1) \subset H^3(Z_3, \mathbf{Q})$, Kondo's abelian variety is isogeneous to a subvariety of the intermediate jacobian of the quartic threefold Z_3. It is well-known that

$$H^3(Z_C, \mathbf{C}) = H^{2,1}(Z_C) \oplus H^{1,2}(Z_C), \qquad \text{and} \quad h^{2,1}(Z_C) = 30.$$

In particular, the intermediate Jacobian J_C of Z_C is a principally polarized abelian variety. Lemma 4.6 implies that

$$J_C \sim J(C)^3 \times A_C^2 \times A_K^7$$

since now $V' \cong \mathbf{Q}(-1)^7 \subset NS_0(Y_2)_{\mathbf{Q}}$ is a trivial Hodge structure. Here A_K is an elliptic curve with CM by K.

5. Correspondences

5.1. In case the Hodge substructure $V \subset H_0^k(Y, \mathbf{Q})$ has a half twist we have an inclusion $V_{1/2}(-1) \subset H^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q})$ (cf. Lemma 3.2) and

Proposition 1.6 gives an inclusion $V \hookrightarrow V_{1/2} \otimes K_{-1/2}$, hence $V(-1) \subset V_{1/2}(-1) \otimes K_{-1/2}$. Thus we have inclusions:

$$V(-1) \hookrightarrow V_{1/2}(-1) \otimes K_{-1/2} \hookrightarrow H_0^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}) \\ \hookrightarrow H^{k+2}(Y_k \times Y_1^2, \mathbf{Q}).$$

Since also $V \subset H_0^k(Y_k, \mathbf{Q})$, the Hodge conjecture predicts that there is a cycle, i.e., a correspondence, on the product of Y_k and $Y_k \times Y_1^2$ which induces the inclusion of Hodge structures $V(-1) \hookrightarrow H_0^{k+2}(Y_k \times Y_1^2, \mathbf{Q})$. More precisely, there should be a cycle Γ on the product $Y_k \times Y_k \times Y_1^2$ such that the restriction of the map

$$\rho_{[\Gamma]} : H^k(Y_k, \mathbf{Q})(-1) \longrightarrow H^{k+2}(Y_k \times Y_1^2, \mathbf{Q}) \\ x \longrightarrow \pi_{2*}\left(\pi_1^*(x).[\Gamma]\right)$$

induces the inclusion

$$V(-1) \hookrightarrow H_0^{k+2}(Y_k \times Y_1^2, \mathbf{Q}).$$

Here π_1 and π_2 are the two projections

$$\pi_1 : Y_k \times Y_k \times Y_1^2 \longrightarrow Y_k \qquad \pi_2 : Y_k \times Y_k \times Y_1^2 \longrightarrow Y_k \times Y_1^2$$

and $[\Gamma]$ is the cohomology class of Γ. The map π_{2*} is the Poincaré dual of push-forward on homology.

We show that such a cycle exists.

5.2. Theorem. *Let α be the automorphism of V defined in 2.1. Let S be the Hodge substructure of $V \otimes K_{-1/2} \otimes K_{-1/2}$ defined by:*

$$S := \{w \in V \otimes K_{-1/2} \otimes K_{-1/2} : \ (\alpha \otimes \zeta \otimes 1)w = w, \ (1 \otimes \zeta \otimes \zeta)w = w\}.$$

Then $S \subset V_{1/2}(-1) \otimes K_{-1/2}$ and $S \cong V(-1)$. Moreover, there is a cycle $\Gamma \subset Y_k \times (Y_k \times Y_1^2)$ such that

$$\rho_{[\Gamma]} : H^k(Y_k, \mathbf{Q})(-1) \longrightarrow H^{k+2}(Y_k \times Y_1^2, \mathbf{Q})$$

induces the isomorphism

$$V(-1) \longrightarrow S \cong V(-1) \subset V_{1/2}(-1) \otimes K_{-1/2} \subset H^{k+2}(Y_k \times Y_1^2, \mathbf{Q}).$$

Proof. By Proposition 1.6 and Theorem 3.4, the $(\alpha \otimes \zeta)$-invariant subspace in $V \otimes K_{-1/2}$ is $V_{1/2}(-1)$, so $S \subset V_{1/2}(-1) \otimes K_{-1/2}$. Similarly, the $(\zeta \otimes \zeta)$-invariant subspace of $K_{-1/2} \otimes K_{-1/2}$ is $K_0(-1) \cong K(-1)$, a trivial Hodge structure of weight 2. For dimension reasons it follows that $S \cong V(-1)$.

Explicitly, let $\gamma = \alpha \otimes \zeta \otimes 1 \in End(V \otimes K(-1))$. Then $\gamma^d = 1$ hence $(\gamma - 1)(\gamma^{d-1} + \ldots + 1) = 0$ and the map $v \otimes c \mapsto \sum_{i=0}^{d-1} \gamma^i(v \otimes c)$ is the projection onto the subspace of γ-invariants. The isomorphism $V(-1) \to S$ is thus given by

$$v \longmapsto \sum_{i=0}^{d-1} \alpha^i(v) \otimes (\zeta^i \otimes 1)(c) \quad \subset V \otimes K(-1)$$

where $c \in K(-1) \subset K_{-1/2} \otimes K_{-1/2}$ is arbitrary but non-zero.

The existence of Γ follows from the fact that the Hodge substructure $K(-1) \subset K_{-1/2} \otimes K_{-1/2} \hookrightarrow H^1(Y_1) \otimes H^1(Y_1)$ is trivial, i.e., generated by $(1,1)$ classes. □

5.3. Using Shioda's work, we obtain a similar correspondence with Z_{k+1} instead of $Y_k \times Y_1$.

5.4. Theorem. *There is a cycle U on $Y_k \times Z_{k+1} \times Y_1$ such that the map*

$$\rho_{[U]} : H^k(Y_k, \mathbf{Q})(-1) \longrightarrow H^{k+2}(Z_{k+1} \times Y_1, \mathbf{Q})$$

induces the embedding

$$V(-1) \hookrightarrow V_{1/2}(-1) \otimes K_{-1/2} \hookrightarrow H^{k+1}(Z_{k+1}, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}).$$

Proof. The cycle Γ on the product of Y_k and $Y_k \times Y_1^2$ induces the map

$$\rho_{[\Gamma]} : H^k(Y_k, \mathbf{Q})(-1) \longrightarrow H^{k+2}(Y_k \times Y_1 \times Y_1, \mathbf{Q})$$

which induces the embedding

$$\begin{aligned} V(-1) \xrightarrow{\cong} S \cong V(-1) \subset V_{1/2}(-1) \otimes K_{-1/2} \\ \subset H^k(Y_k, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}). \end{aligned}$$

The work of Shioda ([S] Theorem II) provides a correspondence (a blow up of $Y_k \times Y_1$) which induces a map (as in Proposition 3.5)

$$\mu : H^k(Y_k) \otimes H^1(Y_1) \longrightarrow H^{k+1}(Z_{k+1}, \mathbf{Q}).$$

This map embeds the subspace $\left(H^k(Y_k) \otimes H^1(Y_1)\right)^\beta$ into $H^{k+1}(Z_{k+1}, \mathbf{Q})$. The composition of $\rho_{[\Gamma]}$ with $\mu \times 1$ then induces the desired map. In other words, the cycle U is the image of Γ by Shioda's correspondence. □

5.5. Kuga-Satake varieties. For a polarized Hodge structure V of weight 2 with $\dim V^{2,0} = 1$ Kuga and Satake [KS] defined an abelian variety $J_{KS}(V)$ with the property that

$$V \hookrightarrow H^1(J_{KS}(V), \mathbf{Q})^{\otimes 2}.$$

Voisin [V2] observed that if V is of CM-type for an imaginary quadratic field K, then $J_{KS}(V)$ has abelian subvarieties of dimension 1 and n, where $\dim_{\mathbf{Q}} V = 2n$. More precisely (cf. [vG]), there is an inclusion of Hodge structures:

$$K_{-1/2} \oplus V_{1/2} \hookrightarrow H^1(J_{KS}(V), \mathbf{Q}).$$

For such a Hodge structure V the inclusion of V into $H^1(J_{KS}(V), \mathbf{Q})^{\otimes 2}$ is now simply a consequence of Proposition 1.6 which gives an inclusion:

$$V \hookrightarrow V_{1/2} \otimes K_{-1/2} \qquad (\subset H^1(J_{KS}(V), \mathbf{Q}) \otimes H^1(J_{KS}(V), \mathbf{Q})).$$

In case $V \subset H^2(Y, \mathbf{Q})$ for a surface Y, as in 5.1 the Hodge conjecture predicts that there is a cycle, a ‘strict’ Kuga-Satake correspondence, on the product of Y and $J_{KS}(V)^2$ which induces

$$\begin{array}{ccc} V & \hookrightarrow & V_{1/2} \otimes K_{-1/2} \\ \downarrow & & \downarrow \\ H^2(Y, \mathbf{Q}) & \longrightarrow & H^1(J_{KS}(V), \mathbf{Q}) \otimes H^1(J_{KS}(V), \mathbf{Q}). \end{array}$$

5.6. Kuga-Satake correspondences. To consider Kuga-Satake varieties we determine the cases in which $V(q)$ has weight two and $\dim V(q)^{2,0} = 1$. This is the case if and only if $k - 2q = 2$ and $h_0^{k-q,q}(Y_k)(1) = 1$, cf. Lemma 2.7. By Lemma 2.4, the dimension of $H_0^{k-q,q}(Y_k)(1)$ is 1 only in two cases:

$$d(q+1) - k - 2 = (d-2)(k+1) \qquad \text{or} \qquad d(q+1) - k - 2 = 0.$$

Using $k - 2q = 2$, the first case gives $d \leq 1$ and hence is not possible. The second case gives $d = 2 + \frac{4}{k}$. Since $k \geq 2$, we only have two possibilities:

$$k = 2,\ d = 4 \qquad \text{or} \qquad k = 4,\ d = 3.$$

The case $k = 2$ was done in [vG] (in a different way), so we will consider only the case of the cubic fourfold here. Note however that the argument below can easily be adapted to the case $k = 2$.

Since $k = 4$, we have $q = 1$. The intermediate jacobian $J(Z_5)$ of the cubic 5-fold Z_5 is an abelian variety. Any abelian subvariety $J(V(1)_{1/2})$ of $J_{KS}(V(1))$ with $H^1(J(V(1)_{1/2}), \mathbf{Q}) \cong V(1)_{1/2}$ is isogeneous to the abelian subvariety of $J(Z_5)$ defined by $V_{1/2}(-1) \subset H^5(Z_5, \mathbf{Q})$. The correspondence U in Theorem 5.6 is thus a Kuga-Satake correspondence in the sense that it induces the inclusion (of Hodge structures of weight 6):

$$V(-1) \hookrightarrow V_{1/2}(-1) \otimes K_{-1/2} \hookrightarrow H^5(Z_5, \mathbf{Q}) \otimes H^1(Y_1, \mathbf{Q}).$$

To obtain a 'strict' Kuga-Satake correspondence (of weight two Hodge structures) we first need to find a surface S with an inclusion of weight two Hodge structures $V(1) \subset H^2(S, \mathbf{Q})$ and a correspondence U_0 on $S \times Y_4$ such that $\rho_{[U_0]}$ induces an isomorphism on the Hodge substructure V:

$$\begin{array}{cccc} \rho_{[U_0]} : & H^2(S, \mathbf{Q})(-1) & \longrightarrow & H^4(Y_4, \mathbf{Q}) \\ & \uparrow & & \uparrow \\ & V & \xrightarrow{\cong} & V. \end{array}$$

By [I], we can take S to be the surface parametrizing the lines incident to a fixed general line in Y_4 and for U_0 we can take the universal line over S.

We also need a correspondence $\mathcal{F}$, a family of surfaces over a curve B,

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{m_2} & Z_5 \\ m_1 \downarrow & & \\ B & & \end{array}$$

such that the image of the associated Abel-Jacobi map

$$(m_2)_* m_1^* : H^1(B, \mathbf{Q})(-2) \longrightarrow H^5(Z_5, \mathbf{Q})$$

contains the Hodge substructure $V_{1/2}(-1)$ of $H^5(Z_5, \mathbf{Q})$. Then the weight one Hodge structure $V(1)_{1/2}$ is a Hodge substructure of $H^1(B, \mathbf{Q})$. The correspondence $\mathcal{F}$ also defines the map which induces

$$\begin{array}{cccc} \rho_{[\mathcal{F}]} : & H^5(Z_5, \mathbf{Q}) & \longrightarrow & H^1(B, \mathbf{Q})(-2) \\ & \uparrow & & \uparrow \\ & V(-1)_{1/2} & \xrightarrow{\cong} & V(1)_{1/2}(-2). \end{array}$$

Collino [C] proved that if $\mathcal{F}$ is the family of planes in a *general* cubic fivefold, then the associated Abel-Jacobi map is an isomorphism. Our cubic fivefolds Z_5 are, however, very special.

The composition of the correspondences U_0, U and $\mathcal{F} \times \Delta$, where Δ is the diagonal on $Y_1 \times Y_1$ (so $\mathcal{F} \times \Delta$ is a cycle on the product of $Z_5 \times Y_1$ and $B \times Y_1$) then gives a strict Kuga-Satake correspondence:

$$\begin{array}{ccc} H^2(S, \mathbf{Q}) & \longrightarrow & H^1(J_{KS}(V(1)), \mathbf{Q}) \otimes H^1(J_{KS}(V(1)), \mathbf{Q}) \\ \uparrow & & \uparrow \\ V(1) & \hookrightarrow & V(1)_{1/2} \otimes K_{-1/2}. \end{array}$$

Note however that we do not know a curve B and a correspondence $\mathcal{F}$ with the desired properties.

5.7. Voisin's example. The cohomology of a cubic fourfold with equation $x_5^3 + x_4^3 = G(x_0, \ldots, x_3)$ was considered by Voisin [V2]. She shows that a (strict) Kuga-Satake correspondence exists, taking S to be a K3 surface associated to a plane in Y_4 and using the variety of lines on the cubic threefold Y_3 defined by $x_4^3 = G(x_0, \ldots, x_3)$ to obtain $\mathcal{F}$.

5.8. Moduli. In Sect. 4.7 we related the half twist to Kondo's proof of the birational isomorphism between the moduli space of plane quartics and a ball quotient which parametrizes weight one Hodge structures with an action of the field $\mathbf{Q}(\sqrt{-1})$. Using the folowing lemma, we obtain a similar result for the moduli space of cubic threefolds.

5.9. Lemma. *The eigenvalues of $x \in K$ on the tangent space to $J(V_{1/2})$ at the origin are x with multiplicity 1 and $\bar{x}$ with multiplicity 10.*

Proof. By definition, we have $V_{1/2}^{1,0}(2) = H_0^{3,1}(Y_4, \mathbf{C})_+ \oplus H_0^{2,2}(Y_4, \mathbf{C})_-$ on which $x \in K$ acts via diagonal matrices $diag(\sigma(x), \bar{\sigma}(x))$. Since $V_{1/2}^{1,0}(2)$ is the dual of the tangent space to the origin of $J(V_{1/2})$, the lemma follows if we choose the right embedding $K \subset \mathbf{C}$. $\square$

5.10. Ball quotients. The moduli space of Hodge structures of weight one and of type (1,10) for the field K is the 10-ball. Also cubic threefolds depend on 10 moduli. By Lemma 4.3 above, a general cubic threefold is sent to a general abelian variety in this 10-dimensional family which is simple, and, in particular, it is not (directly) related to the 5-dimensional intermediate jacobian of the cubic 3-fold.

In this context one needs to work with $\mathbf{Z}$-Hodge structures. One can take $H^1(J(V_{1/2}), \mathbf{Z}) = H_0^4(Y_4, \mathbf{Z})$, and the polarization E on $H^1(J(V_{1/2}), \mathbf{Z})$ is determined by the polarization Q on $H_0^4(Y_4, \mathbf{Z})$, which was determined by Hassett [H] following work of Beauville and Donagi [BD], and the action of the order three automorphism of Y_4 on the polarized Hodge structure $H_0^4(Y_4, \mathbf{Z})$. Unfortunately, this action (i.e. its conjugacy class in $O(Q)(\mathbf{Z})$) is not known.

Independently, Allcock, Carlson and Toledo have extended their results (private communication): they can show that the moduli spaces of cubic threefolds and that of all Del-Pezzo surfaces are ball quotients. For cubic threefolds, they show that the moduli space of the Hodge structures $H^4(Y_4)$ is an arithmetic quotient of a 10-ball (if, instead, one uses half-twists as above, this is immediate since $V_{1/2}$ has weight 1) and (using Voisin's Torelli theorem [V1]) that the period map $X_3 \mapsto H^4(Y_4)$ is injective. The case of Del-Pezzo surfaces of degree 2 is in the paper [CT] by Carlson and Toledo and is equivalent to Kondo's construction. The aim of Allcock, Carlson and Toledo is to use the ball quotients to determine the fundamental groups of the corresponding moduli spaces and their monodromy representations.

6. Surfaces

6.1. We consider surfaces $Y_2 \subset \mathbf{P}^3$ of degree d. The Hodge substructure $V \subset H^2(Y_2, \mathbf{Q})$ has a half twist if and only if $3 < d < 8$ (see Corollary 2.9). We discussed the case $d = 4$ in 4.7. Here we briefly discuss the cases $4 < d < 8$.

6.2. Quintic surfaces. Since d is prime we have $V = H_0^k(Y_k, \mathbf{Q})$. Theorem 2.6 implies that V has a half twist if and only if $k = 5q+2$ or $k = 5q+3$. In the last case $h_0^{k-q,q} = 1$, but if $k = 5q+2$ then:

$$\begin{aligned} h_0^{k-q,q}(Y_k) &= \dim H^0\left(\mathcal{O}_{\mathbb{P}^{k+1}}(5(q+1) - (5q+2) - 2)\right) / J_{5(q+1)-(5q+2)-2} \\ &= h^0\left(\mathcal{O}_{\mathbb{P}^{k+1}}(1)\right) = k+2 \end{aligned}$$

and the eigenspaces have dimensions (cf. Lemma 2.4):

$$h_0^{k-q,q}(1) = h^0(\mathcal{O}_{\mathbb{P}^k}(1)) = k+1, \qquad h_0^{k-q,q}(2) = h^0(\mathcal{O}_{\mathbb{P}^k}(0)) = 1.$$

For the quintic surface Y_2, we have $V_{1/2}(-1) \subset H^3(Z_3, \mathbf{Q})$, and since $H^{3,0}(Z_3) \neq 0$ we cannot expect to see $V_{1/2}$ easily. However, the following geometric construction seems to induce the inclusion $V = H_0^2(Y_2, \mathbf{Q}) \subset V_{1/2} \otimes H^1(A_K)$.

Consider a general line l in $\mathbf{P}^2$. Its inverse image under the cover $Y_2 \to \mathbf{P}^2$ is isomorphic to a plane curve $C = C_l$ defined by an equation of type $y^5 = f_5(x)$ from some quintic polynomial $f_5 \in \mathbf{C}[x]$. The moduli space of such curves is 2-dimensional. We can decompose the cohomology of C with respect to the automorphism of order 5 (cf. 2.4) and obtain:

$$h^{1,0}(C)(1) = 3, \quad h^{1,0}(C)(2) = 2, \quad h^{1,0}(C)(3) = 1, \quad h^{1,0}(C)(4) = 0.$$

The moduli space of abelian 6-folds with this type of automorphism, let's call it $\mathcal{M}$, also has dimension 2 (since the space of invariants for the automorphism in $S^2H^{1,0}(C)$ is also 2-dimensional, see also [dJN] for this example). In $\mathcal{M}$ we have curves parametrizing abelian varieties isogeneous to products $A_2 \times A_4$ where A_2 is the jacobian of $C_2 :\ y^2 = x^5 - 1$ (and $h^{1,0}(C_2)(1) = h^{1,0}(C_2)(2) = 1$, the other $h^{1,0}(i)$'s being zero) and A_4 is an abelian 4-fold with automorphism of order 5 with eigenvalues ζ, ζ, ζ^2, ζ^3 on $H^{1,0}(A_4)$. In particular, we can take $A_K = A_2$.

We have a rational map

$$(\mathbf{P}^2)^* \longrightarrow \mathcal{M}, \qquad l \longmapsto Jac(C_l).$$

The inverse image Z of a curve in $\mathcal{M}$ as above carries a family of curves $\mathcal{C}$ which maps onto Y_2:

$$\begin{array}{ccc} \mathcal{C} \longrightarrow Y_2 & \qquad & \mathcal{C}_l = C_l \hookrightarrow Y_2 \\ \downarrow & & \downarrow \\ Z & & \{l\} \end{array}$$

The Jacobians of the C_l's have a constant factor A_2 and one expects that the pull-back of V lies in $H^1(A_2, \mathbf{Q}) \otimes H^1(Z, \mathbf{Q})$ and that $V_{1/2} \subset H^1(Z, \mathbf{Q})$.

6.3. Sextic surfaces. Here the action of the automorphism α_2 splits the 105-dimensional primitive cohomology in 3 subspaces:

$$H_0^2(Y_2, \mathbf{Q}) = V_6 \oplus V_3 \oplus V_2, \qquad V := V_6,$$

where the eigenvalues of α_k^* on the V_i are primitive i-th roots of unity and, by definition, $V = V_6$. From Proposition 2.6 we see that V has a half twist. From the formulae from Sect. 2 it follows that

$$\dim V = 42, \quad \dim V^{2,0} = h^{2,0}(Y_k)(1) = 6,$$

$$\dim V^{1,1} = h^{1,1}(Y_k)(1) + h^{1,1}(Y_k)(5) = 15 + 15 = 30.$$

Moreover, in this case V_2, which has CM by the field of cube roots of unity, also has a half twist:

$$\dim V_2 = 42, \quad \dim V_2^{2,0} = h^{2,0}(Y_k)(2) = 3,$$

$$\dim V_2^{1,1} = h^{1,1}(Y_k)(2) + h^{1,1}(Y_k)(4) = 18 + 18 = 36.$$

We do not know a geometric interpretation for the half twists of V_2 and V_3.

6.4. Septic surfaces. Since 7 is a prime number, $V = H_0^2(Y_2, \mathbf{Q})$. The main problem here is to find a correspondence which induces the inclusion $V \hookrightarrow V_{1/2} \otimes K_{-1/2}$. In this case we can take $K_{-1/2} = H^1(C, \mathbf{Q})$ where $C : \ y^2 = x^7 - 1$ ($\zeta \in K$ acts on the curve via $(x, y) \mapsto (x, \zeta y)$). Since $V_{1/2}$ has weight one it may be identified with a Hodge substructure of the H^1 of some curve C'. Of course, we also have $V_{1/2}(-1) \subset H^3(Z_3, \mathbf{Q})$ but it seems difficult to describe this Hodge substructure geometrically (via an Abel-Jacobi map).

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