TWISTOR LINES IN THE PERIOD DOMAIN OF COMPLEX TORI

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Abstract. As in the case of irreducible holomorphic symplectic manifolds, the period domain *Compl* of compact complex tori of even dimension $2n$ contains twistor lines. These are special 2-spheres parametrizing complex tori whose complex structures arise from a given quaternionic structure. In analogy with the case of irreducible holomorphic symplectic manifolds, we show that the periods of any two complex tori can be joined by a *generic* chain of twistor lines. Furthermore, we show that twistor lines are holomorphic submanifolds of *Compl*, of degree $2n$ in the Plücker embedding of *Compl*.

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Introduction

Let M be a Riemannian manifold of real dimension $4m$ with metric g. Then M is called *hyperkähler* with respect to g (see [8, p. 548]) if there exist complex structures I, J and K on M, such that I, J, K are covariantly constant and are isometries of the tangent bundle TM with respect to g, satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K.$$

We call the ordered triple (I, J, K) *a hyperkähler structure on M compatible with g*.

A hyperkähler structure (I, J, K) gives rise to a sphere S^2 of complex structures on M:

$$S^2 = \{aI + bJ + cK | a^2 + b^2 + c^2 = 1\}.$$

We call the family $\mathcal{M} = \{(M, \lambda) | \lambda \in S^2\} \to S^2$ a *twistor family over the twistor sphere* S^2. The family $\mathcal{M}$ can be endowed with a complex structure, so that it becomes a complex manifold and the fiber $\mathcal{M}_\lambda$ is biholomorphic to the complex manifold (M, λ), see [8, p. 554]. For every $\lambda = aI + bJ + cK \in S^2$, the closed alternating form $g(\lambda \cdot, \cdot)$ determines a Kähler class in $H^{1,1}((M, \lambda), \mathbb{R})$.

The known examples of compact hyperkähler manifolds are even-dimensional complex tori and irreducible holomorphic symplectic manifolds (*IHS manifolds*).

For these examples there exist well-defined period domains, carrying the structure of a complex manifold, and every twistor family $\mathcal{M}$ determines an embedding of the base S^2 into the corresponding period domain as a 1-dimensional complex submanifold. The image of such an embedding is called a *twistor line*.

The period of a hyperkähler manifold is called *generic*, if the corresponding manifold has trivial Néron-Severi group. A path of twistor lines is an ordered sequence $S_1, \ldots, S_m$ of twistor spheres such that $S_i \cap S_{i+1}$ is non-empty if $1 \leq i \leq m-1$. We call such a path generic, if the periods at intersections of successive lines in the path are generic.

Let A be a complex torus of dimension $2n$. The period domain *Compl* is an open subset of the Grassmanian $G(2n, 4n)$, whose points are $2n$-dimensional complex planes, realizing the real weight 1 Hodge structures on the complex $4n$-dimensional vector space $T_{0,\mathbb{R}}A \otimes \mathbb{C}$. The open subset consists of those $2n$-planes in $T_{0,\mathbb{R}}A \otimes \mathbb{C}$ that do not intersect the real subspace $V_{\mathbb{R}} := T_{0,\mathbb{R}}A \subset V_{\mathbb{C}} := T_{0,\mathbb{R}}A \otimes \mathbb{C}$. Explicitly, a complex structure $I \colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ corresponds to the $2n$-plane $(\mathbb{1} - iI)V_{\mathbb{R}} \in Gr(2n, 4n) = Gr(2n, V_{\mathbb{C}})$ where $\mathbb{1}$ denotes the identity map. As a homogeneous space, *Compl* is the orbit of I under the conjugation action of $GL(V_{\mathbb{R}})$: $Compl \cong GL(V_{\mathbb{R}})/G_I$, where $G_I \cong GL_{2n}(\mathbb{C})$ is the stabilizer of I. This orbit is endowed with a complex manifold structure such that the above embedding $I \mapsto (\mathbb{1} - iI)V_{\mathbb{R}} \in Gr(2n, 4n)$ is biholomorphic. The period domain *Compl* consists of two connected components, corresponding to the components $GL^+(V_{\mathbb{R}})$ and $GL^-(V_{\mathbb{R}})$ of $GL(V_{\mathbb{R}})$.

Our main result is

Theorem 1. (1) *Any twistor sphere on a complex torus embeds into Compl as a complex 1-dimensional submanifold. The degree of twistor lines in $G(2n, 4n)$ with respect to the Plücker embedding is $2n$.*

(2) *In each of the two connected components of Compl any two periods can be connected by a generic path of twistor lines.*

We recall that an IHS manifold is a simply connected compact Kähler manifold M with $H^0(M, \Omega^2_M)$ generated by an everywhere non-degenerate holomorphic 2-form σ.

Examples of IHS manifolds include $K3$ surfaces and, more generally, Hilbert schemes of points on $K3$ surfaces.

In the case of IHS manifolds it is known that any two periods can be connected by a path of twistor lines (see [5], which is an exposition of Verbitsky [12]). Moreover, such a path can be chosen generic (in [5] and [12] the genericity of a path merely means that the twistor lines in the path contain *some* generic periods, but, in fact, the proof of Proposition 3.7 in [5] establishes the seemingly stronger kind of genericity that we have formulated). In [12] generic twistor path connectivity was used to prove the surjectivity of the corresponding period mapping, which was a part of the Torelli theorem for IHS manifolds proved there.

Although the surjectivity of the period mapping in the case of complex tori is clear, twistor path connectivity is of interest for deforming sheaves. Assume M is a hyperkähler manifold (not necessarily simply connected) with Riemannian metric $g(\cdot, \cdot)$ and a fixed complex structure I, let us denote by h the Kähler class on M represented by the form $h(\cdot, \cdot) = g(I\cdot, \cdot)$. Then, by definition, we have a sphere of complex structures on M, the corresponding twistor family $\mathcal{M} \to S^2$, and a Kähler class represented by the form $\omega_\lambda(\cdot, \cdot) = g(\lambda\cdot, \cdot)$ on the fiber $M_\lambda = (M, \lambda)$ for each

$\lambda \in S^2$, such that $M_I = M$ and $\omega_I = h$. By definition, the class ω_λ (considered up to multiplication by a positive scalar) is *the Kähler class on* M_λ.

Recall that a vector bundle on M is called h-slope-polystable if it is isomorphic to a direct sum of h-slope-stable bundles with equal slopes. The following theorem was proved in [9, Thm. 3.17, Thm. 3.19].

Theorem. *Let F be an h-slope-polystable vector bundle over a hyperkähler manifold M. If the Chern classes $c_1(F)$ and $c_2(F)$ remain of Hodge type for all the complex structures λ on M belonging to the sphere S^2, then the bundle F extends to a vector bundle $\mathcal{F}$ over $\mathcal{M}$. Furthermore, for all $\lambda \in S^2$, the restriction $\mathcal{F}|_{M_\lambda}$ is an ω_λ-slope-polystable bundle.*

This theorem was used in [6] to prove that every rational Hodge isometry between two $K3$-surfaces is algebraic.

A complex torus M of even dimension $2n$ has many different hyperkähler structures. Fix an action of the algebra of quaternions $\mathbb{H}$ (generated by three complex structures I, J, K on M) on the real vector space $V_{\mathbb{R}} = T_{0,\mathbb{R}}M$, which determines a sphere $S = S^2$ of complex structures on M. There exists a (non-unique) metric $g(\cdot,\cdot)$ on $V_{\mathbb{R}}$ such that I, J, K are orthogonal operators with respect to g (see Proposition 1.2 for details). The form $h(\cdot,\cdot) = g(I\cdot,\cdot)$, associated to the metric $g(\cdot,\cdot)$ determines a Kähler class on M. Verbitsky's theorem can be applied to such data to deform h-slope-stable bundles over twistor paths in the moduli space of complex tori.

A purely geometric motivation behind the study of the connectivity, besides the application to deforming sheaves, is discussed in Remark 2 below.

For the case of IHS manifolds, the proof of the connectivity relies on the realization of the period domain as the Grassmanian of oriented positive real 2-planes in the second cohomology, where positivity is with respect to the Beauville-Bogomolov bilinear form, again see [5]. This bilinear form provides a very convenient tool for investigating the local topology of this period domain.

For complex tori, however, we do not have such a realization of their period domain and cannot use a similar argument. Here, instead, we need to use the (less refined) structure of the period domain of complex tori as a homogeneous space (which, certainly, the period domain of an IHS manifold is, as well). This homogeneous structure allows us to proceed with proving the twistor path connectivity in steps that are, in their broad strokes, parallel to the steps of the proof of the twistor path connectivity for the period domains of IHS manifolds.

Remark 1. As shown by Beauville [2] (using results of Cheeger and Gromoll), a general compact hyperkähler manifold M has a finite étale cover $\widetilde{M}$ isomorphic to the product of a complex torus and a finite number of irreducible hyperkähler manifolds. Since the irreducible hyperkähler manifolds are simply connected, one can easily see that the Néron-Severi group of $\widetilde{M}$ is isomorphic to the direct sum of the Néron-Severi groups of its factors. Twistor families for M give rise to twistor families for $\widetilde{M}$ and its factors. Vice-versa, twistor families for (any of) the factors of $\widetilde{M}$ give rise, in a generally non-unique way, to twistor families for $\widetilde{M}$ and M. One can then deduce the generic twistor path connectivity of the moduli space of M from the generic twistor path connectivity of the moduli spaces of complex tori and those of irreducible hyperkähler manifolds.

Remark 2. There is a relation between twistor path connectivity and rational connectedness, that is, the connectedness of points of a complex manifold by chains of rational curves (for the latter see, for example, [10]). The Grassmanian $G(2n, 4n)$ being a rational variety ($G(2n, 4n) \overset{\sim}{\dashrightarrow} \mathbb{P}^{4n^2}$), is certainly rationally connected. However, rational connectedness is a weaker property than twistor path connectivity. Indeed, the variety of lines in $\mathbb{P}^{4n^2}$, passing through a fixed point, has complex dimension $4n^2 - 1$ (and the dimension of the variety of rational curves of degree $d > 1$ in $\mathbb{P}^{4n^2}$, passing through a fixed point, is even larger), thus its real dimension is $8n^2 - 2$. On the other hand, our dimension count, in Corollary 1.5, of the space of all twistor lines, passing through a fixed point in the period domain, is $4n^2 - 1$. Thus, through a given point, there are "half as many" twistor lines as general rational curves, and the problem of twistor path connectivity may be roughly considered as a "sub-Riemannian" version of rational connectedness.

The plan of the paper is as follows.

In Section 1 we describe our basic set-up, the complex-analytic structure of $Compl$ considered as a submanifold in $End(V_{\mathbb{R}})$ and show that the twistor spheres $S^2 \subset Compl$ are complex submanifolds (Corollary 1.8). We define the union C_I of all twistor spheres passing through a given period I and show that G_I acts transitively on the set of twistor spheres containing I. The sets C_I will serve as the main tool in the proof of twistor path connectivity.

In Section 2 we provide an argument, illustrated by a picture, that the set of periods reachable from a given one I by means of all possible triples of consecutive twistor spheres contains an open neighborhood of the initial period. Then, the connectedness of the period domain allows us to conclude that any two periods can be connected by a path of twistor lines. The three spheres argument is essentially due to the transversality formulated in its most general form in Proposition 2.5, and it is somewhat analogous to the "three lines argument" in [5, Prop. 3.7]. We also show that C_I is a real analytic subset of $Compl$.

In Section 3 we prove the generic connectivity part of Theorem 1. The idea of the proof is to show that the space of triples of consecutive twistor spheres connecting a fixed pair of periods is not the union of its two subspaces for which the first or, respectively, the second, of the two joint points belongs to the locus of tori with nontrivial Néron-Severi group in the period domain. Again, the transversality, stated in Proposition 2.5, constitutes the main tool for proving generic connectivity.

In Section 4 we prove that the degree of twistor lines in $G(2n, 4n)$ with respect to the Plücker embedding is $2n$. Here we use the fact that the group $GL(V_{\mathbb{R}})$ acts transitively on the set of all twistor lines together with an explicit computation on an explicit example.

In Section 5 we gather the calculations needed in the example of Section 4.

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1. The space of twistor spheres

1.1. Let A be a complex torus of complex dimension $2n$. Denote by $V_{\mathbb{R}}$ the real tangent space $T_{\mathbb{R},0}A$ and by V the complex tangent space $T_{\mathbb{C},0}A \subset T_{\mathbb{R},0}A \otimes \mathbb{C}$, so that $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2\dim_{\mathbb{C}} V = 4n$. As we said above, the group $G := GL(V_{\mathbb{R}})$ acts transitively via conjugation on the period domain, that is, the set of complex structures on $V_{\mathbb{R}}$: $g \cdot I = {}^{g}I := gIg^{-1}$ for $g \in GL(V_{\mathbb{R}})$.

Let $I\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ be the complex structure operator on $V_{\mathbb{R}}$ induced by scalar multiplication by i on V. Then we can write $Compl = G \cdot I$, $Compl$ is diffeomorphic to the homogeneous space $G/G_I \cong G/GL(V) \cong GL_{4n}(\mathbb{R})/GL_{2n}(\mathbb{C})$. It carries the structure of a complex manifold, see [4, p. 31] and Proposition 1.7.

Assume that $J\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ is a complex structure anticommuting with I. Then I and J determine a twistor sphere

$$S(I,J) := \{aI + bJ + cK | a^2 + b^2 + c^2 = 1\},$$

where $K = IJ$. In general, for two complex structures I_1, I_2, not necessarily anticommuting, such that $I_1 \neq \pm I_2$, and such that they are contained in the same twistor sphere S, we will also denote this sphere by $S(I_1, I_2)$. Our notation is justifed by the following lemma, whose proof is an exercise that we leave to the reader.

Lemma 1.1. *Every twistor sphere S is uniquely determined by any pair of non-proportional complex structures $I_1, I_2 \in S$.*

1.2. Let J be a complex structure that anti-commutes with I. Then $V_{\mathbb{R}}$ splits, in a non-unique way, as a direct sum of 4-dimensional subspaces of the form $\langle v, Iv, Jv, IJv\rangle$ for nonzero vectors $v \in V_{\mathbb{R}}$, and the union of the specified bases of the 4-subspaces forms a basis of $V_{\mathbb{R}}$. In this basis the matrix of J has a block-diagonal form with the following $4{\times}4$ blocks on the diagonal

$$\left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right).$$

Proposition 1.2. *Given a triple of complex structures (I, J, K) on A satisfying the quaternionic identities, there exist a (non-unique) metric g on A such that (I, J, K) is a hyperkähler structure with respect to g.*

Proof. Choose a basis of $V_{\mathbb{R}}$ as in Paragraph 1.2 and define a metric $g(\cdot,\cdot)$ on $V_{\mathbb{R}}$ by declaring this basis to be orthonormal. Then I, J and K are isometries with respect to $g(\cdot,\cdot)$ and g is Kähler with respect to all three complex structures. □

1.3. By the definition of $Compl$, the group G acts transitively on it:

$$g \in G\colon J \mapsto {}^{g}J = gJg^{-1}.$$

In particular, G acts on the set of all twistor spheres $S(I,J)$ in $Compl$:

$$g \cdot S(I,J) = S({}^{g}I, {}^{g}J).$$

For $g \in G_I$ we have $g \cdot S(I,J) = S(I, {}^{g}J)$. We have

Proposition 1.3. *The group G_I acts transitively on the set N_I of complex structures anticommuting with I.*

Proof. Let J be a complex structure that anti-commutes with I. The group $G_I \cong GL(V) < GL^+(V_{\mathbb{R}}) = GL^+_{4n}(\mathbb{R})$ acts transitively on the set of bases as in Paragraph 1.2, hence also on the set of J anti-commuting with I. □

1.4. Therefore, given a complex structure $J \in N_I$, $N_I = G_I \cdot J \cong G_I/G_{I,J}$ is the orbit of J under G_I, where $G_{I,J}$ is the stabilizer group of J in G_I. Since $G_{I,J}$ is the subgroup of elements of $G_I = GL(V)$ commuting with J, that is, preserving the quaternionic structure on $V_{\mathbb{R}}$ determined by I and J, we have $G_{I,J} \cong GL(V, \mathbb{H})$ which we will also denote by $G_{\mathbb{H}}$. So $N_I \cong GL(V)/GL(V, \mathbb{H})$ and we deduce

Corollary 1.4. *The set N_I is a real submanifold of Compl of dimension $4n^2$.*

Proof. The dimension of the orbit as a complex manifold is $\dim_{\mathbb{C}} GL(V) - \dim_{\mathbb{C}} GL(V, \mathbb{H}) = (2n)^2 - 2n^2 = 2n^2$. The real dimension is thus equal to $4n^2$. □

1.5. Let $S = S(I, J)$ for $J \in N_I$ be a twistor sphere. Define $G_{I,S} \subset G_I$ to be the stabilizer of S as a set, i.e., the set of elements g of G_I such that $g \cdot S \subset S$. For any $g \in G_{I,S}$, the complex structure ${}^gJ \in S$ also anticommutes with I, so ${}^gJ$ is of the form $aJ + bK, a^2 + b^2 = 1$. Setting $a = \cos t, b = \sin t$ we have $aJ + bK = e^{\frac{tI}{2}} J e^{-\frac{tI}{2}}$, where $e^{sI} = \cos s\, \mathbb{1} + \sin s\, I \in G_I$ realizes, via the conjugation action, the rotations of S around $\{\pm I\}$. Conversely, if $g \in G_I$ and ${}^gJ \in S$, then $g \in G_{I,S}$. The set of $g \in G_{I,S}$ such that ${}^gJ = J$ is the quaternionic subgroup $G_{I,J} = G_{\mathbb{H}} \subset G_{I,S}$. Explicitly, we have $G_{I,S} = \langle e^{tI}, t \in \mathbb{R}\rangle \times G_{\mathbb{H}}$, where $\langle e^{tI}, t \in \mathbb{R}\rangle \cong S^1$ (which is a subgroup of the center of G_I). This tells us, in particular, that $\dim_{\mathbb{R}} G_{I,S} = \dim_{\mathbb{R}} G_{\mathbb{H}} + 1 = 4n^2 + 1$.

Let M_I be the set of all twistor spheres in *Compl* containing I. The natural map $N_I \to M_I$ identifies two complex structures J_1 and J_2 whenever they belong to the same twistor sphere through I, i.e., $S(I, J_1) = S(I, J_2)$. More precisely, they belong to the great circle in $S := S(I, J_1)$ consisting of elements anticommuting with I. Hence, for the S^1-action $J \in N_I \mapsto e^{tI}J = e^{\frac{tI}{2}} J e^{-\frac{tI}{2}}$ on N_I defined above, we have $N_I/S^1 = M_I$. Therefore Corollary 1.4 immediately implies

Corollary 1.5. *The set M_I is a real manifold of dimension $4n^2 - 1$.*

1.6. **The twistor cone of I.** Define the set $C_I := \bigcup_{S \in M_I} S \subset Compl$ as the union of all twistor spheres containing I. All spheres in this union contain the complex structures I and $-I$. We will sometimes refer to the set C_I as a (twistor) cone. Proposition 1.3 immediately implies

Corollary 1.6. *The group G_I acts transitively on $M_I \cong G_I/G_{I,S}$ so that $C_I = \bigcup_{g \in G_I} g \cdot S(I, J)$.*

We will give an explicit local parametrization of C_I in the next section and prove that the cone C_I is a real-analytic subset of *Compl* of dimension $4n^2 + 1$ (Proposition 2.6).

1.7. We now describe the complex structure on the tangent bundle of the orbit $Compl = G \cdot I$. Then we will see that the tangent bundle TS^2 of an arbitrary twistor sphere $S^2 \subset Compl$ is a subbundle of the restricted tangent bundle $TCompl|_{S^2}$, invariant under the complex structure of $TCompl|_{S^2}$. This will imply the well-known fact that the twistor sphere S^2 is a complex submanifold in *Compl*.

Proposition 1.7. *The submanifold $Compl \subset End(V_{\mathbb{R}})$ is a complex manifold. Its complex structure l_I is given by left multiplication by I on $T_I Compl \subset End(V_{\mathbb{R}})$.*

The complex structure of $Compl$ is induced by that of $Gr(2n, 4n)$ via the embedding $I \mapsto (\mathbb{1} - iI)V_{\mathbb{R}}$. The proof of Proposition 1.7 is a technical exercise that we leave to the reader.

Proof. Denoting the differential of the embedding $Compl \ni I \mapsto (\mathbb{1} - iI)V_{\mathbb{R}} \in Gr(2n, 4n)$ by φ we have the following commutative diagram

$$\begin{array}{ccc} T_I Compl \ni X & \xrightarrow{\varphi} & \frac{d}{dt}\Big|_{t=0} (\mathbb{1} - i\, e^{tX} I)V_{\mathbb{R}} \in T_{(\mathbb{1}-iI)V_{\mathbb{R}}} Gr(2n, 4n) \\ \Big\downarrow l_I & & \Big\downarrow i\times \\ T_I Compl \ni Y & \xrightarrow{\varphi} & \frac{d}{dt}\Big|_{t=0} (\mathbb{1} - i\, e^{tY} I)V_{\mathbb{R}} \in T_{(\mathbb{1}-iI)V_{\mathbb{R}}} Gr(2n, 4n) \end{array}$$

where l_I denotes the complex structure operator on $T_I Compl$ and '$i\times$' denotes the multiplication by i on

$$\mathbb{C}^{4n^2} \cong Hom((\mathbb{1} - iI)V_{\mathbb{R}}, V_{\mathbb{C}}/(\mathbb{1} - iI)V_{\mathbb{R}}) = T_{(\mathbb{1}-iI)V_{\mathbb{R}}} Gr(2n, 4n),$$

so that $\varphi \circ l_I = i\varphi$.

We note that $T_I Compl \cong T_e G / T_e G_I$ and, as $T_e G_I$ consists of all operators in $End(V_{\mathbb{R}})$ commuting with I, the tangent space $T_I Compl$ can be identified with the subspace of operators in $End(V_{\mathbb{R}})$ anticommuting with I. Indeed, every operator $X \in End(V_{\mathbb{R}})$ can be written as a sum of an operator anticommuting with I and an operator commuting with I, $X = \frac{1}{2}(X - X^I) + \frac{1}{2}(X + X^I)$, where $X^I = IXI^{-1}$. This allows us to immediately assume that X and Y in the above diagram anticommute with I.

Now we evaluate

$$\varphi(X) = \frac{d}{dt}\bigg|_{t=0} (\mathbb{1} - i\, e^{tX} I)V_{\mathbb{R}} = \{(\mathbb{1} - iI)v \mapsto -i(XI - IX)v = -2iXIv \,|\, v \in V_{\mathbb{R}}\},$$

which, after multiplying by i becomes $i\varphi(X) = \{(\mathbb{1} - iI)v \mapsto 2XIv \,|\, v \in V_{\mathbb{R}}\}$ (here we slightly abuse notation by writing, instead of the actual $\varphi(X), i\varphi(X)$, their representatives in $Hom((\mathbb{1} - iI)V_{\mathbb{R}}, V_{\mathbb{C}})$).

Now, considering $\varphi(Y) = \{(\mathbb{1} - iI)v \mapsto -2iYIv = -2YiIv \,|\, v \in V_{\mathbb{R}}\}$ as a vector in $Hom((\mathbb{1} - iI)V_{\mathbb{R}}, V_{\mathbb{C}}/(\mathbb{1} - iI)V_{\mathbb{R}})$, we may write $\varphi(Y) = \{(\mathbb{1} - iI)v \mapsto -2Yv \,|\, v \in V_{\mathbb{R}}\}$. In order to have the equality $\varphi \circ l_I = i\varphi$, setting $Y = l_I(X)$ we write

$$\varphi(Y) = i\varphi(X) = \{(\mathbb{1} - iI)v \mapsto 2XIv = -2IXv \,|\, v \in V_{\mathbb{R}}\}.$$

In order for the latter equality to be true it is necessary and sufficient that $Y = IX$, that is, the map l_I is the left multiplication by I on $T_I Compl$. □

Corollary 1.8. *The twistor spheres $S^2 \subset Compl$ are complex submanifolds.*

Proof. The proof is based on the simple observation that the tangent space of $S^2 = S(I, J)$ at the point I, for $I, J, K = IJ$ satisfying the quaternionic identities, is the 2-plane $\langle J, K\rangle_{\mathbb{R}} \subset T_I Compl$ and this plane is obviously invariant under left multiplication by I. Thus, TS^2 is a complex subbundle of $TCompl|_{S^2}$ and thus $S^2 \subset Compl$ is a complex submanifold. □

Remark 1.9. As was pointed out to us by the referee, there is an alternative proof of Corollary 1.8 that follows from considering $S \subset Compl$ as a subset in $Gr(2n, V_{\mathbb{C}})$. Namely, denoting by $\mathbb{H}$ the algebra of quaternions and fixing a representation $\mathbb{H} \to End(V_{\mathbb{R}})$ defined by $S = S(I, J)$, we obtain a structure of an $\mathbb{H}$-module on $V_{\mathbb{R}}$. This $\mathbb{H}$-module is of the form $\mathbb{H} \otimes V'$ for an n-dimensional $\mathbb{R}$-vector space V'. The eigenspace $V^{1,0} \subset V_{\mathbb{C}}$ for a complex structure induced by the action of $\mathbb{H}$ is of the form $\mathbb{H}^{1,0} \otimes V'_{\mathbb{C}}$, where $V'_{\mathbb{C}} = V' \otimes \mathbb{C}$ and $\mathbb{H}^{1,0}$ is the corresponding eigenspace in $\mathbb{H} \otimes \mathbb{C}$. Taking the tensor product with $V'_{\mathbb{C}}$ defines a complex analytic embedding $i \colon Gr(2, \mathbb{H} \otimes \mathbb{C}) \hookrightarrow Gr(2n, V_{\mathbb{C}})$. Thus, every twistor line in $Gr(2n, V_{\mathbb{C}})$ is the image of a twistor line in $Gr(2, \mathbb{H} \otimes \mathbb{C})$ under some embedding i as above. Now, the twistor lines in the quadric (under the Plücker embedding) $Gr(2, \mathbb{H} \otimes \mathbb{C})$ are known to be obtained as linear subspace sections, thus they are complex analytic submanifolds. Hence, our $S \subset Gr(2n, 4n)$ is a complex analytic submanifold.

2. Twistor path connectivity of *Compl*

The main result of this section is Theorem 2.3. Before proving it we need to introduce a certain mapping and prove an important technical result about it (Proposition 2.1).

2.1. Let I, J, K be a triple of complex structures belonging to a twistor sphere S. Consider the smooth mapping

$$\begin{array}{rccc} \Phi\colon & G_J \times G_K & \longrightarrow & Compl, \\ & (g_1, g_2) & \longmapsto & {}^{g_1 g_2}I, \end{array}$$

where, as before, the action on *Compl* is by conjugation: $g \cdot I = {}^{g}I = gIg^{-1}$. The mapping Φ clearly sends $G_{\mathbb{H}} \times G_{\mathbb{H}}$ to I, so that its differential $d_{(e,e)}\Phi$ factors through

$$\widetilde{d_{(e,e)}\Phi} : T_eG_J/T_eG_{\mathbb{H}} \oplus T_eG_K/T_eG_{\mathbb{H}} \to T_I Compl.$$

Proposition 2.1. *Suppose I, J, K is a quaternionic triple. The mapping*

$$\widetilde{d_{(e,e)}\Phi} : T_eG_J/T_eG_{\mathbb{H}} \oplus T_eG_K/T_eG_{\mathbb{H}} \to T_I Compl$$

is injective. Hence, since the two spaces have the same dimension, it is an isomorphism.

Proof. By the definition of $\widetilde{d_{(e,e)}\Phi}$, its restrictions to the above direct summands are injective. Let us show that it is injective on the direct sum. Consider $X \in T_eG_J, Y \in T_eG_K$ and the vector $\widetilde{d_{(e,e)}\Phi}(X + T_eG_{\mathbb{H}}, Y + T_eG_{\mathbb{H}})$, which is

$$d_{(e,e)}\Phi(X+Y) = \left.\frac{d}{dt}\right|_{t=0} (e^{tX}e^{tY} \cdot I) = (X+Y)I - I(X+Y) \in T_I Compl.$$

Assume that this vector is zero, that is, $X + Y$ commutes with I:

$$I(X+Y) = (X+Y)I. \tag{1}$$

Then the conjugate $(X+Y)^J = J^{-1}(X+Y)J = X^J + Y^J = X - JYJ$ must also commute with I. Using that Y commutes with K we obtain

$$X - JYJ = X - JYKI = X - JKYI = X - IYI.$$

The commutation with I is expressed now by $I(X - IYI) = (X - IYI)I$, or

$$IX + YI = XI + IY,$$

which gives

$$I(X - Y) = (X - Y)I.$$

Adding the last equality to (1) side by side gives that $XI = IX$, hence $YI = IY$, which implies $X, Y \in T_eG_{\mathbb{H}}$. This proves the required injectivity of $\widetilde{d_{(e,e)}\Phi}$. □

Corollary 2.2. *Suppose I, J, K is a quaternionic triple. The mapping Φ is a submersion at $(e, e) \in G_J \times G_K$, that is*

$$d_{(e,e)}\Phi(T_eG_J \oplus T_eG_K) = T_I Compl \cong \mathbb{R}^{8n^2}.$$

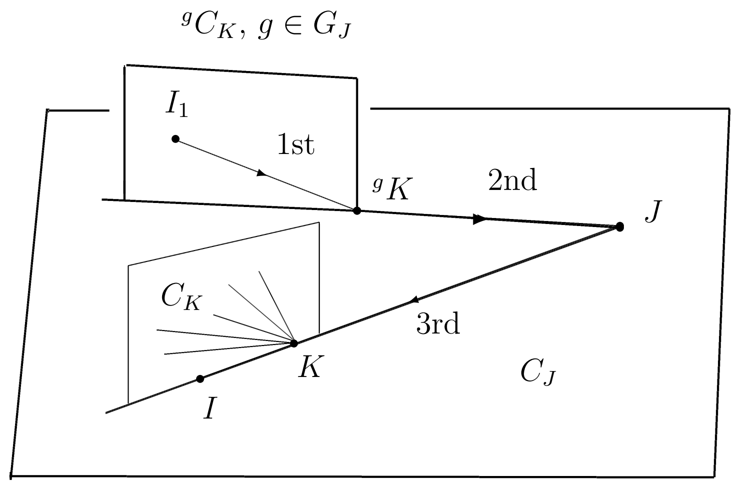
Proof. As $\dim_{\mathbb{R}} T_eG_J/T_eG_{\mathbb{H}} = \dim_{\mathbb{R}} T_eG_K/T_eG_{\mathbb{H}} = 4n^2$, the statement that Φ is a submersion follows from the fact that $d_{(e,e)}\Phi$ factors through $\widetilde{d_{(e,e)}\Phi}$ and the fact that the mapping $\widetilde{d_{(e,e)}\Phi} : T_eG_J/T_eG_{\mathbb{H}} \oplus T_eG_K/T_eG_{\mathbb{H}} \to T_I Compl$ is an isomorphism by Proposition 2.1. □

Theorem 2.3. *Given a complex structure $I \in End(V_{\mathbb{R}})$, there is a neighborhood of I in the space of complex structures on $V_{\mathbb{R}}$ such that, for any complex structure I_1 in this neighborhood, there is a twistor path consisting of three spheres joining I to I_1. Consequently, each connected component of $Compl$ is twistor path connected.*

Proof. Choose a complex structure J, anticommuting with I, and consider the sphere $S = S(J, I)$ and the cone C_J. By Lemma 1.1, the complex structures $K = IJ$ and I span the sphere $S = S(K, I) = S(J, I)$. We can then form the cone C_K whose intersection with C_J contains S. See Picture 1 below where the cones C_J and C_K are depicted by transversal planes and the sphere S lying in their intersection is depicted by a line.

We first show that the images of C_K under the action of G_J ("rotation of C_K around J") sweep out an open neighborhood of I in $Compl$. Since Φ is a submersion by Corollary 2.2, there exist neighborhoods $U_{e,J} \subset G_J$ and $U_{e,K} \subset G_K$ of e such that the set $\Phi(U_{e,J} \times U_{e,K})$ contains an open neighborhood of I. By definition, the cone C_K contains the orbit $G_K \cdot I$. Hence the union $\bigcup_{g \in G_J} {}^gC_K$ contains the image of Φ and consequently it contains an open neighborhood of I.

Now the three twistor spheres connecting I to an arbitrary point I_1 in this neighborhood are found as illustrated in the following picture.



Picture 1.

Finally we conclude that each of the two connected components of $Compl$ is twistor path connected. ☐

2.2. Another immediate consequence of the injectivity of $\widetilde{d_{(e,e)}\Phi}$ proved in Proposition 2.1 is the following

Corollary 2.4. *For a quaternionic triple I, J, K, the triple intersection of the submanifolds $G_I/G_{\mathbb{H}}, G_J/G_{\mathbb{H}}$ and $G_K/G_{\mathbb{H}}$ of the homogeneous space $G/G_{\mathbb{H}}$ at $eG_{\mathbb{H}}$ is transversal.*

The following generalization of this transversality is one of the main ingredients of the proof of connectivity by generic twistor paths in Section 3.

Proposition 2.5. *Let I_1, I_2, I_3 be complex structures belonging to the same twistor sphere S. The submanifolds $G_{I_1}/G_{\mathbb{H}}, G_{I_2}/G_{\mathbb{H}}, G_{I_3}/G_{\mathbb{H}}$ in $G/G_{\mathbb{H}}$ intersect transversally (as a triple) if and only if I_1, I_2, I_3 are linearly independent as vectors in $End(V_{\mathbb{R}})$.*

Proof. Choose anticommuting complex structures I, J in S, and set $K = IJ$. By Corollary 2.4,

$$T_eG/T_eG_{\mathbb{H}} = V_I \oplus V_J \oplus V_K, \tag{2}$$

where we set $V_I := T_eG_I/T_eG_{\mathbb{H}}, V_J := T_eG_J/T_eG_{\mathbb{H}}, V_K := T_eG_K/T_eG_{\mathbb{H}}$.

We shall prove that $T_eG/T_eG_{\mathbb{H}}$ also decomposes into the direct sum of its subspaces $V_i := T_eG_{I_i}/T_eG_{\mathbb{H}}, i = 1, 2, 3$. Put $I_i = a_iI + b_iJ + c_iK, i = 1, 2, 3$. Assume, on the contrary, that for certain vectors $X \in V_1, Y \in V_2$ and $Z \in V_3$ we have $X + Y + Z = 0$. Let $X := X_I + X_J + X_K$ be the decomposition of X into the sum of its components in the respective subspaces of (2), and do similarly for Y and Z. Then for X the commutation relation $[X, I_1] = 0$ can be written as

$$a_1[X_J + X_K, I] + b_1[X_I + X_K, J] + c_1[X_I + X_J, K] = 0.$$

Note that in the above expression, the term $[X_J, I]$, for example, anticommutes with both I, J, hence commutes with $K = IJ$, and an analogous commutation relation holds for the other terms as well. Hence we can decompose the expression on the left side of the above equality with respect to (2):

$$(b_1[X_K, J] + c_1[X_J, K]) + (a_1[X_K, I] + c_1[X_I, K]) + (a_1[X_J, I] + b_1[X_I, J]) = 0.$$

From here we conclude that $b_1[X_K, J] + c_1[X_J, K] = 0, a_1[X_K, I] + c_1[X_I, K] = 0$ and $a_1[X_J, I] + b_1[X_I, J] = 0$. Perturbing the quaternionic triple I, J, K, we may assume that all $a_i, i = 1, 2, 3$, are nonzero. Then we can use the last two equalities to express

$$[X_J, I] = -\frac{b_1}{a_1}[X_I, J], \quad [X_K, I] = -\frac{c_1}{a_1}[X_I, K]. \tag{3}$$

Note that $F_J := [\cdot, J] \colon V_I \to V_K, F_K := [\cdot, K] \colon V_I \to V_J$ and $F_I := [\cdot, I] \colon V_J \to V_K$ are isomorphisms of the respective vector spaces. Then, using (3), we can write

$$X_J = -\frac{b_1}{a_1} F_I^{-1} \circ F_J(X_I), \quad X_K = -\frac{c_1}{a_1} F_I^{-1} \circ F_K(X_I),$$

so that

$$X = X_I + \left(-\frac{b_1}{a_1} F_I^{-1} \circ F_J(X_I)\right) + \left(-\frac{c_1}{a_1} F_I^{-1} \circ F_K(X_I)\right).$$

Using $a_2, a_3 \neq 0$, we obtain similar expressions for Y and Z. Since F_I, F_J, F_K are isomorphisms, the equality $X + Y + Z = 0$ can now be written as

$$\begin{pmatrix} 1 & 1 & 1 \\ -\frac{b_1}{a_1} & -\frac{b_2}{a_2} & -\frac{b_3}{a_3} \\ -\frac{c_1}{a_1} & -\frac{c_2}{a_2} & -\frac{c_3}{a_3} \end{pmatrix} \begin{pmatrix} X_I \\ Y_I \\ Z_I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This has a nontrivial solution if and only if the columns of the matrix, i.e., I_1, I_2, I_3, are linearly dependent. ☐

2.3. We can now prove that the cone C_I has a real analytic structure. Define the incidence correspondence

$$\mathcal{S}_I := \{(S, J) \mid J \in S\} \subset M_I \times Compl.$$

Then $\mathcal{S}_I$ is an S^2-bundle over M_I and C_I is the image of $\mathcal{S}_I$ by the projection to $Compl$:

$$\begin{array}{ccccc} N_I & \hookrightarrow & \mathcal{S}_I & \xrightarrow{pr_2} & C_I \subset Compl \\ & \searrow & \downarrow{\scriptstyle pr_1} & & \\ & & M_I. & & \end{array}$$

The projection $\mathcal{S}_I \to M_I$ has two sections σ_+ and σ_-, given by $+I$ and $-I$ respectively.

Proposition 2.6. *The real-analytic map $pr_2 : \mathcal{S}_I \to C_I$ is a diffeomorphism away from the images of $\sigma_\pm$ and contracts these images to points. Therefore the cone C_I is a real-analytic subset of $Compl$ of dimension $4n^2 + 1$, smooth away from the points $\pm I$.*

Proof. First note that pr_2 clearly contracts the images of $\sigma_\pm$. Also, it is injective away from $\pm I$ by Lemma 1.1. To see that it is also an immersion away from $\pm I$, let J be in $C_I \setminus \{\pm I\}$, not necessarily anticommuting with I. Define the following mapping

$$\Phi\colon (T_eG_I/T_eG_{\mathbb{H}}) \times \mathbb{R} \to C_I,$$
$$(X,t) \mapsto e^X e^{tK} J e^{-tK} e^{-X},$$

where $K \in S(I,J) \setminus S^1$ for $S^1 = \langle I, J\rangle_{\mathbb{R}} \cap S(I,J)$. Then the restriction of Φ on a small enough neighborhood of $(0,0) \in (T_eG_I/T_eG_{\mathbb{H}}) \times \mathbb{R}$ defines a parametrization of C_I around J.

Here the subgroup $e^{tK}, t \in \mathbb{R}$, rotates the sphere $S = S(I,J)$ around the axis $\{\pm K\}$ and, together with the rotation subgroup $e^{tI} \subset G_{I,S} \subset G_I$, sweeps out in S, via the above action, a neighborhood of any point of S other than $\pm I, \pm K$. Proposition 2.5 provides that K may be chosen arbitrarily in $S \setminus S^1$, which in turn gives us that C_I is a manifold, smooth away from $\pm I$, of dimension $\dim_{\mathbb{R}}(G_I/G_{I,S}) + \dim_{\mathbb{R}} S = (4n^2-1)+2 = 4n^2+1$. The fact that the points $\pm I$ are indeed singular points of the cone C_I is easy to prove. ∎

3. Connectivity by generic twistor paths

Recall that a period in *Compl* is *generic* if the corresponding complex torus has trivial Néron-Severi group. A twistor path in *Compl* is called *generic*, if its successive twistor spheres intersect at generic periods. In this section we prove the connectivity part of Theorem 1, i.e.,

Proposition 3.1. *Any two periods in the period domain Compl can be connected by a generic twistor path.*

In this section, with the exception of Lemma 3.3 and its proof, we do not assume that the complex structures I, J, K (with or without subscripts) anticommute.

3.1. Outline of the proof.

Define $\mathcal{T}$ to be the closure, in $Compl \times Compl \times Compl$, of the set of triples (I,J,K) that are linearly independent and belong to the same twistor sphere. Denote by

$$\begin{array}{rccl} pr_1\colon & Compl \times Compl \times Compl & \longrightarrow & Compl, \\ pr_{23}\colon & Compl \times Compl \times Compl & \longrightarrow & Compl \times Compl \end{array}$$

the respective projections. For $(I_1, J_1, K_1) \in \mathcal{T}$, we defined, in Paragraph 2.1, the mapping Φ_{I_1,J_1,K_1}:

$$\begin{array}{rccl} \Phi_{I_1,J_1,K_1}\colon & G_{J_1} \times G_{K_1} & \longrightarrow & Compl, \\ & (g_1, g_2) & \longmapsto & g_1 g_2 I_1 g_2^{-1} g_1^{-1} = {}^{g_1 g_2} I_1. \end{array}$$

Proposition 2.5 tells us that, when I_1, J_1, K_1 are linearly independent, Φ_{I_1,J_1,K_1} is a submersion near $(e,e) \in G_{J_1} \times G_{K_1}$. In other words, there is a neighborhood $U_{e,G} \subset G = GL^+(V_{\mathbb{R}})$ of $e \in G$ such that the map Φ_{I_1,J_1,K_1} is submersive on $U_{e,J_1} \times U_{e,K_1}$, where $U_{e,J_1} := U_{e,G} \cap G_{J_1}$ and $U_{e,K_1} := U_{e,G} \cap G_{K_1}$ (and the image is, thus, a neighborhood of I_1 in *Compl*).

Let I_2 be an arbitrary point in the image of Φ_{I_1,J_1,K_1} and let $(g_1,g_2) \in U_{e,J_1} \times U_{e,K_1}$ be such that $I_2 = {}^{g_1 g_2} I_1$. With this notation, the three twistor spheres connecting I_1

to I_2 are: $S_1 := S(I_1, J_1, K_1)$, $S := {}^{g_1}S_1 = S({}^{g_1}I_1, {}^{g_1}J_1 = J_1, {}^{g_1}K_1)$ and $S_2 := {}^{g_1g_2}S_1 = S(I_2, {}^{g_1g_2}J_1, {}^{g_1g_2}K_1 = {}^{g_1}K_1)$, with the joint points J_1 and ${}^{g_1}K$.

We are going to show that, for a fixed I_1, there is a neighborhood $U_{I_1} \subset Compl$ of I_1 such that for any $I_2 \in U_{I_1}$, we can choose a generic $J \in C_I$, a $K \in S(I, J)$ and find $(g_1, g_2) \in \Phi^{-1}_{I,J,K}(I_1)$ as above such that ${}^{g_1g_2}K$ is also generic.

We begin by proving, in Lemma 3.4, that the set of non-generic periods in C_{I_1} is a countable union of proper analytic subsets, i.e., J can be chosen generic.

Next, for I_2 close to I_1, and with S_1, S, S_2 as above, connecting I_1 to I_2, the initial sphere S_1 together with the choice of $J, K \in S_1$, uniquely determines the final sphere S_2 together with the pair of periods ${}^{g_1g_2}J, {}^{g_1g_2}K$.

To justify this uniqueness we first need to control the fibers of the maps $\Phi_{I,J,K}$ in a neighborhood of (I_1, J_1, K_1), which we do in Lemma 3.5. This allows us to introduce, in Paragraphs 3.4 and 3.7, two maps $\Psi^{I_1\to I_2}$ and $\Psi^{I_2\to I_1}$ which, roughly speaking, switch (S_1, J, K) and $(S_2, {}^{g_1g_2}J, {}^{g_1g_2}K)$.

We then show in Lemma 3.9, after shrinking our various domains, that the composition of $\Psi^{I_1\to I_2}$ and $\Psi^{I_2\to I_1}$ is the identity. Corollary 3.10 then shows that this implies the irreducibility of the set of triples (S_1, S, S_2) joining I_1 and I_2 mentioned in the introduction, which gives that J and ${}^{g_1g_2}K$ can both be chosen generic.

Thus the chain of three twistor spheres connecting I_1 to I_2 for every I_2 in some neighborhood of I_1 can be chosen in such a way that the periods at the intersections are generic. For arbitrary I_1 and I_2, we connect them by a path in $Compl$ consisting of generic triple subchains.

3.2. Let us first show that there are generic periods $J \in C_I$. Dimension-wise this is not trivial because $\dim_{\mathbb{R}} C_I = 4n^2 + 1$, whereas the real dimension of the locus of, for example, abelian varieties in $Compl$ is $4n^2 + 2n$. For an alternating form Ω on $V_{\mathbb{R}}$ we denote by $Compl_\Omega$ the locus of periods in $Compl$ at which Ω represents a class of Hodge (1,1)-type, that is

$$Compl_\Omega = \{I \in Compl \,|\, \Omega(I\cdot, I\cdot) = \Omega(\cdot,\cdot)\}.$$

If we fix a basis of $V_{\mathbb{R}}$ and switch to matrix descriptions, then the condition $\Omega(I\cdot, I\cdot) = \Omega(\cdot,\cdot)$ simply becomes ${}^tI\,\Omega I = \Omega$, where I and Ω also denote the matrices of the corresponding complex structure and alternating form. The locus of marked complex tori with nontrivial Néron-Severi group is

$$\mathcal{L}_{NS} = \bigcup_{0\neq[\Omega]\in H^2(A,\mathbb{Q})} Compl_\Omega,$$

where A is a fixed complex torus.

Lemma 3.2. *For any alternating form Ω and any twistor sphere S, the intersection $S \cap Compl_\Omega$ is either finite or all of S.*

Lemma 3.3. *For any nonzero alternating form Ω, the cone C_I is not contained in $Compl_\Omega$.*

Lemma 3.3 immediately implies

Lemma 3.4. *For every $I \in Compl$ the set of non-generic periods in C_I, that is $C_I \cap \mathcal{L}_{NS}$, is a countable union of closed subsets of C_I none of which contains an open neighborhood (in C_I) of any of its points.*

Proof of Lemma 3.2. Follows from the fact that $Compl_\Omega$ and S are complex analytic subsets of $Compl$.

The twistor sphere S is analytic by Corollary 1.8. The subset $Compl_\Omega$ is a complex analytic subvariety in $Compl$ as it is the locus where Ω belongs to the fiber of a holomorphic subbundle of the Hodge bundle on $Compl$ arising from the Hodge filtration (alternatively one can say that $Compl_\Omega$ corresponds to a Schubert cycle determined by the line $\mathbb{C}[\Omega] \subset H^2(A, \mathbb{C})$ in the respective flag Grassmanian realization of the period domain). □

Proof of Lemma 3.3. We shall prove the following equivalent statement.

For any J anti-commuting with I and any nonzero alternating form Ω on $V_\mathbb{R}$ there is a neighborhood $U_\Omega \subset C_I$ of J such that the locus $Compl_\Omega$ intersects U_Ω along a real-analytic subvariety of positive codimension.

If $J \notin Compl_\Omega$ there is nothing to prove. Assume $J \in Compl_\Omega$.

As in Paragraph 1.2, choose a basis of $V_\mathbb{R}$ in which the matrices of J and I are block-diagonal with 4×4 blocks

$$\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

respectively. Such a basis is the union of sets of vectors of the form $\{v, Jv, Iv, JIv\}$. We denote the matrices of I, J and Ω in this basis by the same letters.

Consider the orbit of J under the conjugation action of G_I: $G_I \cdot J \cong G_I/G_\mathbb{H}$. Let

$$\begin{array}{rccc} \Psi\colon & G_I & \longrightarrow & Compl, \\ & g & \longmapsto & {}^gJ = gJg^{-1}, \end{array}$$

be the evaluation map of the action. Put $G_\Omega := \Psi^{-1}(Compl_\Omega)$, that is,

$$G_\Omega = \{g \in G_I \,|\, {}^t({}^gJ)\Omega({}^gJ) = \Omega\},$$

(note that G_Ω need not be a subgroup in G_I). Let $g(\tau)$ be any curve in G_Ω with tangent vector $X := g'(0) \in T_eG_\Omega$ at $e = g(0) \in G_\Omega$. Then, differentiating the constant function ${}^t({}^{g(\tau)}J)\Omega({}^{g(\tau)}J)$ at $\tau = 0$ we obtain

$$-\,{}^tX\,{}^tJ\Omega J + {}^tJ\,{}^tX\Omega J + {}^tJ\Omega XJ - {}^tJ\Omega JX = 0.$$

The left hand side may be simplified, given that ${}^tJ\Omega J = \Omega$ and ${}^tJ = J^{-1} = -J$, to

$$\begin{aligned} &-\,{}^tX\Omega + {}^tJ\,{}^tXJ\,{}^tJ\Omega J + {}^tJ\Omega J\,{}^tJXJ - \Omega X = \\ &= -\,{}^tX\Omega + {}^tJ\,{}^tXJ\Omega + \Omega\,{}^tJXJ - \Omega X \\ &= {}^t(X^J - X)\Omega + \Omega(X^J - X), \end{aligned}$$

where

$$X^J := J^{-1}XJ = JXJ^{-1}.$$

So, denoting $Y := X^J - X$, we have the equality

$$ {}^tY\Omega + \Omega Y = 0, \tag{4}$$

where Y commutes with I and anticommutes with J. Note that for any $X \in T_eG_I$, $X = \frac{1}{2}(X + X^J) + \frac{1}{2}(X - X^J)$, where $X + X^J \in T_eG_I$ commutes with J and $X - X^J \in T_eG_I$ anticommutes with J. The tangent space $T_eG_\mathbb{H}$ is the subspace of elements of

T_eG_I that commute with J. Hence, the subspace of Y's in T_eG_I anticommuting with J maps isomorphically onto the quotient space $V_I := T_eG_I/T_eG_{\mathbb{H}} \cong T_J\,(G_I \cdot J)$ under the quotient map $T_eG_I \to V_I$. So we need to check that for a nonzero Ω the space of solutions to (4), which is naturally identified with $T_J\,(G_I \cdot J \cap Compl_\Omega)$, has dimension strictly less than $\dim_{\mathbb{R}} T_J\,(G_I \cdot J) = \dim_{\mathbb{R}} V_I = 4n^2$ (i.e., not all of the orbit $G_I \cdot J$ lies in $Compl_\Omega$).

Now conjugate equation (4) by I to obtain

$$^tY\Omega^I + \Omega^I Y = 0.$$

Adding and subtracting this from (4) we obtain

$$^tY(\Omega + \Omega^I) + (\Omega + \Omega^I)Y = 0 \quad \text{and} \quad ^tY(\Omega - \Omega^I) + (\Omega - \Omega^I)Y = 0.$$

So we may assume that Ω is either I-invariant or I-anti-invariant in equation (4).

Case of I-invariant Ω. As Ω is J-invariant, it determines a skew-symmetric operator $\Omega\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$, commuting with J. So we may choose an Ω-invariant plane $P = \langle v, Jv\rangle \subset V_{\mathbb{R}}$ corresponding to a complex eigenvector $v - iJv$ of $\Omega\colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ such that the matrix of $\Omega|_P$ is

$$\begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}.$$

The complex structure I provides another such plane $IP = \langle Iv, JIv\rangle$, which is also Ω-invariant and orthogonal to P, so that on $P \oplus IP = \langle v, Jv, Iv, JIv\rangle$ the matrices of Ω, J and I are 4×4-block-diagonal with the following blocks on the diagonal

$$\left(\begin{array}{cc|cc} 0 & -\lambda & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda \\ 0 & 0 & -\lambda & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right).$$

The condition that Y commutes with I and anticommutes with J tells us that Y has a 4×4-block structure with blocks of the form

$$\left(\begin{array}{cc|cc} a_1 & a_2 & b_1 & b_2 \\ a_2 & -a_1 & b_2 & -b_1 \\ \hline -b_1 & b_2 & a_1 & -a_2 \\ b_2 & b_1 & -a_2 & -a_1 \end{array}\right).$$

Noting that $\Omega = JD = DJ$ for a diagonal matrix D commuting with J, we can rewrite (4) as

$$DY = {}^tYD,\ Y^I = Y,\ Y^J = -Y. \tag{5}$$

For notational convenience we write the matrix Y in terms of its 2×2-blocks $Y_{k,l}$, $Y = (Y_{k,l})$, $1 \leqslant k, l \leqslant 2n$, and denote by $\mathbb{1}_{2\times 2}$ the 2×2 identity matrix. If at least one λ_i, $1 \leqslant i \leqslant n$, in D is nonzero we get for all $1 \leqslant j \leqslant n$ the equalities of 4×4-blocks

$$\begin{pmatrix} \lambda_i \mathbb{1}_{2\times 2} & 0 \\ 0 & -\lambda_i \mathbb{1}_{2\times 2} \end{pmatrix} \cdot \begin{pmatrix} Y_{2i-1,2j-1} & Y_{2i-1,2j} \\ Y_{2i,2j-1} & Y_{2i,2j} \end{pmatrix} =$$
$$= \begin{pmatrix} ^tY_{2j-1,2i-1} & ^tY_{2j,2i-1} \\ ^tY_{2j-1,2i} & ^tY_{2j,2i} \end{pmatrix} \cdot \begin{pmatrix} \lambda_j \mathbb{1}_{2\times 2} & 0 \\ 0 & -\lambda_j \mathbb{1}_{2\times 2} \end{pmatrix}.$$

These matrix equalities completely determine all $n-1$ off-diagonal 4×4-entries of Y in the i-th "fat" row of 4×4-blocks in terms of the off-diagonal 4×4-entries of the i-th "fat" column, $1 \leqslant i \leqslant n$. So the codimension of the space of solutions of (5) is at least $4(n-1)$ (precise lower bound that is reached in the least restrictive case $\lambda_j = \lambda_i$ for all j). For the diagonal 4×4-entry, $i = j$, we obtain $b_2 = 0$ in $Y_{2i-1,2i}$, so that the codimension is at least $4n-3$.

Case of I-anti-invariant Ω: This is done similarly and leads to the same codimension bound $\geqslant 4n-3$. Alternatively, one could note that, if $C_I \subset Compl_\Omega$, then, in particular, $\pm I \in Compl_\Omega$, so that Ω is I-invariant, and this is the only case we need consider.

Now, by Lemma 3.2, either a twistor sphere in C_I entirely lies in some $Compl_\Omega$ or its intersection with $\mathcal{L}_{NS}$ contains only finitely many points of each $Compl_\Omega$. If $I \notin \mathcal{L}_{NS}$ then no twistor sphere in C_I is contained in $\mathcal{L}_{NS}$. The codimension estimate above then allows us to conclude that, for every nonzero Ω, the subset $C_I \cap Compl_\Omega$ is of codimension at least $(4n-3)+2 = 4n-1 > 0$ in C_I. If $I \in \mathcal{L}_{NS}$, the lower bound for the codimension is still at least $4n-3 > 0$. The proof is now complete. □

3.3. The transversality of the triple intersection of $G_{I_1}/G_{\mathbb{H}}, G_{J_1}/G_{\mathbb{H}}, G_{K_1}/G_{\mathbb{H}}$ at $eG_{\mathbb{H}}$, which is equivalent to the direct sum decomposition $T_eG/T_eG_{\mathbb{H}} = T_eG_{I_1}/T_eG_{\mathbb{H}} \oplus T_eG_{J_1}/T_eG_{\mathbb{H}} \oplus T_eG_{K_1}/T_eG_{\mathbb{H}}$, is preserved if we perturb $(I_1, J_1, K_1) \in \mathcal{T}$ a little. In other words, there is a compact neighborhood $U_{I_1,J_1,K_1} \subset \mathcal{T}$ of (I_1, J_1, K_1) and a compact neighborhood $U_{e,G} \subset G$ such that $\Phi_{I,J,K}\colon U_{e,J} \times U_{e,K} \to Compl$ is a submersion onto its image for all $(I, J, K) \in U_{I_1,J_1,K_1}$. Moreover, there is a compact neighborhood $U_{I_1} \subset Compl$ of I_1 which is contained in the image $\Phi_{I,J,K}(U_{e,J} \times U_{e,K})$ for all $(I, J, K) \in U_{I_1,J_1,K_1}$. We will always assume that for each neighborhood $U_{e,G}$ we made a choice of such $U_{I_1} = U_{I_1}(U_{e,G})$. Note that every $I_2 \in U_{I_1}$ is a regular value of $\Phi_{I,J,K}$ for all $(I, J, K) \in U_{I_1,J_1,K_1}$.

Lemma 3.5. *There exists a neighborhood $U_{e,G}$ such that for all $I_2 \in U_{I_1}$ and for all $(I, J, K) \in U_{I_1,J_1,K_1}$, the full preimage $\Phi^{-1}_{I,J,K}(I_2)$ is an $8n^2$-dimensional submanifold in $U_{e,J} \times U_{e,K}$ of the form*

$$\{(f_1h_1, h_1^{-1}f_2h_2) \mid h_1, h_2 \in G_{\mathbb{H}}\} \cap (U_{e,J} \times U_{e,K}), \tag{6}$$

where (f_1, f_2) is a pair in $U_{e,J} \times U_{e,K}$ such that $\Phi_{I,J,K}(f_1, f_2) = I_2$.

Proof of Lemma 3.5. The fact that $\Phi^{-1}_{I,J,K}(I_2) \cap (U_{e,J} \times U_{e,K})$ consists of a finite number of $8n^2$-dimensional manifolds follows from the regularity of I_2.

While the part of the fiber in (6) may have been easily guessed, the fact that for a small enough $U_{e,G}$ this is the whole fiber follows from Proposition 2.5. Indeed, assuming that we have $(f_1, f_2), (g_1, g_2) \in \Phi^{-1}_{I,J,K}(I_2) \subset G_J \times G_K$, we see that $f_2^{-1}f_1^{-1}g_1g_2 \in G_I$. Setting $g_I = f_2^{-1}f_1^{-1}g_1g_2$ and $g_J = f_1^{-1}g_1 \in G_J$ we have the equality

$$f_2g_I = g_Jg_2.$$

The left side of the equality lies in G_KG_I and the right side lies in G_JG_K. If we restrict ourselves to $\Phi^{-1}_{I,J,K}(I_2) \cap (U_{e,G} \times U_{e,G})$ for a small enough neighborhood $U_{e,G} \subset G$ then Proposition 2.5 tells us that, for every element in the product $U_{e,J}U_{e,K}U_{e,I}$, each of its three factors is uniquely determined up to a $G_{\mathbb{H}}$-correction.

So from our equality $f_2 g_I = g_J g_2$ we obtain $g_I, g_J \in G_{\mathbb{H}}$, which, after setting $h_1 := g_J = f_1^{-1} g_1$ and $h_2 := g_I$, implies that $g_1 = f_1 h_1$ and $g_2 = g_J^{-1} f_2 g_I = h_1^{-1} f_2 h_2$.

Since $U_{I_1}, U_{I_1,J_1,K_1}, U_{e,G}$ are compact and $U_{e,G}$ is independent of the choice of $(I,J,K) \in U_{I_1,J_1,K_1}$, there is a universal upper bound for the number of connected components of $\Phi_{I,J,K}^{-1}(I_2) \cap (U_{e,J} \times U_{e,K})$, for all $I_2 \in U_{I_1}$ and all $(I,J,K) \in U_{I_1,J_1,K_1}$. Therefore we can shrink the compact neighborhood $U_{e,G}$ so that the fibers $\Phi_{I,J,K}^{-1}(I_2) \cap (U_{e,J} \times U_{e,K})$ for all $I_2 \in U_{I_1}$ and all $(I,J,K) \in U_{I_1,J_1,K_1}$ contain only the component specified in (6). □

Regarding the proof of Lemma 3.5, we note the following.

Remark 3.6. It is not hard to see that the fiber $\Phi_{I,J,K}^{-1}(I_2)$ in Lemma 3.5, as a topological subspace of $G \times G$, depends continuously on $I_2 \in U_{I_1}$ and $(I,J,K) \in U_{I_1,J_1,K_1}$.

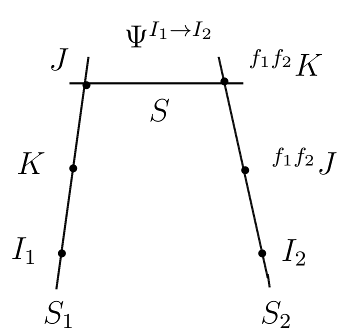
Remark 3.7. In general, it is possible that $g \in U_{e,G}$ is not uniquely representable as a triple product of elements in the larger sets G_J, G_K, G_I and thus we cannot say if the whole fiber $\Phi_{I,J,K}^{-1}(I_2) \subset G_J \times G_K$ consists of just one $G_{\mathbb{H}} \times G_{\mathbb{H}}$-orbit as in Lemma 3.5. This is why we possibly need to shrink $U_{e,G}$.

3.4. Recall that, for any I, $M_I = G_I/G_{I,S}$ parametrizes the twistor lines through I (see Paragraph 1.5). For all I, put $U_{I_1,J_1,K_1}(I) := pr_{23}(pr_1^{-1}(I) \cap U_{I_1,J_1,K_1})$. Then $U_{I_1,J_1,K_1}(I_1)$ is a neighborhood of (J_1,K_1) in $C_{I_1} \times_{M_{I_1}} C_{I_1} = pr_{23}(pr_1^{-1}(I_1) \cap \mathcal{T})$. Consider the map

$$\begin{array}{rccl} \Psi^{I_1 \to I_2}\colon & U_{I_1,J_1,K_1}(I_1) & \longrightarrow & C_{I_2} \times_{M_{I_2}} C_{I_2} = pr_{23}(pr_1^{-1}(I_2) \cap \mathcal{T}), \\ & (S(J,K),J,K) & \longmapsto & (S({}^{f_1f_2}J, {}^{f_1f_2}K), {}^{f_1f_2}K, {}^{f_1f_2}J), \end{array}$$

where $(f_1, f_2) \in \Phi_{I_1,J,K}^{-1}(I_2) \cap (U_{e,J} \times U_{e,K})$, and we use, in an obvious way, the triple notation of the kind $(S(J,K),J,K)$ for the elements of the fiber products above. Note the switched order of ${}^{f_1f_2}K, {}^{f_1f_2}J$. The role of this change of order will be clarified later.

Lemma 3.5 guarantees that the mapping $\Psi^{I_1 \to I_2}$ is well-defined, as its value at $(S(J,K),J,K)$ is uniquely determined by the fiber $\Phi_{I_1,J,K}^{-1}(I_2) \cap (U_{e,J} \times U_{e,K})$, so it does not depend on the choice of a particular point in the fiber.



Picture 2: For fixed I_1 and I_2 any pair $(J,K) \in C_{I_1} \times_{M_{I_1}} C_{I_1}$ near (J_1,K_1) determines a unique pair $({}^{f_1f_2}K, {}^{f_1f_2}J) \in C_{I_2} \times_{M_{I_2}} C_{I_2}$.

3.5. Next, for each $(I,J,K) \in U_{I_1,J_1,K_1}$ consider the mapping $\Phi_{I,K,J}$ (note that we switched J and K in the subscript). By shrinking the original U_{I_1,J_1,K_1} and $U_{e,G}$ if needed, we can find a compact neighborhood $V_{e,G} \subset G$ such that for each $(I,J,K) \in U_{I_1,J_1,K_1}$ we have

(a) $\Phi_{I,K,J}\colon V_{e,K} \times V_{e,J} \to Compl$ is a submersion onto its image;

(b) every fiber of this mapping is of the form described in Lemma 3.5; and

(c) the image $\Phi_{I,K,J}(V_{e,K} \times V_{e,J})$ contains $U_{I_1} \subset \bigcap_{(I,J,K)\in U_{I_1,J_1,K_1}} \Phi_{I,J,K}(U_{e,J} \times U_{e,K})$

(see Paragraph 3.3).

By Lemma 3.5, conditions (a) and (b) are satisfied. We need only to comment on (c). By Lemma 3.5, for the original triple $(I_1,J_1,K_1) \in U_{I_1,J_1,K_1}$, we can find $V_{e,G}$ such that $\Phi_{I_1,K_1,J_1}\colon V_{e,K_1} \times V_{e,J_1} \to Compl$, where $V_{e,K_1} := V_{e,G} \cap G_{K_1}, V_{e,J_1} := V_{e,G} \cap G_{J_1}$, satisfies (a) and (b). Shrinking $U_{e,G}$ and, thus, U_{I_1}, if needed, we can satisfy (c) for Φ_{I_1,K_1,J_1}. Now shrinking U_{I_1,J_1,K_1} and again $U_{e,G}$, if needed, we can satisfy conditions (a), (b) and (c) for all $(I,J,K) \in U_{I_1,J_1,K_1}$.

3.6. Now introduce $V_{I_1,K_1,J_1} := \{(I,K,J) \,|\, (I,J,K) \in U_{I_1,J_1,K_1}\}$ and $V_{I_1,K_1,J_1}(I) := pr_{23}(pr_1^{-1}(I) \cap V_{I_1,K_1,J_1})$.

Then, for all (I,K,J) in the interior of V_{I_1,K_1,J_1}, the set $pr_1(V_{I_1,K_1,J_1})$ is a neighborhood of I in $Compl$ and $V_{I_1,K_1,J_1}(I)$ is a neighborhood of $(K,J) \in C_I \times_{M_I} C_I$. Note that, due to Condition (c) in Paragraph 3.5, for all $I \in U_{I_1} \cap pr_1(V_{I_1,K_1,J_1})$ and for all $(K,J) \in V_{I_1,K_1,J_1}(I)$, the image $\Phi_{I,K,J}(U_{e,K} \times U_{e,J})$ contains the neighborhood U_{I_1}.

3.7. Choose $I_2 \in U_{I_1} \cap pr_1(V_{I_1,K_1,J_1})$ and K,J such that $(I_2,K,J) \in V_{I_1,K_1,J_1}$. Conditions (a),(b) and (c) in Paragraph 3.5 allow us to define, analogously to $\Psi^{I_1\to I_2}$, the map

$$\begin{array}{rccc} \Psi^{I_2\to I_1}\colon & V_{I_1,K_1,J_1}(I_2) & \longrightarrow & C_{I_1} \times_{M_{I_1}} C_{I_1}, \\ & (S(J,K),K,J) & \longmapsto & (S({}^{d_1d_2}J, {}^{d_1d_2}K), {}^{d_1d_2}J, {}^{d_1d_2}K), \end{array}$$

for $(d_1,d_2) \in \Phi^{-1}_{I_2,K,J}(I_1)$ (again, note the reversed order of J and K in the subscript).

The period ${}^{f_1f_2}K$ in Picture 2 above will play the role of the "rotation center" for $\Phi_{I_2,{}^{f_1f_2}K,{}^{f_1f_2}J}$ (here $J,K \in C_{I_1}$), similar to the role that J plays for $\Phi_{I_1,J,K}$. This explains why we switched J and K.

Below we will impose restrictions on the domain of $\Psi^{I_1\to I_2}$ in order for the image of this map to be contained in the domain of $\Psi^{I_2\to I_1}$, so that we can compose them.

We begin by choosing a compact neighborhood U_{J_1,K_1} of (J_1,K_1) in $U_{I_1,J_1,K_1}(I_1)$, which can at first be all of $U_{I_1,J_1,K_1}(I_1)$. We will later modify U_{J_1,K_1}, without changing the original U_{I_1,J_1,K_1}.

Lemma 3.8. *For fixed V_{I_1,K_1,J_1}, we can shrink $U_{e,G}$ and U_{J_1,K_1} so that for arbitrary $I_2 \in U_{I_1}$,*

$$\Psi^{I_1\to I_2}(U_{J_1,K_1}) \subset V_{I_1,K_1,J_1}(I_2).$$

Proof. As in Paragraph 3.5, this follows from the fact that the mapping $\Psi^{I_2\to I_1}$ depends continuously on I_2 (see Remark 3.6), and that

$$\lim_{I_2\to I_1} \Psi^{I_2\to I_1} = (12)\colon U_{J_1,K_1} \to V_{I_1,K_1,J_1}(I_1),$$

$$(S(J,K),J,K) \mapsto (S(J,K),K,J),$$

the latter mapping is trivially defined on the whole U_{J_1,K_1}, so that the sizes of the domains $V_{I_1,K_1,J_1}(I_2)$ of $\Psi^{I_2\to I_1}$'s are bounded away from zero, when I_2 is close to I_1.

As before, we can further shrink $U_{e,G}$ (and hence U_{I_1}), if needed, so that properties (a), (b), (c) in Paragraph 3.5 hold independently of the point $I_2 \in U_{I_1}$. □

3.8. Possibly shrinking $U_{e,G}$, we can and will assume that it is invariant under taking inverses, $g \mapsto g^{-1}$.

Lemma 3.9. *Possibly further shrinking $U_{e,G}$ and U_{J_1,K_1}, satisfying the conclusion of Lemma 3.8, we have for all $I_2 \in U_{I_1}$*

$$\Psi^{I_2\to I_1} \circ \Psi^{I_1\to I_2} = Id|_{U_{J_1,K_1}}.$$

Proof. For all $(f_1, f_2) \in \Phi^{-1}_{I_1,J,K}(I_2) \cap (U_{e,G} \times U_{e,G})$ and all $(S(J,K), J, K) \in U_{J_1,K_1}$, we want the neighborhoods $V_{e,\,f_1f_2K} = V_{e,G} \cap G_{f_1f_2K}, V_{e,\,f_1f_2J} = V_{e,G} \cap G_{f_1f_2J}$ to contain, respectively, the neighborhoods ${}^{f_1f_2}U_{e,K} = f_1f_2U_{e,K}f_2^{-1}f_1^{-1}$ and ${}^{f_1f_2}U_{e,J} = f_1f_2U_{e,J}f_2^{-1}f_1^{-1}$, so that, in particular, $V_{e,\,f_1f_2K} \times V_{e,\,f_1f_2J}$ contains the pair

$$(d_1, d_2) = (f_1f_2 \cdot f_2^{-1} \cdot f_2^{-1}f_1^{-1}, f_1f_2 \cdot f_1^{-1} \cdot f_2^{-1}f_1^{-1}) = (f_1f_2^{-1}f_1^{-1}, f_1f_2 \cdot f_1^{-1} \cdot f_2^{-1}f_1^{-1}).$$

Here the invariance of $U_{e,G}$ under taking inverses is used. The pair (d_1, d_2) certainly belongs to the preimage $\Phi^{-1}_{I_2,\,f_1f_2K,\,f_1f_2J}(I_1)$ as the product of its entries is $f_1f_2^{-1}f_1^{-1} \cdot f_1f_2 \cdot f_1^{-1} \cdot f_2^{-1}f_1^{-1} = f_2^{-1}f_1^{-1}$.

Note that, for $U_{e,G}$ small enough, the neighborhoods ${}^{f_1f_2}U_{e,K} \times {}^{f_1f_2}U_{e,J}$ will also be uniformly small for all $(f_1, f_2) \in \Phi^{-1}_{I_1,J,K}(I_2) \cap (U_{e,G} \times U_{e,G})$, so that the fiber of $\Phi^{-1}_{I_2,\,f_1f_2K,\,f_1f_2J}(I_1)$ in ${}^{f_1f_2}U_{e,K} \times {}^{f_1f_2}U_{e,J}$ consists of a unique connected component of the form described in Lemma 3.5. Then the pair (d_1, d_2) is contained in this "good" part of the fiber $\Phi^{-1}_{I_2,\,f_1f_2K,\,f_1f_2J}(I_1)$ and we can use (d_1, d_2) to evaluate $\Psi^{I_2\mapsto I_1}$ at $(S({}^{f_1f_2}J, {}^{f_1f_2}K), {}^{f_1f_2}J, {}^{f_1f_2}K)$. Thus

$$\Phi_{I_2,\,f_1f_2K,\,f_1f_2J}(d_1, d_2) = I_1$$

and

$${}^{d_1d_2f_1f_2}J = J, {}^{d_1d_2f_1f_2}K = K,$$

so that

$$\Psi^{I_2\mapsto I_1}(S({}^{f_1f_2}J, {}^{f_1f_2}K), {}^{f_1f_2}K, {}^{f_1f_2}J) = (S(J,K), J, K),$$

where, certainly, $S(J,K) = S(I_1, J, K)$, proving that the composition $\Psi^{I_2\to I_1} \circ \Psi^{I_1\to I_2}$ is the identity on U_{J_1,K_1}.

In order to ensure that $V_{e,\,f_1f_2K} \times V_{e,\,f_1f_2J}$ contains (d_1, d_2), we assume, shrinking $U_{e,G}$ and U_{J_1,K_1} if necessary, but not changing $V_{e,G}$ and the previously fixed V_{I_1,K_1,J_1}, that for all $(S(J,K), J, K) \in U_{J_1,K_1}$ and for all points $(f_1, f_2) \in \Phi^{-1}_{I_1,J,K}(I_2) \cap (U_{e,G} \times U_{e,G})$, the neighborhoods $V_{e,\,f_1f_2K}, V_{e,\,f_1f_2J}$ contain, respectively, the neighborhoods ${}^{f_1f_2}U_{e,K}$ and ${}^{f_1f_2}U_{e,J}$. □

Corollary 3.10. *Let U_{I_1} be defined by $U_{e,G}$ ($U_{e,G}$ satisfying Lemma 3.9). For arbitrary $I_2 \in U_{I_1}$, both joint points $J \in C_{I_1}$ and ${}^{f_1f_2}K \in C_{I_2}$ of a triple of twistor spheres connecting I_1 and I_2, can be chosen generic.*

Proof. Define

$$\begin{array}{rccc} pr_K\colon & V_{I_1,K_1,J_1}(I_2) & \longrightarrow & C_{I_2}\subset Compl, \\ & (S(J,K),K,J) & \longmapsto & K. \end{array}$$

This projection is a submersion onto its image. By Lemma 3.4, the locus $\mathcal{L}_{NS}$ intersects C_{I_2} in a countable union of closed submanifolds of positive codimension in C_{I_2}. As the mapping pr_K is a submersion onto its image, the preimage $pr_K^{-1}(\mathcal{L}_{NS}\cap C_{I_2})$ is also a countable union of closed submanifolds of positive codimension in $V_{I_1,K_1,J_1}(I_2)$. Similarly, for

$$\begin{array}{rccc} pr_J\colon & U_{J_1,K_1} & \longrightarrow & C_{I_1}\subset Compl, \\ & (S(J,K),J,K) & \longmapsto & J, \end{array}$$

$pr_J^{-1}(\mathcal{L}_{NS}\cap C_{I_1})\subset U_{J_1,K_1}$ is a countable union of closed submanifolds of positive codimension. The mapping $\Psi^{I_2\to I_1}$ is real-analytic, so the closure of $\Psi^{I_2\to I_1}(pr_J^{-1}(\mathcal{L}_{NS}\cap C_{I_2}))$ in U_{J_1,K_1} does not contain interior points. Therefore

$$U_{J_1,K_1}\neq pr_J^{-1}(\mathcal{L}_{NS}\cap C_{I_1})\cup\Psi^{I_2\to I_1}(pr_K^{-1}(\mathcal{L}_{NS}\cap C_{I_2})). \tag{7}$$

Since, by Lemma 3.9, $\Psi^{I_2\to I_1}\circ\Psi^{I_1\to I_2}=Id|_{U_{J_1,K_1}}$, the inequality (7) tells us that the image of the mapping $\Psi^{I_1\to I_2}$ is not contained in $pr_K^{-1}(\mathcal{L}_{NS}\cap C_{I_2})$. Thus we may find a pair $(J,K)\in U_{J_1,K_1}$ such that $J=pr_J(S(J,K),J,K)\notin\mathcal{L}_{NS}\cap C_{I_1}$ and ${}^{f_1f_2}K=pr_K(\Psi^{I_1\to I_2}(S(J,K),J,K))\notin\mathcal{L}_{NS}\cap C_{I_2}$, that is, both periods are generic. □

4. The degree of twistor lines

In this section we show that twistor lines in $Compl$ have degree $2n$ in the Plücker embedding. We first show that the group $G=GL(V_{\mathbb{R}})$ acts transitively on the set of twistor lines in $Compl$ and then compute the degree of an explicit twistor line.

Lemma 4.1. *The group $G=GL(V_{\mathbb{R}})$ acts transitively on the set of twistor lines in $Compl$.*

Proof. Given two twistor spheres $S_1=S(I_1,J_1)$ and $S_2=S(I_2,J_2)$, there is an element $g\in G$ sending I_1 to I_2, hence sending S_1 to a twistor sphere through I_2. The lemma now follows from Corollary 1.6. □

4.1. To construct our example, consider the affine chart in the Grassmannian $G(2n,4n)$ of normalized period matrices $(\mathbb{1}|Z)$, where $\mathbb{1}$ is, in general, the $2n\times 2n$ identity matrix and Z now denotes a non-degenerate $2n\times 2n$ complex matrix. Let us fix a basis of $V_{\mathbb{R}}$ and write the matrix of an arbitrary complex structure $I\colon V_{\mathbb{R}}\to V_{\mathbb{R}}$ in the following block form

$$I=\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

for $2n\times 2n$ real matrices A,B,C,D. Then the relation

$$(\mathbb{1}\,|\,Z)I=(i\mathbb{1}\,|\,iZ),$$

gives the matrix equations

$$A+ZC=i\mathbb{1},\quad B+ZD=iZ.$$

Assume that C is invertible so that the first equation allows us to write $Z=(i\mathbb{1}-A)C^{-1}$. The condition that I is a complex structure will then guarantee that the second equation is automatically satisfied.

4.2. **The case** $n = 1$**.** Momentarily assume $n = 1$ and consider the twistor sphere $S = S(I, J)$ where I and J have the respective matrices

$$\left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

and put $K = IJ$. So for $\lambda \in S$,

$$\lambda = aI + bJ + cK = \left(\begin{array}{cc|cc} 0 & -a & -b & -c \\ a & 0 & -c & b \\ \hline b & c & 0 & -a \\ c & -b & a & 0 \end{array}\right).$$

Assume additionally that $b^2 + c^2 \neq 0$, that is, $\lambda \in S \setminus \{\pm I\}$. Here

$$A = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, C = \begin{pmatrix} b & c \\ c & -b \end{pmatrix}, C^{-1} = \frac{1}{b^2 + c^2} \begin{pmatrix} b & c \\ c & -b \end{pmatrix}.$$

Then

$$Z = (i\mathbb{1} - A)C^{-1} = \frac{1}{b^2 + c^2} \begin{pmatrix} ac + ib & -ab + ic \\ -ab + ic & -ac - ib \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix},$$

where Z clearly satisfies the equations

$$z_1 + z_4 = 0,\ z_2 - z_3 = 0,\ \det Z = z_1 z_4 - z_2 z_3 = 1.$$

4.3. Now, for a general n, we can construct a twistor line in the period domain of complex $2n$-dimensional tori, which, in the affine chart of $G(2n, 4n)$ above, corresponds to the locus of matrices $(\mathbb{1}|Z)$ where Z is the block-diagonal matrix with the same 2×4-block

$$\left(\begin{array}{cc|cc} 1 & 0 & u & v \\ 0 & 1 & v & -u \end{array}\right), u^2 + v^2 = -1,$$

on the diagonal.

4.4. The degree of the curve in the example is $2n$ under the Plücker embedding $G(2n, 4n) \hookrightarrow \mathbb{P}^{\binom{4n}{2n}-1}$. Indeed, the Plücker coordinates in the above affine chart are given by the maximal minors of the matrix $(\mathbb{1}|Z)$. The twistor line S in our example is contained in the plane $P(S)$ with parameters u, v in the part given by the affine chart. Let W be the homogeneous coordinate given by the minor formed by all $\mathbb{1}_{2\times 2}$-blocks and let U and V be any homogeneous coordinates such that after restricting to $P(S)$ we get $u = \frac{U}{W}$ and $v = \frac{V}{W}$.

Let us consider from now on the plane $P(S)$ as a projective (complete) 2-plane in $Gr(2n, 4n)$ with coordinates U, V, W. Then the minor formed by the (u, v)-blocks restricts to $P(S)$ as $(U^2 + V^2)^n$. Rewriting the equation $u^2 + v^2 = -1$ of our twistor line S in homogeneous coordinates we get $U^2 + V^2 + W^2 = 0$, so that, restricting the polynomial $(U^2 + V^2)^n$ to S, we see that it vanishes precisely when $U^2 + V^2 = -W^2$ vanishes, that is, only at the points $\pm I$ of S outside of our affine chart. Each of the two factors in the expansion $(U^2 + V^2)^n = (U + iV)^n (U - iV)^n$ vanishes to order n at the respective point, so the total order of vanishing is $n + n = 2n$ which is the degree of the image of S under the Plücker embedding.

Corollary 4.2. *Twistor lines have degree $2n$ in the Plücker embedding of $Compl$.*

Proof. Follows from Lemma 4.1 and Paragraph 4.4. □

Remark 4.3. There is an alternative proof of the above corollary following the lines explained in Remark 1.9. Namely, we have the embedding $i\colon Gr(2, \mathbb{H}\otimes\mathbb{C}) \hookrightarrow Gr(2n, V_{\mathbb{C}}), \mathbb{H}^{1,0} \mapsto \mathbb{H}^{1,0}\otimes V'_{\mathbb{C}}$. Let $e_1, \ldots, e_n$ be some basis of $V'_{\mathbb{C}}$. Then, in terms of the Plücker embeddings of the respective Grassmanians, we have $i(u\wedge v) = u\otimes e_1\wedge v\otimes e_1\wedge\cdots\wedge u\otimes e_n\wedge v\otimes e_n$, which induces an isomorphism $i^*(\mathcal{O}_{Gr(2n,V_{\mathbb{C}})}(1)) \cong \mathcal{O}_{Gr(2,\mathbb{H}\otimes\mathbb{C})}(n)$ of the sheaves on the quadric $Gr(2, \mathbb{H}\otimes\mathbb{C})$, thus justifying that the degree of $S\subset Gr(2n, 4n)$ under the Plücker embedding is $2n$.

5. Appendix

Let A be a 2-dimensional complex torus with period belonging to the twistor line S constructed in Paragraph 4.2.

For any $\Omega\in Hom(\wedge^2\Gamma, \mathbb{Q})$ such that $S\cap Compl_{\Omega}$ is infinite, by Lemma 3.2, the whole twistor line S is contained in $Compl_{\Omega}$. Below we determine all Ω such that $S\subset Compl_{\Omega}$: these are specified by the invariance conditions $\Omega(I\cdot, I\cdot) = \Omega(J\cdot, J\cdot) = \Omega(\cdot,\cdot)$ and form a 3-dimensional subspace in $Hom(\wedge^2\Gamma, \mathbb{Q})$. The invariance conditions mean that the first Riemann bilinear relation is satisfied.

On the other hand, the second Riemann bilinear relation does not hold: these Ω determine $(1,1)$-classes in the cohomology of tori in this twistor line whose hermitian forms are always indefinite. Thus, none of the classes determined by these Ω is Kähler. For the formulation of the Riemann bilinear relations see, for example, [7, Ch. 2].

5.1. **The first bilinear relation.** Let Q be the matrix of the alternating form corresponding to an I, J-invariant cohomology class in $H^2(A,\mathbb{Q})$, written in the basis in which the matrices of I, J are as in the previous section. The I, J-invariance then translates into the commutation relations $QI = IQ$ and $QJ = JQ$. A general skew-symmetric such Q has the form

$$Q = \begin{pmatrix} 0 & -b & c & -d \\ b & 0 & d & c \\ -c & -d & 0 & b \\ d & -c & -b & 0 \end{pmatrix}, \quad b, c, d\in\mathbb{Q}.$$

Such Q, by definition, determines a rational class of Hodge type $(1,1)$ for all tori with periods in $S(I,J)$, so that the first bilinear relation $\Omega Q^{-1}\,{}^t\Omega = 0$ is automatically guaranteed by the choice of Q.

5.2. **The second bilinear relation $-i\Omega Q^{-1}\,{}^t\overline{\Omega} > 0$.** For Q as above it is easy to find Q^{-1}. Indeed, note that

$$Q^2 = -(b^2+c^2+d^2)\mathbb{1}_{4\times 4},$$

where $\mathbb{1}_{4\times 4}$ is the 4×4 identity matrix, so that $Q^{-1} = -\frac{1}{(b^2+c^2+d^2)}Q$ and $-i\Omega Q^{-1}\,{}^t\overline{\Omega} > 0$ is equivalent to $i\Omega Q\,{}^t\overline{\Omega} > 0$. We have

$$\Omega Q = \begin{pmatrix} -uc+vd & -b-ud-vc & c-vb & -d+ub \\ b-vc-ud & -vd+uc & d+ub & c+vb \end{pmatrix},$$

so that $\Omega Q\,{}^t\overline{\Omega}$ is equal to

$$\begin{pmatrix} (\overline{u}-u)c+(v-\overline{v})d+(u\overline{v}-\overline{u}v)b & (\overline{u}-u)d+(\overline{v}-v)c-(1+|u|^2+|v|^2)b \\ (\overline{u}-u)d+(\overline{v}-v)c+(1+|u|^2+|v|^2)b & (u-\overline{u})c+(\overline{v}-v)d+(u\overline{v}-\overline{u}v)b \end{pmatrix}.$$

Setting $u=u_1+iu_2$ and $v=v_1+iv_2$, we compute

$$i\Omega Q\,{}^t\overline{\Omega}=\begin{pmatrix} 2(u_2c-v_2d)+2(u_1v_2-u_2v_1)b & 2(u_2d+v_2c)-i(1+|u|^2+|v|^2)b \\ 2(u_2d+v_2c)+i(1+|u|^2+|v|^2)b & -2(u_2c-v_2d)+2(u_1v_2-u_2v_1)b \end{pmatrix}.$$

Now the determinant $\det i\Omega Q\,{}^t\overline{\Omega}$ is

$$\det i\Omega Q\,{}^t\overline{\Omega}=(4b^2(u_1v_2-u_2v_1)^2-4(u_2c-v_2d)^2)-(b^2(1+|u|^2+|v|^2)^2+4(v_2c+u_2d)^2)=$$
$$=b^2(4(u_1v_2-u_2v_1)^2-(1+|u|^2+|v|^2)^2)-4(u_2c-v_2d)^2-4(v_2c+u_2d)^2.$$

Let us show that indeed $4(u_1v_2-u_2v_1)^2-(1+|u|^2+|v|^2)^2\leqslant 0$ for all $u,v\in\mathbb{C}$ such that $u^2+v^2=-1$. This would prove that $i\Omega Q\,{}^t\overline{\Omega}>0$ never holds for the periods in our twistor line.

The complex equation $u^2+v^2=-1$ is equivalent to the two real equations $u_2^2+v_2^2=u_1^2+v_1^2+1$ and $u_1u_2+v_1v_2=0$. Introducing the vectors

$$X=\begin{pmatrix}u_1\\v_1\end{pmatrix},Y=\begin{pmatrix}u_2\\v_2\end{pmatrix}.$$

these equalities can be written as $|Y|^2=|X|^2+1$ and $X\perp Y$. The term $(u_1v_2-u_2v_1)^2$ above is the square of the dot product of X and the result of rotation of Y by $\frac{\pi}{2}$, so that $X\perp Y$ implies that $(u_1v_2-u_2v_1)^2=|X|^2|Y|^2$. Now the equality $|Y|^2=|X|^2+1$ allows us to write $1+|u|^2+|v|^2=|X|^2+|Y|^2$ and we have

$$4|X|^2|Y|^2-(|X|^2+|Y|^2)^2=-(|X|^2-|Y|^2)^2=-1<0.$$

So, finally we obtain $\det i\Omega Q\,{}^t\overline{\Omega}<0$ and none of the Q above determines a Kähler class.

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