

# SYMMETRIC CORRESPONDENCES WITH DECOMPOSABLE MINIMAL EQUATION

ELHAM IZADI AND HERBERT LANGE

ABSTRACT. We study symmetric correspondences with completely decomposable minimal equation on smooth projective curves  $C$ . The Jacobian of  $C$  then decomposes correspondingly. For all positive integers  $g$  and  $\ell$ , we give series of examples of smooth curves  $C$  of genus  $n^\ell(g-1)+1$  with correspondences satisfying minimal equations of degree  $\ell+1$  such that the Jacobian of  $C$  has at least  $2^\ell$  isogeny components.

## 1. INTRODUCTION

A correspondence of a smooth projective curve  $C$  into itself is a divisor on the product  $C \times C$ . Correspondences were extensively studied by the Italian geometers of the 19th century. Their importance lies in the fact that they determine endomorphisms of the Jacobian  $J(C)$  of  $C$ . In fact, the ring of equivalence classes of these correspondences is isomorphic to the ring of endomorphisms of  $J(C)$ .

One way to use this is given by the well-known bigonal, trigonal and tetragonal (or, more generally,  $n$ -gonal) constructions, where one uses correspondences to detect isomorphisms between certain Prym varieties and Jacobians. Another way is to construct Prym-Tyurin varieties, that is, roughly speaking, abelian subvarieties of Jacobians to which the restriction of the canonical polarization is a multiple of a principal polarization.

Here we study symmetric completely decomposable correspondences, that is correspondences which satisfy a minimal equation over the rationals all of whose zeros are rational numbers. It is easy to see (Proposition 2.3) that such a correspondence induces a decomposition up to isogeny of the corresponding Jacobian into a product of abelian subvarieties which are given by the zeros of the minimal polynomial.

We give a series of examples for this: Consider a smooth projective curve  $X$  of genus  $g \geq 1$  with an étale cover  $f : Y \rightarrow X$  of degree  $n \geq 2$ . Given a set  $\{\sigma_1, \dots, \sigma_\ell\}$  of automorphisms of  $X$  for any positive integer  $\ell$ , we let  $C$  be the  $\ell$ -fold fibre product of  $Y$  with itself over  $X$  for the compositions  $\sigma_i f : Y \rightarrow X$  for  $i = 1, \dots, \ell$ :

$$C := Y_{\sigma_1} \times_X Y_{\sigma_2} \times_X \cdots \times_X Y_{\sigma_\ell},$$

where  $Y_{\sigma_i} := Y$  but with the étale cover  $\sigma_i \circ f : Y_{\sigma_i} \rightarrow X$ . The curve  $C$  is always smooth and we show in Section 4 that it is irreducible if and only if the monodromy group of the cover  $f : Y \rightarrow X$  is  $\ell$ -transitive.

---

*Date:* August 14, 2021.

Part of this work was done while the second author was visiting the University of California San Diego. He wishes to thank UCSD for its support and hospitality.

The curve  $C$  admits an effective correspondence  $D$  of degree  $\ell(n-1)$  given by

$$D := \{((\sigma_1 x)_{j_1}, \dots, (\sigma_\ell x)_{j_\ell}), ((\sigma_1 x)_{k_1}, \dots, (\sigma_\ell x)_{k_\ell}) \in C \times C \mid k_i = j_i \text{ except for exactly one index } i\}.$$

Note that for  $\ell = 1$  and  $\sigma_1 = id$  we have  $C = Y$  and  $D$  is the well known correspondence

$$D = \{(y_1, y_2) \mid y_1 \neq y_2, f(y_1) = f(y_2)\} \subset Y^2.$$

Our main result is:

**Theorem 1.1.** *The correspondence  $D$  is completely decomposable and satisfies the equation*

$$\prod_{r=0}^{\ell} (D - nr + \ell) = 0.$$

We also identify the eigen-abelian subvarieties of the Jacobian  $JC$  for the eigenvalues  $nr - \ell$  for all  $r$  (Corollary 3.4). We compute their dimensions (Lemma 3.7) and show that  $C$  has principally polarized quotients  $(B, \Xi)$  of dimension  $d = (n-1)^\ell(g-1)$  or  $(n^\ell - (n-1)^\ell)(g-1) + 1$  such that the image of  $C$  in  $B$  has cohomology class equal to  $\ell! n^\ell \frac{[\Xi]^{d-1}}{(d-1)!}$  (Lemmas 3.5 and 3.7).

It follows from our results that the Jacobian of  $C$  is isogenous to the product of at least  $2^\ell$  abelian varieties. To our knowledge, up to now, examples of such curves were only obtained from curves with sufficiently large automorphism groups (see, e.g., [CLRR09]). Starting with an arbitrary curve  $X$  of genus  $\geq 2$  and any étale cover  $Y \rightarrow X$  of degree  $n \geq 2$ , we can construct the curve  $C$  with  $\sigma_1 = \dots = \sigma_\ell = Id$ . In this way we obtain a family of dimension  $3g - 3$  of curves  $C$  of genus  $n^\ell(g-1) + 1$  with at least  $2^\ell$  isogeny factors. When  $n = 2$ , all these isogeny factors have dimension  $g-1$ . One may ask whether they are all isogenous to the Prym variety of the cover  $f : Y \rightarrow X$ .

In Section 2 we define completely decomposable correspondences and derive the corresponding decomposition of the Jacobian. Section 3 contains the proof of the main theorem and its consequences. Finally, Section 4 contains the statement and proof of the irreducibility criterion for  $C$ .

## 2. CORRESPONDENCES ON A CURVE $C$

Let  $C$  be a smooth projective curve over an algebraically closed field of characteristic 0. A *correspondence of  $C$  of bidegree  $(d_1, d_2) \in \mathbb{Z}^2$*  is given by a divisor  $D$  of  $C \times C$  such that the projections  $p_i : D \rightarrow C$  are of degree  $d_i$  for  $i = 1$  and  $2$ . Here the degree  $d_i$  is defined as follows: If  $D = \sum_k a_k D_k$  with reduced and irreducible curves  $D_k \subset C \times C$ , then  $d_i = \sum_k a_k \deg(p_i : D_k \rightarrow C)$ .

The transposed correspondence of  $D$  is by definition the correspondence

$$D^t := \tau^* D$$

where  $\tau$  denotes the natural involution of  $C \times C$  which swaps the factors. A correspondence is called *symmetric* if  $D^t = D$ . For a symmetric correspondence we have  $d_1 = d_2$  and  $d := d_1$  is called its *degree*.

Any effective correspondence of bidegree  $(d_1, d_2)$  gives an algebraic map

$$(2.1) \quad D : C \rightarrow C^{(d_1)}, \quad x \mapsto D(x) = p_{2*} p_1^*(x).$$

Here, as usual,  $C^{(d_1)}$  denotes the  $d_1$ -th symmetric power of  $C$ . Since any divisor on  $C \times C$  can be written in a unique way as the difference of 2 effective divisors, it is clear how to extend this definition to any correspondence. Often correspondences are defined via such maps. If we define

$$D \left( \sum_k a_k x_k \right) := \sum_k a_k D(x_k),$$

for a correspondence  $D$  and any divisor  $\sum_k a_k x_k$  of  $C$ , it is clear how to define the  $n$ -fold power  $D^n$  of  $D$  as a group homomorphism from the group of divisors  $\text{Div } C$  to itself. Hence  $\mathbb{Z}[D] \subset \text{End}(\text{Div } C)$  is a  $\mathbb{Z}$ -module.

**Definition 2.1.** Assume that  $\mathbb{Z}[D] \subset \text{End}(\text{Div } C)$  is finitely generated. Let  $\varphi$  be the generator of the ideal of polynomials of  $\mathbb{Q}[X]$  vanishing on  $D$  such that

- (1)  $\varphi$  has integer coefficients and
- (2) the leading coefficient of  $\varphi$  is positive and minimal among the positive leading coefficients of the generators with integer coefficients.

We call  $\varphi$  the minimal polynomial of  $D$ .

We say  $\varphi$  is *completely decomposable* if all its zeros are integers, i.e., if

$$\varphi = a \prod_{i=1}^k (X - a_i) \quad \text{with } a, a_i \in \mathbb{Z} \quad \text{for all } i.$$

Two correspondences  $D_1$  and  $D_2$  are equivalent if there exist line bundles  $L$  and  $M$  on  $C$  such that (see [BL04, Section 11.5])

$$\mathcal{O}_{C^2}(D_1) \cong \mathcal{O}_{C^2}(D_2) \otimes p_1^* L \otimes p_2^* M.$$

The set of equivalence classes of correspondences is a ring (see [BL04, Exercise 11.12.14]) which, according to [BL04, Theorem 11.5.1], is isomorphic to the endomorphism ring  $\text{End}(JC)$  of the Jacobian of  $C$ . Hence, associating to  $D$  its equivalence class defines a natural map

$$\gamma : \mathbb{Z}[D] \longrightarrow \text{End}(JC).$$

We denote the image of  $D$  by  $\gamma_D$ .

**Definition 2.2.** The minimal polynomial  $\psi$  of  $\gamma_D$  is the generator of the ideal of polynomials of  $\mathbb{Q}[X]$  vanishing on  $\gamma_D$  such that

- (1)  $\psi$  has integer coefficients and
- (2) the leading coefficient of  $\psi$  is positive and minimal among the positive leading coefficients of the generators with integer coefficients.

Clearly  $\psi$  is a divisor of  $\varphi$ , hence also completely decomposable with pairwise different zeros. We may choose the indices in such a way that

$$\psi(D) = b \prod_{i=1}^{\ell} (D - a_i)$$

for  $\ell \leq k$  and  $b \mid a$ . For all  $i = 1, \dots, \ell$  we define  $A_i$  to be the component of the identity of the kernel of the endomorphism  $\gamma_D - a_i$ :

$$A_i := \ker(\gamma_D - a_i)^0 \subset JC.$$

**Proposition 2.3.** *Suppose the effective correspondence  $D$  of  $C$  admits a completely decomposable minimal polynomial. Then, with the notation above, the addition map*

$$\alpha : A_1 \times \cdots \times A_\ell \rightarrow JC$$

*is an isogeny.*

*Proof.* The map  $\alpha$  is an isogeny if and only if its differential is an isomorphism of tangent spaces at 0. Now, the tangent space of  $A_i$ , as a subspace of the tangent space of  $JC$  at 0, is the corresponding eigenspace of the differential of  $\gamma_D$  at 0. Since the  $a_i$  are pairwise distinct, the tangent spaces of the  $A_i$  give a decomposition of the tangent space of  $JC$ . So the differential of  $\alpha$  is given by the eigenspace decomposition of the tangent space and is thus an isomorphism.  $\square$

**Remark 2.4.** Whereas a correspondence  $D$  of  $C$  does not necessarily admit a minimal polynomial equation, its image  $\gamma_D \in \text{End}(JC)$  does. Hence an analogous result to Proposition 2.3 is valid in greater generality, expressing it for endomorphisms instead of correspondences. We chose the above form, since this is exactly what we need for our applications.

### 3. ÉTALE COVERS OF CURVES WITH AUTOMORPHISMS

3.1. Let  $X$  be a smooth curve of genus  $g$  with an étale cover

$$f : Y \longrightarrow X,$$

of degree  $n \geq 2$ . For an automorphism  $\sigma$  of  $X$ , denote

$$Y_\sigma := Y, \quad \text{but with the covering} \quad f_\sigma := \sigma \circ f : Y \longrightarrow X.$$

Given a set  $\{\sigma_1, \dots, \sigma_\ell\}$  of automorphisms of  $X$ , let  $C_{\sigma_1, \dots, \sigma_\ell}$  denote the fiber product:

$$(3.1) \quad C_{\sigma_1, \dots, \sigma_\ell} := Y_{\sigma_1} \times_X \cdots \times_X Y_{\sigma_\ell}.$$

Note that the automorphisms are allowed to be the identity and need not be distinct. If we denote the fibre of  $f$  over  $x \in X$  by

$$f^{-1}(x) = \{x_1, \dots, x_n\},$$

then, as a set,

$$C_{\sigma_1, \dots, \sigma_\ell} = \{((\sigma_1 x)_{j_1}, \dots, (\sigma_\ell x)_{j_\ell}) \mid 1 \leq j_i \leq n\} \subset Y^\ell.$$

**Remark 3.1.** Since  $f$  is étale, so are  $\sigma_i f$  and the projections  $q_i : C_{\sigma_1, \dots, \sigma_\ell} \rightarrow Y_{\sigma_i}$  for  $i = 1, \dots, \ell$ . Hence  $C_{\sigma_1, \dots, \sigma_\ell}$  is smooth. In general it is not irreducible. In Section 4 we show that  $C$  is irreducible if and only if the monodromy group of the cover  $f : Y \rightarrow X$  is  $\ell$ -transitive.

For  $((\sigma_1 x)_{j_1}, \dots, (\sigma_\ell x)_{j_\ell}) \in C_{\sigma_1, \dots, \sigma_\ell}$ , define

$$(3.2) \quad D((\sigma_1 x)_{j_1}, \dots, (\sigma_\ell x)_{j_\ell}) := \sum_{k \neq j_1} ((\sigma_1 x)_k, \dots, (\sigma_\ell x)_{j_\ell}) + \cdots + \sum_{k \neq j_\ell} ((\sigma_1 x)_{j_1}, \dots, (\sigma_\ell x)_k).$$

**Lemma 3.2.**  *$D$  is a fixed-point free effective symmetric correspondence on  $C_{\sigma_1, \dots, \sigma_\ell}$  of degree  $\ell(n - 1)$ .*

*Proof.* As a curve in  $C \times C$ , we have

$$D := \{((\sigma_1 x)_{j_1}, \dots, (\sigma_\ell x)_{j_\ell}), ((\sigma_1 x)_{k_1}, \dots, (\sigma_\ell x)_{k_\ell}) \in C_{\sigma_1, \dots, \sigma_\ell} \times C_{\sigma_1, \dots, \sigma_\ell} \mid k_i = j_i \text{ except for exactly one index } i\}.$$

This description shows that  $D$  is an effective symmetric correspondence of degree  $\ell(n-1)$  of  $C_{\sigma_1, \dots, \sigma_\ell}$ . Since  $f : Y \rightarrow X$  is étale, the correspondence  $D$  is fixed-point free.  $\square$

3.2. From now on, to alleviate the notation, we write

$$C := C_{\sigma_1, \dots, \sigma_\ell}.$$

Also, for each subset  $J = \{i_1, \dots, i_r\} \subset \{1, \dots, \ell\}$ , we put  $C_J := C_{\sigma_{i_1}, \dots, \sigma_{i_r}}$  and, for any two subsets  $I, J$  such that  $I \subset J$ , we let

$$\pi_J : C \twoheadrightarrow C_J, \quad \pi_{I,J} : C_J \twoheadrightarrow C_I$$

be the natural projections. For  $J = \emptyset$ , we have  $C_J = C_\emptyset = X$  and  $\pi_\emptyset : C \rightarrow X$  is the unique map to  $X$ . For  $J = \{i\}$ , we have  $C_J = Y_{\sigma_i}$  and  $\pi_J = \pi_{\{i\}} : C \rightarrow Y_{\sigma_i}$  is the  $i$ -th projection of the fiber product. Note that the correspondence  $D$  induces endomorphisms of all the subquotients  $\pi_J^* JC_J / \sum_{I \subset J} \pi_I^* JC_I$  of  $JC$ . We have

**Theorem 3.3.** *For each  $r \in \{0, 1, \dots, \ell\}$  and each subset  $J = \{i_1, \dots, i_r\} \subset \{1, \dots, \ell\}$  of cardinality  $r$ , the endomorphism induced by the correspondence  $D$  on the subquotient  $\pi_J^* JC_J / \sum_{I \subset J} \pi_I^* JC_I$  is equal to multiplication by  $n(\ell - r) - \ell$ .*

*Proof.* Without loss of generality, we may assume that  $J = \{1, \dots, r\}$ . Consider a general point of  $C_J$ . We can choose the indices in such a way that this is  $((\sigma_1 x)_1, \dots, (\sigma_r x)_1)$ . We have

$$\pi_J^* ((\sigma_1 x)_1, \dots, (\sigma_r x)_1) = \sum_{1 \leq i_{r+1}, \dots, i_\ell \leq n} ((\sigma_1 x)_1, \dots, (\sigma_r x)_1, (\sigma_{r+1} x)_{i_{r+1}}, \dots, (\sigma_\ell x)_{i_\ell})$$

and

$$\begin{aligned} D\pi_J^* ((\sigma_1 x)_1, \dots, (\sigma_r x)_1) = & \sum_{1 \leq i_{r+1}, \dots, i_\ell \leq n} \left( \sum_{2 \leq k \leq n} ((\sigma_1 x)_k, \dots, (\sigma_r x)_1, (\sigma_{r+1} x)_{i_{r+1}}, \dots, (\sigma_\ell x)_{i_\ell}) + \dots \right. \\ & \dots + \sum_{2 \leq k \leq n} ((\sigma_1 x)_1, \dots, (\sigma_r x)_k, (\sigma_{r+1} x)_{i_{r+1}}, \dots, (\sigma_\ell x)_{i_\ell}) \\ & + \sum_{1 \leq k \leq n, k \neq i_{r+1}} ((\sigma_1 x)_1, \dots, (\sigma_r x)_1, (\sigma_{r+1} x)_k, \dots, (\sigma_\ell x)_{i_\ell}) + \dots \\ & \left. \dots + \sum_{1 \leq k \leq n, k \neq i_\ell} ((\sigma_1 x)_1, \dots, (\sigma_r x)_1, (\sigma_{r+1} x)_{i_{r+1}}, \dots, (\sigma_\ell x)_k) \right), \end{aligned}$$

hence

$$\begin{aligned}
(D+r)\pi_J^*((\sigma_1x)_1, \dots, (\sigma_r x)_1) &= \sum_{1 \leq k, i_{r+1}, \dots, i_\ell \leq n} \left( ((\sigma_1x)_k, \dots, (\sigma_r x)_1, (\sigma_{r+1}x)_{i_{r+1}}, \dots, (\sigma_\ell x)_{i_\ell}) + \dots \right. \\
&\quad \left. \dots + ((\sigma_1x)_1, \dots, (\sigma_r x)_k, (\sigma_{r+1}x)_{i_{r+1}}, \dots, (\sigma_\ell x)_{i_\ell}) \right) \\
&+ \sum_{1 \leq i_{r+1}, \dots, i_\ell \leq n} \left( \sum_{1 \leq k \leq n, k \neq i_{r+1}} ((\sigma_1x)_1, \dots, (\sigma_r x)_1, (\sigma_{r+1}x)_k, \dots, (\sigma_\ell x)_{i_\ell}) + \dots \right. \\
&\quad \left. \dots + \sum_{1 \leq k \leq n, k \neq i_\ell} ((\sigma_1x)_1, \dots, (\sigma_r x)_1, (\sigma_{r+1}x)_{i_{r+1}}, \dots, (\sigma_\ell x)_k) \right).
\end{aligned}$$

Now note that

$$\begin{aligned}
&\sum_{1 \leq k, i_{r+1}, \dots, i_\ell \leq n} \left( ((\sigma_1x)_k, \dots, (\sigma_r x)_1, (\sigma_{r+1}x)_{i_{r+1}}, \dots, (\sigma_\ell x)_{i_\ell}) + \dots \right. \\
&\quad \left. \dots + ((\sigma_1x)_1, \dots, (\sigma_r x)_k, (\sigma_{r+1}x)_{i_{r+1}}, \dots, (\sigma_\ell x)_{i_\ell}) \right)
\end{aligned}$$

belongs to  $\sum_{I \subset J} \pi_I^* \text{Pic } C_I$ , while

$$\begin{aligned}
&\sum_{1 \leq i_{r+1}, \dots, i_\ell \leq n} \left( \sum_{1 \leq k \leq n, k \neq i_{r+1}} ((\sigma_1x)_1, \dots, (\sigma_r x)_1, (\sigma_{r+1}x)_k, \dots, (\sigma_\ell x)_{i_\ell}) + \dots \right. \\
&\quad \left. \dots + \sum_{1 \leq k \leq n, k \neq i_\ell} ((\sigma_1x)_1, \dots, (\sigma_r x)_1, (\sigma_{r+1}x)_{i_{r+1}}, \dots, (\sigma_\ell x)_k) \right) \\
&= (n-1)(l-r)\pi_J^*((\sigma_1x)_1, \dots, (\sigma_r x)_1).
\end{aligned}$$

□

**Corollary 3.4.** *D satisfies the following completely decomposable equation*

$$(3.3) \quad \prod_{r=0}^{\ell} (D - nr + \ell) = 0$$

where we identify any integer  $r$  with  $r$  times the identity correspondence (i.e.,  $r$  times the diagonal). Furthermore, for each  $r$ , the eigen-abelian subvariety  $A_{n(\ell-r)-\ell}$  of  $JC$  for the eigenvalue  $n(\ell-r) - \ell$  is contained in  $\sum_{\#J=r} \pi_J^* JC_J$  and its projection to the quotient  $\sum_{\#J=r} \pi_J^* JC_J / \sum_{\#I \leq r-1} \pi_I^* JC_I$  is an isogeny.

In particular,

- (1) for  $r = 0$ , we have  $A_{n\ell-\ell} = \pi_{\emptyset}^* JX$ ,
- (2)  $A_{n(\ell-1)-\ell}$  is isogenous to

$$\left( \sum_{i=1}^{\ell} \pi_{\{i\}}^* JY \right) / \pi_{\emptyset}^* JX,$$

- (3) and

$$\sum_{r=1}^{\ell} A_{n(\ell-r)-\ell}$$

is the Prym variety of the cover  $\pi_\emptyset : C \rightarrow X$ .

3.3. It follows in particular that  $A_{-\ell}$  is the “new” part of  $JC$ , meaning the part that is complementary to the images of Jacobians of curves  $C_{\sigma_{i_1}, \dots, \sigma_{i_r}}$  for  $r < \ell$  which, by the above, is equal to  $\sum_{r=0}^{\ell-1} A_{n(\ell-r)-\ell}$ . Let  $A'_{-\ell}$  be the quotient of  $JC$  by the abelian subvariety  $\sum_{r=0}^{\ell-1} A_{n(\ell-r)-\ell}$ . Let  $\mu : A_{-\ell} \hookrightarrow JC \twoheadrightarrow A'_{-\ell}$  be the restriction of the polarization of  $JC$  to  $A_{-\ell}$  and let  $m$  be a positive integer such that the kernel of  $\mu$  is contained in the  $m$ -torsion subgroup  $A_{-\ell}[m]$  of  $A_{-\ell}$ . Choose a subgroup  $K$  of  $A'_{-\ell}[m]$ , maximal isotropic with the respect to the Riemann form of  $\mu$ . Then the polarization  $\mu$  induces a principal polarization  $\Xi$  on the quotient  $B := A'_{-\ell}/K$ . By [Wel87, Proposition 1.7], the image of an Abel embedding of  $C$  in  $B$  has cohomology class  $m \frac{\Xi^{d-1}}{(d-1)!}$  where  $d$  is the dimension of  $B$ . Note that the intersection  $A_{-\ell} \cap \sum_{r=1}^{\ell} A_{nr-\ell}$  is contained in  $A_{-\ell}[m]$  (see, e.g. [Wel87, p. 88]). We have

**Lemma 3.5.** *We have*

$$A_{-\ell} \cap \sum_{r=1}^{\ell} A_{nr-\ell} \subset A_{-\ell}[\ell!n^\ell].$$

In particular, we can choose  $B$  so that the image of an Abel embedding of  $C$  in  $B$  has cohomology class  $\ell!n^\ell \frac{\Xi^{d-1}}{(d-1)!}$

*Proof.* Suppose that  $x \in A_{-\ell} \cap \sum_{r=1}^{\ell} A_{nr-\ell}$ . So we can write

$$x = \sum_{r=1}^{\ell} x_r$$

where  $x_r \in A_{nr-\ell}$ . Successively applying the endomorphisms  $D + \ell - rn$  for  $r = 0, \dots, \ell$  gives the series of equations

$$\begin{array}{rcccc} 0 = nx_1 & +2nx_2 & +3nx_3 + \dots & +(\ell-1)nx_{\ell-1} & +\ell nx_\ell \\ 0 = & 1 \cdot 2n^2x_2 & +2 \cdot 3n^2x_3 + \dots & +(\ell-2)(\ell-1)n^2x_{\ell-1} & +(\ell-1)\ell n^2x_\ell \\ \vdots & & & & \\ 0 = & & & 1 \cdot \dots \cdot (\ell-1)n^{\ell-1}x_{\ell-1} & +2 \cdot \dots \cdot \ell n^{\ell-1}x_\ell \\ 0 = & & & & 1 \cdot \dots \cdot \ell n^\ell x_\ell \end{array}$$

So we obtain  $\ell!n^\ell x_\ell = (\ell-1)!n^\ell x_{\ell-1} = 0$ . Going from bottom to top and multiplying the  $r$ -th equation by  $(\ell-r)!$ , we obtain

$$r!(\ell-r)!n^\ell x_r = 0$$

for all  $r = 1, \dots, \ell$ . In particular,

$$\ell!n^\ell x_r = \ell!n^\ell x = 0$$

for all  $r = 1, \dots, \ell$ . □

**Hypothesis 3.6.** From now on in this section, we assume that the curve  $C_{\sigma_1, \dots, \sigma_\ell}$  is irreducible.

3.4. For any integer  $r$ , put

$$g_r := n^r(g - 1) + 1.$$

Since  $C$  is an étale cover of degree  $n^\ell$  of  $X$ , it has genus  $g_\ell$ . For  $r = 0, \dots, \ell$ , denote

$$d_r := \dim A_{rn-\ell},$$

the dimension of the eigen-abelian subvariety of  $JC$  for the eigenvalue  $rn - \ell$ . In particular,

$$d = d_0 = \dim A_{-\ell}, \quad d_\ell = g, \quad \text{and} \quad \sum_{r=0}^{\ell} d_r = g_\ell.$$

By [BL04, Prop. 11.5.2 p. 334] and Lemma 3.2, we have the trace formula

$$\sum_{r=0}^{\ell} (nr - \ell) d_r = l(n - 1).$$

Simple manipulations give the following relations

$$(3.4) \quad d_0 + \dots + d_{\ell-1} = g_\ell - g = (n^\ell - 1)(g - 1),$$

$$(3.5) \quad d_1 + 2d_2 + \dots + (\ell - 1)d_{\ell-1} = \ell(g_{\ell-1} - g) = \ell(n^{\ell-1} - 1)(g - 1).$$

When  $\ell = 2$ , we immediately compute

$$(3.6) \quad d_0 = (n - 1)^2(g - 1), \quad d_1 = 2(n - 1)(g - 1).$$

Theorem 3.3 also shows that for each  $r \in \{0, \dots, \ell\}$  and each subset  $J = \{i_1, \dots, i_{\ell-r}\} \subset \{1, \dots, \ell\}$  of cardinality  $\ell - r$ , the eigen-abelian subvariety of  $JC_J$  for the eigenvalue  $-(\ell - r)$  (for the correspondence on  $C_J$ ) is exactly the part of  $JC_J$  that maps into the eigen-abelian subvariety of  $JC$  for the eigenvalue  $nr - \ell$ . Therefore, if we denote  $d_s(C_J)$  the dimension of the eigen-abelian subvariety of  $JC_J$  for the eigenvalue  $ns - (\ell - r)$ , we have

$$d_r \leq \sum_{\#J=\ell-r} d_0(C_J).$$

**Lemma 3.7.** *For all  $\ell \geq 2$ , all  $r \in \{0, \dots, \ell\}$ , and for all subsets  $J \subset \{1, \dots, \ell\}$  of cardinality  $\ell - r$ , the dimension  $d_s(C_J)$  only depends on  $r$ . Writing  ${}_{\ell-r}d_s := d_s(C_J)$  for some (or any)  $J$ , we have, for all  $r \in \{0, \dots, \ell - 1\}$ ,*

$$d_r = {}_{\ell}d_r = \binom{\ell}{r} (n - 1)^{\ell-r} (g - 1).$$

*In particular,  ${}_{\ell}d_r = \binom{\ell}{r} {}_{\ell-r}d_0$  and  $A_{nr-\ell}$  is isogenous to the product of the eigen-abelian subvarieties of  $JC_J$  for the eigenvalue  $-(\ell - r)$ , where  $J$  runs over all subsets of cardinality  $\ell - r$  of  $\{1, \dots, \ell\}$ .*

*Proof.* We proceed by induction. We first note that the first step of the induction, for  $\ell = 2$ , was done in (3.6). Assume now  $\ell \geq 3$  and that the statement holds for all  $k \leq \ell - 1$ . As we saw above, for  $r \in \{0, \dots, \ell\}$ ,

$$d_r = {}_{\ell}d_r \leq \sum_{\#J=\ell-r} d_0(C_J).$$



Hence, for  $r \in \{1, \dots, \ell - 1\}$ , applying the induction hypothesis to the Jacobians of the curves  $C_J$ , for all  $J$  of cardinality  $r$ , we have  $d_0(C_J) = \ell_{-r}d_0 = (n - 1)^{\ell-r}(g - 1)$ , and

$$\ell d_r \leq \binom{\ell}{r} (n - 1)^{\ell-r}(g - 1).$$

Now we compute

$$\sum_{r=1}^{\ell-1} r \binom{\ell}{r} (n - 1)^{\ell-r}(g - 1) = \ell(g - 1) \sum_{r=1}^{\ell-1} \binom{\ell-1}{r-1} (n - 1)^{\ell-1} = \ell(n^{\ell-1} - 1)(g - 1),$$

which is equal to the sum  $\ell d_1 + 2\ell d_2 + \dots + (\ell - 1)\ell d_{\ell-1}$  by (3.5). Since, in addition, all the  $\ell d_r$  and  $\binom{\ell}{r}(n - 1)^{\ell-r}(g - 1)$  are positive integers and  $\ell d_r \leq \binom{\ell}{r}(n - 1)^{\ell-r}(g - 1)$ , we conclude

$$\ell d_r = \binom{\ell}{r} (n - 1)^{\ell-r}(g - 1)$$

for  $r \in \{1, \dots, \ell - 1\}$ . Now we compute

$$\ell d_0 = (n - 1)^\ell (g - 1)$$

using (3.4). The remainder of the lemma now easily follows.  $\square$

3.5. Note, in particular, that each abelian subvariety  $A_r$  of  $JC$  is isogenous to the product of at least  $\binom{\ell}{r}$  abelian subvarieties. Hence  $JC$  is isogenous to the product of at least  $\sum_{r=0}^{\ell} \binom{\ell}{r} = 2^\ell$  abelian subvarieties.

3.6. For  $n = 2$ , the dimension of all of the isogeny components of  $JC$  obtained from the eigen-abelian subvariety of  $JC_J$  for the eigenvalue  $-(\ell - r)$  (for all  $J$ ) is always  $g - 1$ . One may ask whether these are all isogenous to the Prym variety of the cover  $f : Y \rightarrow X$ .

3.7. For  $\ell = 1$ , the curve  $C = C_{\sigma_1}$  coincides with the étale cover  $Y_{\sigma_1} = Y \xrightarrow{\sigma_1 f} X$  and the correspondence  $D$  is the usual correspondence

$$D = \{(p, q) \mid p \neq q, f(p) = f(q)\} \subset Y^2,$$

residual to the diagonal of  $Y^2$  in the inverse image of the diagonal of  $X^2$ . It satisfies the equation

$$(D + 1)(D - n + 1) = 0$$

and the eigen-abelian subvariety  $A_{-1}$  is the usual Prym variety of the cover  $\sigma_1 f : Y \rightarrow X$ . As we saw in Corollary 3.4,  $A_{n-1} = \pi_{\emptyset}^* JX$ .

3.8. For  $\ell = 2$ , we have

$$C = \{((\sigma_1 x)_i, (\sigma_2 x)_j) \mid i, j = 1, \dots, n\} \subset Y^2$$

and the equation

$$(D + 2)(D - n + 2)(D - 2n + 2) = 0.$$

The abelian subvariety  $A_{-2} + A_{n-2} \subset JC$  is the Prym variety of  $\pi_{\emptyset} : C \rightarrow X$ , and  $A_{2n-2} = \pi_{\emptyset}^* JX$ .

3.9. If  $X$  is general, it has no non-trivial automorphisms, but we can still construct the curve  $C$  for any étale cover  $Y$  and  $\sigma_1 = \dots = \sigma_\ell = Id$ , the identity. In this case each eigen-abelian subvariety  $A_r$  is isogenous to the  $\binom{\ell}{r}$ -th power of an abelian subvariety and  $JC$  has many repeated factors.

#### 4. IRREDUCIBILITY OF THE CURVES $C_{\sigma_1, \dots, \sigma_\ell}$

We investigate the irreducibility of the curve  $C = C_{\sigma_1, \dots, \sigma_\ell}$ . We start with

**Lemma 4.1.** *The curve  $C$  is irreducible if and only if the monodromy group of the cover  $f : Y \rightarrow X$  is  $\ell$ -transitive.*

*Proof.* The datum of a path  $\gamma$  in  $C$  starting at a point  $((\sigma_1 x)_{i_1}, \dots, (\sigma_\ell x)_{i_\ell})$  is equivalent to the data of paths  $\beta_j$  in  $Y$  starting at  $(\sigma_j x)_{i_j}$  such that  $\sigma_1 f(\beta_1) = \dots = \sigma_\ell f(\beta_\ell)$ . Therefore, given a loop  $\alpha$  in  $X$  based at a point  $x$ , lifting it to a path in  $C$  means lifting it to paths  $\beta_j$  starting at lifts  $(\sigma_j x)_{i_j}$  of  $\sigma_j x$  such that  $\alpha = \sigma_1 f(\beta_1) = \dots = \sigma_\ell f(\beta_\ell)$ . Therefore, denoting the permutation associated to a loop by the same symbol, the action of  $\alpha$  on the fiber of  $\pi_\varnothing : C \rightarrow X$  at  $x$  is

$$\alpha(((\sigma_1 x)_{i_1}, \dots, (\sigma_\ell x)_{i_\ell})) = (\sigma_1 \circ \alpha((\sigma_1 x)_{i_1}), \dots, \sigma_\ell \circ \alpha((\sigma_\ell x)_{i_\ell})).$$

Hence, identifying the fundamental groups  $\pi_1(X, \sigma_1 x), \dots, \pi_1(X, \sigma_\ell x)$  via the maps  $\alpha \mapsto \sigma_j \sigma_1^{-1} \circ \alpha$ , we conclude that the permutation action of the fundamental group of  $X$  on the fiber of  $\pi_\varnothing$  at  $x$  is transitive if and only if the action of the fundamental group of  $X$  on the fiber of  $f$  at  $x$  is  $\ell$ -transitive.  $\square$

Note that the above result shows that the irreducibility of  $C$  only depends on the cover  $f : Y \rightarrow X$ .

Clearly,  $\mathfrak{S}_n$  itself is  $n$ -transitive.

Multiply transitive groups fall into six infinite families and four classes of sporadic groups. They are all primitive, i.e., they do not preserve any partitions of  $\{1, \dots, n\}$ .

Below  $q$  is a power of a prime number. We refer to [DM96] for the following.

- For  $\ell \geq 6$ , the only  $\ell$ -transitive groups are the symmetric and alternating groups on  $\ell$  and  $\ell + 2$  letters respectively.
- The solvable 2-transitive groups are subgroups of the group of affine transformations  $AGL_r(p)$  for some integer  $r$  and prime number  $p$  which contain all translations. The list of these groups can be found in the Appendix to [Lie87] and is attributed to Hering [Her85].
- The insoluble 2-transitive groups are almost simple, i.e., they contain a non-abelian simple group and are contained in the automorphism group of that simple group. The insoluble 2-transitive groups can be found in [CKS76] and [PS97].
- The projective special linear groups  $PSL(d, q)$  are 2-transitive except for the special cases  $PSL(2, q)$  with  $q$  even, which are 3-transitive.
- The symplectic groups defined over the field of two elements have two distinct actions which are 2-transitive.
- The field  $K$  of  $q^2$  elements has an involution  $\sigma(a) = a^q$ , which allows a Hermitian form to be defined on a vector space on  $K$ . The unitary group on  $V := K^{\oplus 3}$ , denoted  $U_3(q)$ , preserves the isotropic vectors in  $V$ . The action of the projective special unitary group  $PSU(q)$  is 2-transitive on the isotropic vectors.

- The Suzuki group of Lie type  $Sz(q)$  is the automorphism group of an  $S(3, q+1, q^2+1)$  Steiner system, an inversive plane of order  $q$ , and its action is 2-transitive.
- The Ree group of Lie type  $R(q)$  is the automorphism group of an  $S(2, q+1, q^3+1)$  Steiner system, a unital of order  $q$ , and its action is 2-transitive.
- The Mathieu groups  $M_{12}$  and  $M_{24}$  are the only 5-transitive groups besides  $S_5$  and  $A_7$ . The groups  $M_{11}$  and  $M_{23}$  are 4-transitive, and  $M_{22}$  is 3-transitive.
- The projective special linear group  $PSL(2, 11)$  has another 2-transitive action related to the Witt geometry  $W_{11}$ .
- The Higman-Sims group  $HS$  is 2-transitive.
- The Conway group  $Co_3$  is 2-transitive.
- Other 3-transitive groups include  $PSL(2, 7) : 2$  acting on 8 items, as generated by the permutations  $(a, b, c, d)(e, f, g, h)$ ,  $(a, f, c)(d, e, g)$ , and  $(e, f)(d, h)(b, c)$ ; and  $PSL(2, 11) : 2$  acting on 12 items, as generated by the permutations  $(g, b, c, i, d)(j, e, h, f, l)$ ,  $(a, b, c)(d, e, f)(g, h, i)(j, k, l)$ , and  $(a, i)(d, g)(e, j)(h, k)(c, f)$ .

Choosing covers with the above monodromy groups will produce many examples of curves  $C$  that are irreducible.

#### REFERENCES

- [BL04] C. Birkenhake and H. Lange, *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer, Berlin, 2004.
- [CLRR09] Angel Carocca, Herbert Lange, Rubí E. Rodríguez, and Anita M. Rojas, *Prym-Tyurin varieties using self-products of groups*, J. Algebra **322** (2009), no. 4, 1251–1272.
- [CKS76] Charles W. Curtis, William M. Kantor, and Gary M. Seitz, *The 2-transitive permutation representations of the finite Chevalley groups*, Trans. Amer. Math. Soc. **218** (1976), 1–59.
- [DM96] John D. Dixon and Brian Mortimer, *Permutation groups*, Graduate texts in Math., vol. 163, Springer, Berlin, 1996.
- [Her85] Christoph Hering, *Transitive linear groups and linear groups which contain irreducible subgroups of prime order, II*, J. of Algebra **93** (1985), 151–164.
- [Lie87] Martin W. Liebeck, *The affine permutation groups of rank 3*, Proc. London Math. **54** (1987), no. 3, 477–516.
- [PS97] Cheryl E. Praeger and Leonard H. Soicher, *Low rank representations and graphs for sporadic groups*, Cambridge University Press, Cambridge, UK, 1997.
- [Wel87] G. E. Welters, *Curves of twice the minimal class on principally polarized abelian varieties*, Nederl. Akad. Wetensch. Indag. Math. **49** (1987), no. 1, 87–109.

E. IZADI, DEPARTMENT OF MATHEMATICS, UC SAN DIEGO, USA  
*E-mail address:* eizadi@math.ucsd.edu

H. LANGE, DEPARTMENT MATHEMATIK DER UNIVERSITÄT ERLANGEN, GERMANY  
*E-mail address:* lange@mi.uni-erlangen.de