THE IRREDUCIBILITY OF THE PRIMAL COHOMOLOGY OF THE THETA DIVISOR OF AN ABELIAN FIVEFOLD

ELHAM IZADI AND JIE WANG

Abstract. We prove that the primal cohomology of the theta divisor of a very general principally polarized abelian fivefold is an irreducible Hodge structure of level 2.

Contents

Introduction 1
1. Prym varieties associated to a Lefschetz pencil 3
2. Numerical calculations 5
3. General facts about the Clemens-Schmid exact sequence 11
4. Local monodromy representations near $N_0$ 13
5. Local monodromy near the boundary $\Delta$ 15
6. Global monodromy 18
References 21

Introduction

Let $A$ be a principally polarized abelian variety of dimension $g \geq 4$ with smooth theta divisor $\Theta$. By the Lefschetz hyperplane theorem and Poincaré Duality (see, e.g., [IW15]) the cohomology of $\Theta$ is determined by that of $A$ except in the middle dimension $g - 1$. The primitive cohomology of $\Theta$, in the sense of Lefschetz, is

$$H_{pr}^{g-1}(\Theta) := \text{Ker} \left( H^{g-1}(\Theta, \mathbb{Z}) \xrightarrow{\cup \theta} H^{g+1}(\Theta, \mathbb{Z}) \right).$$

The primal cohomology of $\Theta$ is defined as (see [IW15] and [ITW])

$$K := \text{Ker}(j_* : H^{g-1}(\Theta, \mathbb{Z}) \rightarrow H^{g+1}(A, \mathbb{Z}))$$

where $j : \Theta \rightarrow A$ is the inclusion. This is a Hodge substructure of $H_{pr}^{g-1}(\Theta, \mathbb{Z})$ of rank $g! - \frac{1}{g+1} \binom{2g}{g}$ and level $g - 3$ while the primitive cohomology $H_{pr}^{g-1}(\Theta, \mathbb{Z})$ has full level $g - 1$.
The primal cohomology is therefore a good test case for the general Hodge conjecture. The general Hodge conjecture predicts that $\mathbb{K}_\mathbb{Q} := \mathbb{K} \otimes \mathbb{Q}$ is contained in the image, via Gysin push-forward, of the cohomology of a smooth (possibly reducible) variety of pure dimension $g - 3$ (see [IW15]). This conjecture was proved in [IS95] and [ITW] in the cases $g = 4$ and $g = 5$. When $g = 4$, it also follows from the proof of the Hodge conjecture in [IS95] that for $(A, \Theta)$ generic, $\mathbb{K}$ is an irreducible Hodge structure (isogenous to the third cohomology of a smooth cubic threefold).

When $g = 5$, the cohomology of the variety whose cohomology contains $\mathbb{K}$ is no longer irreducible and the irreducibility of $\mathbb{K}$ no longer follows from the proof of the Hodge conjecture.

Our main result is the somewhat unexpected (see [KW, 2.9])

**Theorem 0.1.** For a very general ppav $A$ of dimension 5 with smooth theta divisor $\Theta$. The primal cohomology $\mathbb{K}$ of $\Theta$ is an irreducible Hodge structure of level 2.

As explained in [IW15], the above theorem considerably simplifies the proof of the Hodge conjecture in [ITW]: it is no longer necessary to show that the image of the Abel-Jacobi map in [ITW] contains all of $\mathbb{K}$, only that it intersects $\mathbb{K}$ non-trivially.

If $A$ is replaced by a projective space and $\Theta$ by a smooth hypersurface, then the primitive and the primal cohomology coincide. The primitive cohomology of a general hypersurface is irreducible (see, e.g., [Lam81, 7.3]).

Our strategy, explained below, for proving Theorem 0.1 is to use the Mori-Mukai proof [MM83] of the unirationality of $A_5$.

Let $T$ be an Enriques surface and

\[ f : S \to T \]

the K3 étale double cover corresponding to the canonical class (which is 2-torsion) $K_T \in \text{Pic}(T)$. Let $H$ be a very ample line bundle on $T$ with $H^2 = 10$. A general element in the linear system $|H| \cong \mathbb{P}^5$ is a smooth curve of genus 6 and such smooth curves are parametrized by the Zariski open subset $|H| \setminus D$, where $D$ is the dual variety of the embedding of $T$ in $|H|^*$. For each element $u \in |H| \setminus D$, we obtain a nontrivial étale double cover $D_u := f^{-1}(C_u) \to C_u$. Associating to such a cover its Prym variety $P(D_u, C_u)$ defines a morphism from $|H| \setminus D$ to $A_5$:

\[ |H| \setminus D \to A_5 \]

Mori and Mukai [MM83] showed that as we vary $(T, H)$ in moduli, the family of maps $P_H$ dominates $A_5$.

The ppav $(A, \Theta)$ with singular theta divisor form the Andreotti-Mayer divisor $N_0$ in $A_5$ ([Bea77]). The divisor $N_0$ has two irreducible components $\theta_{null}$ and $N_0'$ ([Deb92],[Mum83]) (as divisors, $N_0 = \theta_{null} + 2N_0'$). The theta divisor of a general point $(A, \Theta) \in \theta_{null}$ has a unique node at a two-torsion point while the theta divisor of a general point in $N_0'$ has two distinct nodes $x$ and $-x$. 
The primal cohomologies of the theta divisors form a variation of (polarized) Hodge structures over $U := |H| \setminus (D \cup \overline{\mathcal{P}^{-1}_H(N_0)})$. Inspired by [Lam81, 7.3], we prove Theorem 0.1 via a detailed study of the monodromy representation

$$\rho : \pi_1(U) \to Aut(K_Q, \langle , \rangle)$$

where $\langle , \rangle$ is the natural polarization on $K_Q$ induced by the intersection pairing on $H^4(\Theta, \mathbb{Q})$.

1. Prym varieties associated to a Lefschetz pencil

1.1. A pencil of double covers. We denote by

$$\tau : S \to S$$

the fixed point free covering involution such that $S/\tau \cong T$. By [Nam85, Prop. 2.3] the invariant subspace of the involution $\tau^*$ acting on the Néron Severi group $NS(S)$ is equal to $f^*(NS(T))$. Since the pullback

$$f^* : NS(T) \to NS(S)$$

is injective, we deduce that $f^*(NS(T))$ is a rank 10 primitive sublattice in $NS(S)$. It follows that the Picard number of $S$ is greater than or equal to 10. By [Nam85, Prop. 5.6], when $T$ is general in moduli,

$$(1.1) \quad NS(S) = f^*NS(T).$$

**Hypothesis:** Throughout this paper, we will assume $T$ satisfies (1.1).

Suppose $l \cong \mathbb{P}^1 \subset |H|$ is a Lefschetz pencil, i.e., it is transverse to the dual variety $\mathcal{D}$. Then the singular curves of the pencil consist of finitely many irreducible nodal curves. Denote by $\mathcal{T} := Bl_{10}T$ (resp. $\tilde{S} := Bl_{20}S$) the blow-up of $T$ (resp. $S$) along the base locus of $l$ (resp. $f^*l$). We obtain a family of étale double covers parametrized by $l$:

$$\begin{array}{ccc}
\tilde{S} & \xrightarrow{i} & \mathcal{T} \\
\downarrow \pi' & & \downarrow \pi \\
& l. & \\
\end{array}$$

**Proposition 1.1.** There are 42 singular fibers in the family $\mathcal{T} \xrightarrow{\pi} l$.

**Proof.** We use the formula

$$\chi_{top}(\mathcal{T}) = \chi_{top}(T) + 10 = \chi_{top}(\mathbb{P}^1)\chi_{top}(C) + N,$$

where $C$ is a smooth fiber in the pencil and $N$ is the number of singular fibers. We obtain $N = 42$. □

Denote by $C_t$ the fiber over $t \in l$ of $\pi$ and $D_t$ the corresponding étale double cover in $\tilde{S}$ and $\{s_i \in l : i = 1, ..., 42\}$ the 42 points where $\pi$ is singular.
Proposition 1.2. For any \( t \in l \), the étale double cover \( D_t \) of \( C_t \) is an irreducible curve.

Proof. Suppose \( D_t \) is reducible for some \( t \). If \( C_t \) is smooth, \( D_t \) must be the trivial cover. If \( C_t \) has one node, \( D_t \) is either the trivial cover or the Wirtinger cover. In either case, the involution \( \iota \) permutes the two components \( D^1_t \) and \( D^2_t \) of \( D_t \). By (1.1), the class of \( D^i_t \) in \( NS(S) \) is \( \iota \) invariant, thus \( D^1_t \) and \( D^2_t \) have the same class in \( NS(S) \) and \( H = 2D^1_t \). However, since \( H^2 = 10 \), the class of \( H \) in \( NS(T) \) is not 2-divisible, a contradiction. \( \square \)

Corollary 1.3. For a singular fiber \( C_{s_i} = C_{pq} := \{ p \sim q \} \) in the pencil \( l \), the étale double cover \( D_{s_i} := D_{pq} \) is obtained by glueing \( p_i \) with \( q_i \) for \( i = 1, 2 \) on a nontrivial étale double cover \( D \) of \( C \), where \( p_i, q_i \in D \) are the inverse images of \( p, q \in C \) respectively.

Proof. The étale double cover \( D_{pq} \) of \( C_{pq} \) is determined by a 2-torsion point in \( \text{Pic}^0(D_{pq}) \). The statement follows immediately from the irreducibility of \( D_{s_i} \) and the exact sequence

\[
1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Pic}^0(D_{pq}) \rightarrow \text{Pic}^0(C) \rightarrow 0 ,
\]

where \( \nu : C \rightarrow C_{pq} \) is the normalization map and the kernel of \( \nu^* \) is generated by the point of order 2 corresponding to the Wirtinger cover. \( \square \)

1.2. The compactified Prym variety. We describe the compactified Prym variety for the cover \( D_{pq} \rightarrow C_{pq} \) as in Corollary 1.3. The semiabelian part \( G_{pq} \) of the Prym variety is the identity component \( \text{Ker}^0(Nm_{pq}) \) of \( \text{Ker}(Nm_{pq}) \subset \text{Pic}^0(D_{pq}) \) in the following commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & \\
1 & \mathbb{C}^* & \rightarrow & \text{Ker}(Nm_{pq}) & \rightarrow & \text{Ker}(Nm) & \rightarrow & 0 & \\
1 & (\mathbb{C}^*)^2 & \rightarrow & \text{Pic}^0(D_{pq}) & \rightarrow & \text{Pic}^0(D) & \rightarrow & 0 & \\
1 & \mathbb{C}^* & \rightarrow & \text{Pic}^0(C_{pq}) & \rightarrow & \text{Pic}^0(C) & \rightarrow & 0 & \\
0 & 0 & 0 & 0 & .
\end{array}
\]

It follows immediately that the group scheme \( G_{pq} \) is a \( \mathbb{C}^* \)-extension of the Prym variety \( (B, \Xi) := \text{Prym}(D, C) \):

\[
1 \rightarrow \mathbb{C}^* \rightarrow G_{pq} \rightarrow B \rightarrow 0 .
\]

Let \( p : P^\nu \rightarrow B \) be the unique \( \mathbb{P}^1 \) bundle containing \( G_{pq} \) and write \( P^\nu \setminus G_{pq} = B_0 \amalg B_\infty \), where \( B_0 \) and \( B_\infty \) are the zero and infinity sections of \( P^\nu \).

The compactified ‘rank one degeneration’ \( P \) is constructed as follows (c.f. [Mum83, §1]).
(1) On $P^\nu$, we have the linear equivalence $B_0 - B_\infty \sim_{\text{lin}} p^{-1}(\Xi - \Xi_b)$ for a unique $b \in B$. Thus 
\[B_0 + p^{-1}\Xi_b \sim_{\text{lin}} B_\infty + p^{-1}\Xi.\]

(2) Let $L^\nu := \mathcal{O}_{P^\nu}(B_0 + p^{-1}\Xi_b)$. Then $L^\nu|_{B_0} \cong \mathcal{O}_B(\Xi)$ and $L^\nu|_{B_\infty} \cong \mathcal{O}_B(\Xi - \Xi_b)$. Via the Leray spectral sequence for $p$, we see that $h^0(P^\nu, L^\nu) = 2$ and $B_0 + p^{-1}\Xi_b, B_\infty + p^{-1}\Xi$ span $|L^\nu|$.

(3) The compactified Prym variety $P$ is constructed from $P^\nu$ by identifying the zero section $B_0 \overset{\sim}{\cong} B$ with the infinity section $B_\infty \overset{p}{\cong} B$ via translation by $b \in B$. We also denote by $\nu : P^\nu \to P$ the normalization morphism.

(4) The line bundle $L^\nu$ descends to a line bundle $L$ on $P$, i.e., $\nu^*L \cong L^\nu$. The linear system $|L|$ is a point.

(5) The theta divisor $\Upsilon \subset P$ is the unique divisor in $|L|$.

**Remark 1.4.** The $\mathbb{P}^1$ bundle $P^\nu \to B$ contains an open subset $P^\nu \setminus B_\infty$ (resp. $P^\nu \setminus B_0$), which is isomorphic to the total space of $N_{B_0/P^\nu} \cong \mathcal{O}_{B_0}(B_0) \cong \mathcal{O}_B(\Xi - \Xi_b)$ (resp. $\mathcal{O}_B(\Xi_b - \Xi)$). We conclude that $P^\nu \cong \mathbb{P}_B(\mathcal{O}_B(\Xi - \Xi_b) \oplus \mathcal{O}_B(\Xi_b - \Xi))$. In particular $G_{pq} \to B$ and $P^\nu \to B$ are topologically trivial $\mathbb{C}^*$ and $\mathbb{P}^1$ bundles, respectively.

**Proposition 1.5.** For a general rank one degeneration, the normalization $\Upsilon^\nu$ of the theta divisor is isomorphic to $Bl_{\Xi \cap \Xi_b}B \subset P^\nu$, the theta divisor $\Upsilon \subset P$ is obtained from $\Upsilon^\nu$ by identifying the proper transforms of $\Xi$ and $\Xi_b$.

**Proof.** Let $\sigma_0, \sigma_\infty$ be elements of $H^0(P^\nu, L^\nu)$, such that $\text{div}(\sigma_0) = B_0 + p^{-1}\Xi_b$ and $\text{div}(\sigma_\infty) = B_\infty + p^{-1}\Xi$. After rescaling, we may assume, under the natural identification $B_0 \overset{p}{\cong} B \overset{\sim}{\cong} B_\infty$, that $\sigma_0|_{B_\infty}$ and $\sigma_\infty|_{B_0}$ differ by translation by $b$. Then $\sigma_0 + \sigma_\infty$ descends to a section of $L$. Since $(\sigma_0 + \sigma_\infty)|_{B_0}$ vanishes precisely on $\Xi$ and $(\sigma_0 + \sigma_\infty)|_{B_\infty}$ vanishes precisely on $\Xi_b$, we conclude that for $u \in B \setminus (\Xi \cap \Xi_b)$, $0 \neq (\sigma_0 + \sigma_\infty)|_{p^{-1}(u)} \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. Thus $\Upsilon^\nu := \text{div}(\sigma_0 + \sigma_\infty)$ maps one-to-one to $B$ away from $\Xi \cap \Xi_b$. On the other hand, the base locus of the pencil $|L^\nu|$ is clearly $p^{-1}(\Xi \cap \Xi_b)$. Thus $\Upsilon^\nu = m[Bl_{\Xi \cap \Xi_b}B]$, for some integer $m$, as divisors in $P^\nu$. Since $(\sigma_0 + \sigma_\infty)|_{B_0}$ is reduced, $m = 1$. \(\square\)

2. **Numerical calculations**

The family of compactified Prym varieties defines a morphism $\rho : l \to \tilde{A}_5$, where $\tilde{A}_5$ is the partial compactification of $A_5$ parametrizing ppav $(A, \Theta)$ of dimension 5 and their rank 1 degenerations. This space is a quasi-projective variety and is essentially the blow-up of the open set $A_3 \Pi A_4$ in the Satake-Baily-Borel compactification $A^*_5$ along its boundary $A_4$ ([Igu67]). The coarse moduli space of $\tilde{A}_5$ is the union of $A_5$ and a divisor $\Delta$ parametrizing rank 1 degenerations. Mumford [Mum83] computed the class of the closure of $\theta_{null}$ and $N'_0$ in $\tilde{A}_5$ to be

\[(\theta_{null}) = 264\lambda - 32\delta,\]

\[(N'_0) = 108\lambda - 14\delta,\]
\[ [N_0] = [\theta_{null}] + 2[N'_0] = 480\lambda - 60\delta, \]

where \( \lambda \) is the first Chern class of the Hodge bundle \( \Lambda \) and \( \delta \) is the class of \( \Delta \).

**Lemma 2.1.** The degree of \( \rho^*\lambda \) is 6.

**Proof.** The pull-back of the Hodge bundle \( \Lambda \) to \( l \) fits in the exact sequence

\[
0 \rightarrow \pi_*\omega_{\tilde{T}/l} \rightarrow \pi'_*\omega_{\tilde{S}/l} \rightarrow \rho^*\Lambda \rightarrow 0,
\]

where \( \omega_{\tilde{T}/l} \) and \( \omega_{\tilde{S}/l} \) are the relative dualizing sheaves. Thus \( c_1(\rho^*\lambda) = c_1(\pi'_*\omega_{\tilde{S}/l}) - c_1(\pi_*\omega_{\tilde{T}/l}) \).

We directly compute that the relative dualizing sheaf \( \omega_{\tilde{T}/l} = K_{\tilde{T}} \otimes \pi^*K_t^{-1} \) has self intersection number \( (\omega_{\tilde{T}/l})^2 = 30 \). Applying Mumford’s relation [ACG11, Chapter 13.7] on \( \overline{M}_6 \), we see that \( c_1(\pi_*\omega_{\tilde{S}/l}) = \frac{30+12}{12} = 6 \). Similarly, we compute \( c_1(\pi'_*\omega_{\tilde{S}/l}) = 12 \) and therefore \( c_1(\rho^*\lambda) = 6 \). \( \square \)

**Corollary 2.2.** In the pencil \( l \), counting with multiplicities, there are 240 fibers with theta divisor singular at a unique two-torsion point and 60 fibers with theta divisor singular at two points.

**Proof.** We directly compute \( l \cdot [\theta_{null}] = l \cdot (264\lambda - 32\delta) = 240 \) and \( l \cdot [N'_0] = l \cdot (108\lambda - 14\delta) = 60 \). \( \square \)

To prove the smoothness of the total spaces \( A \) and \( \Theta \), we first need the following.

**Lemma 2.3.** For \( l \) and \( T \) generic, the image of \( l \) in \( A_5 \) meets \( N_0 \) transversely everywhere.

**Proof.** We need to prove that the image of \( l \) in \( A_5 \) is not tangent to \( N_0 \). Let \( t \) be a point of \( l \) whose image lies in \( N_0 \) and let \( P_t \) be the Prym variety of the cover \( f_t := f|_{D_t} : D_t \rightarrow C_t \). By [Mum74], the singular point of the theta divisor \( \Theta_t \) of \( P_t \) corresponds to an invertible sheaf \( M \) of canonical norm on \( D_t \) such that, either \( h^0(M) \geq 4 \), or \( h^0(M) = 2 \) and there exists an invertible sheaf \( N \) on \( C_t \) with \( h^0(M \otimes f_t^*N^{-1}) > 0 \).

We first eliminate the case \( h^0(M) \geq 4 \). Consider the image of \( l \) in \( R_6 \). By [SV85], the branch divisor of the Prym map \( P : R_6 \rightarrow A_5 \) is \( N'_0 \). The inverse image of \( N'_0 \) in \( R_6 \) is the union of the ramification divisor \( R \) and the anti-ramification divisor \( R' \). By [FGSMV14, Theorem 6.5] we have \( h^0(M) \geq 4 \) if and only if the double cover \( D_t \rightarrow C_t \) belongs to \( R \). Using the formula in [FGSMV14, Corollary 7.3] for the divisor class of \( R \) we compute that the degree of \( R \) on the image of \( l \) is 0. Since \( l \) is generic, it does not lie in \( R \) hence it does not intersect \( R \).

It also follows from the above argument that the Prym map is everywhere of maximal rank on the image of \( l \) in \( R_6 \). In particular, by [DS81], the curve \( C_t \) is not hyperelliptic or trigonal.

We therefore have \( M = f_t^*N(B) \) for an effective divisor \( B \) on \( D_t \) and a line bundle \( N \) on \( C_t \) of degree 4 or 5 such that \( h^0(N) = 2 \). By [FGSMV14, Proposition 7.1], when \( N \) has degree 4, the cover \( D_t \rightarrow C_t \) belongs to the antiramification divisor \( R' \). When \( N \) has degree 5, \( M \) is a singular point of order 2 on \( \Theta_t \), hence the cover \( D_t \rightarrow C_t \) belongs to the inverse image of \( \theta_{null} \) in \( R_6 \). Let \( \alpha \) be the point of order 2 associated to the double cover \( D_t \rightarrow C_t \), in other words, \( \alpha \) is the restriction of the canonical sheaf of \( T \) to \( C_t \). Since \( h^0(M) = 2 \), we have \( h^0(N \otimes \alpha) = 0 \). Furthermore, since \( M \) has canonical norm, we have \( N^2(B) \cong \omega_{C_t} \) where \( B := f_t^*B \).
Let \( q_t \in S^2 H^1(\mathcal{O}_{P_t})^* \) be an equation for the quadric tangent cone to the theta divisor \( \Theta_t \) of \( P_t \) at the singular point \( M \) or \( K_{D_t} \otimes M^{-1} \). Then, using the heat equation, it is easily seen, see, e.g., [Mum75, p. 87], that under the identification \( T_t \mathcal{A}_5 \cong S^2 H^1(\mathcal{O}_{P_t}) \), \( q_t \) is also an equation for the tangent space to \( N_0 \) at \( t \).

Identifying the cotangent space to \( P_t \) with the space \( H^0(K_{C_t} \otimes \alpha) \), the codifferential of the Prym map is identified with the multiplication map

\[
S^2 H^0(K_{C_t} \otimes \alpha) \to H^0(K_{C_t}^2)
\]

(see [Bea77, p. 178]) where we identify the cotangent space to the moduli stack \( \mathcal{R}_6 \) with that of the moduli stack \( \mathcal{M}_6 \) via the natural projection. Since this map is an isomorphism, the quadric \( q_t \) is determined by its zeros on the Prym-canonical image of \( M \).

Assume that either \( p \) or \( q \) is nonzero, a point of the support of \( \overline{B} := f_t s B \). Since \( Q \) contains \( p' \) and \( p'' \), it restricts to a multiple of \( uv = \frac{1}{4}((u+v)^2 - (u-v)^2) \) on \( \langle p' + p'' \rangle \). The restriction of \( Q \) to \( \mathbb{P} H^0(K_{C_t} \otimes \alpha)^* \) is obtained by setting its \( \tau \)-invariant coordinates to 0. Therefore the restriction of \( Q \) to \( \mathbb{P} H^0(K_{C_t} \otimes \alpha)^* \) vanishes on the Prym-canonical image of \( p \) whose equation on \( \langle p' + p'' \rangle \) is \( u - v \).

Therefore the divisor of zeros of \( q_t \) on \( C_t \) is \( \frac{1}{2} f_t s (f_t^* R_N + B + \tau B) = R_N + \overline{B} \in |K_{C_t}^2| \). Next consider the tangent bundle sequence

\[
0 \to T_{C_t} \to T_T|_{C_t} \to \mathcal{O}_{C_t}(C_t) \to 0.
\]

The connecting homomorphism

\[
H^0(\mathcal{O}_{C_t}(C_t)) \to H^1(T_{C_t}) = H^0(K_{C_t}^2)^*
\]

is the Kodaira-Spencer map of the family of curves parametrized by \( |\mathcal{O}_T(C_t)| \). It is given by cup-product with the extension class \( \epsilon \in H^1(T_{C_t}(-C_t)) \) of the tangent bundle sequence. To show that
the image of a generic line \( l \) is not tangent to \( N_0 \), we need to show that the hyperplane defined by \( q_t \) in \( H^1(T_{G_t}) \) does not contain the image of \( H^0(O_{G_t}(C_t)) \). In other words, we need to show that \( q_t \cup \epsilon \in H^0(O_{G_t}(C_t))^* = H^1(\alpha) \) is not zero.

Let \( b \) and \( r_N \) be sections with respective divisors of zeros \( \overline{B} \) and \( R_N \) so that \( q_t = b \cup r_N \). An argument entirely analogous to that on page 252 of [Voi92] shows that \( r_N \cup \epsilon \) is the extension class for the extension

\[
(2.4) \quad 0 \longrightarrow N \longrightarrow E \longrightarrow K_{C_t} \otimes N^{-1} \otimes \alpha \longrightarrow 0
\]

where \( E := F|_{C_t} \) is the restriction of the Lazarsfeld-Mukai bundle \( F \) on \( T \) defined by the natural exact sequence

\[
(2.5) \quad 0 \longrightarrow F^* \longrightarrow H^0(N) \otimes O_T \longrightarrow N \longrightarrow 0.
\]

Using the fact that \( M \) has canonical norm, we obtain \( N^2(\overline{B}) \cong K_{C_t} \) so that \( K_{C_t} \otimes N^{-1} \otimes \alpha \cong N(\overline{B}) \otimes \alpha \). Pulling back sequence (2.4) via multiplication by \( b : N \otimes \alpha \rightarrow N(\overline{B}) \otimes \alpha \), we obtain that \( b \cup r_N \cup \epsilon \) is the extension class for the pulled back extension

\[
(2.6) \quad 0 \longrightarrow N \longrightarrow G \longrightarrow N \otimes \alpha \longrightarrow 0.
\]

Therefore to complete the proof of the lemma, we need to prove that this extension is not split.

Define the torsion free sheaf \( F' \) on \( T \) as the kernel of the composition \( F \rightarrow E \rightarrow O_{\overline{B}} \). Note that by definition we have the exact sequence

\[
(2.7) \quad 0 \longrightarrow F' \longrightarrow F \longrightarrow O_{\overline{B}} \longrightarrow 0.
\]

Dualizing sequence (2.5), we obtain the exact sequence

\[
(2.8) \quad 0 \longrightarrow H^0(N)^* \otimes O_T \longrightarrow F \longrightarrow N^{-1} \otimes O_{C_t}(C_t) \longrightarrow 0.
\]

or

\[
(2.9) \quad 0 \longrightarrow H^0(N)^* \otimes O_T \longrightarrow F \longrightarrow N(\overline{B}) \otimes \alpha \longrightarrow 0.
\]

Twisting (2.9) by \( F^* \) we obtain

\[
(2.10) \quad 0 \longrightarrow H^0(N)^* \otimes F^* \longrightarrow F \otimes F^* \longrightarrow N(\overline{B}) \otimes \alpha \otimes F^* \longrightarrow 0.
\]

From the cohomology of sequence (2.5) we obtain \( h^0(F^*) = h^1(F^*) = 0 \). Similarly, twisting (2.5) with \( \alpha \) and taking cohomology we obtain \( h^0(F^* \otimes \alpha) = h^1(F^* \otimes \alpha) = 0 \). Therefore, the cohomology of sequence (2.10) gives the isomorphism \( H^0(F \otimes F^*) = H^0(N(\overline{B}) \otimes \alpha \otimes F^*) \). Dualizing sequence (2.4), twisting with \( N(\overline{B}) \otimes \alpha \) and taking cohomology we obtain \( h^0(N(\overline{B}) \otimes \alpha \otimes F^*) = 1 \). Therefore \( h^0(F \otimes F^*) = 1 \).

Next we tensor sequence (2.10) with \( \alpha \) and take cohomology to obtain the isomorphism \( H^0(F \otimes F^* \otimes \alpha) = H^0(N(\overline{B}) \otimes F^*) \).
Assume now that sequence (2.6) splits. Then there exists a surjective map \( G \to N \), which implies \( H^0(G^* \otimes N) \neq 0 \). The duals of sequences (2.4) and (2.6), after tensoring with \( N(B) \), are part of the commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \to & \alpha & \to & E^* \otimes N(B) & \to & \mathcal{O}_{C_t}(B) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cong & & \downarrow \\
0 & \to & \alpha(B) & \to & G^* \otimes N(B) & \to & \mathcal{O}_{C_t}(B) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cong & & \downarrow \\
\alpha(B)|_B & \cong & \mathcal{O}_B & & & & & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & & & & & 0.
\end{array}
\]

Since the sections of \( G^* \otimes N \) can be interpreted as the sections of \( G^* \otimes N(B) \) that vanish on \( B \), it follows from the above diagram that we have natural inclusions

\[ H^0(G^* \otimes N) \hookrightarrow H^0(E^* \otimes N(B)) \hookrightarrow H^0(G^* \otimes N(B)). \]

Therefore, if \( H^0(G^* \otimes N) \neq 0 \), we also have \( H^0(N(B) \otimes F^*) = H^0(E^* \otimes N(B)) \neq 0 \).

Summarizing, if sequence (2.6) splits, then \( H^0(F^* \otimes F \otimes \alpha) \neq 0 \). So there exists a nonzero homomorphism

\[ \varphi : F \to F \otimes \alpha. \]

Since \( h^0(F \otimes F^*) = 1 \), the composition \((\varphi \otimes \alpha) \circ \varphi\) is a multiple of the identity. Furthermore, \((\varphi \otimes \alpha) \circ \varphi\) cannot be an isomorphism because otherwise \( \varphi \) would have maximal rank everywhere hence would also be an isomorphism. Therefore \((\varphi \otimes \alpha) \circ \varphi = 0 \). Similarly, \( \varphi \) cannot have maximal rank anywhere since otherwise the same would be true of \( \varphi \otimes \alpha \) and of \((\varphi \otimes \alpha) \circ \varphi\). Therefore the kernel of \( \varphi \) is a subsheaf of rank 1 of \( F \) and the image of \( \varphi \) is a subsheaf of rank 1 of \( F \otimes \alpha \).

Next note that the isomorphism

\[ H^0(F^* \otimes F \otimes \alpha) \xrightarrow{\cong} H^0(F^* \otimes N(B)) = H^0(E^* \otimes N(B)) \]

above is given by composing a homomorphism \( \varphi : F \to F \otimes \alpha \) with the surjection \( F \otimes \alpha \to N(B) \) appearing in sequence (2.9) after twisting with \( \alpha \). Since the image of \( \varphi \in H^0(E^* \otimes N(B)) \) in \( H^0(\mathcal{O}_{C_t}(B)) \) by the map \( H^0(E^* \otimes N(B)) \to H^0(N^{-1} \otimes N(B)) = H^0(\mathcal{O}_{C_t}(B)) \) obtained from sequence (2.4) is nonzero, the composition

\[ N \to E \xrightarrow{\varphi} N(B) \]

is nonzero, hence injective. It follows that the image of \( \varphi \) contains the subsheaf \( N \) of \( N(B) \).
Since $\varphi \in H^0(G^* \otimes N)$, a moment of reflection will show that the composition $F' \hookrightarrow F \overset{\psi}{\rightarrow} F \otimes \alpha$ factors through a homomorphism $\psi : F' \rightarrow F' \otimes \alpha$ whose composition with $F' \otimes \alpha \rightarrow F \otimes \alpha \rightarrow N(\overline{B})$ factors through $N \hookrightarrow N(\overline{B})$. So we have the nonzero composition
\[
\overline{\psi} : F' \overset{\psi}{\rightarrow} F' \otimes \alpha \rightarrow N
\]
which factors through $\varphi : G \rightarrow N$. Since the composition $N \rightarrow G \overset{\varphi}{\rightarrow} N$ is induced by $N \rightarrow E \overset{\varphi}{\rightarrow} N(\overline{B})$, we obtain that $\overline{\psi}$ is surjective. Therefore the image sheaf $\text{Im}(\psi)$ is a torsion free rank 1 sheaf on $T$ whose restriction to $C_t$ is $N$. Let $X$ be a divisor on $T$ representing $c_1(\text{Im}(\psi))$. Then, by, e.g., [BHPdV04, pp. 339-350], $X$ is effective of non-negative self-intersection because $X \cdot C_t$ is positive and $T$ is generic. Furthermore $Y := C_t - 2X$ is also effective since its restriction to $C_t$ is $\overline{B}$ which is effective. Since $h^0(T,Y) \leq h^0(C_t,Y|_{C_t}) = h^0(\overline{B}) = 1$, we have $h^0(Y) \leq 1$ which implies $Y$ has arithmetic genus 1 (since $T$ is generic and does not contain curves of arithmetic genus 0). Therefore $Y^2 = 0$. We have the linear equivalence of effective divisors $C_t \equiv 2X + Y$. Hence $10 = C_t^2 = 4X^2 + 4X \cdot Y$ is a multiple of 4 which is not possible.

To summarize, we have the family of (compactified) Prym varieties and theta divisors
\[
\Theta \rightarrow A \rightarrow l.
\]
This family has 240 fibers where theta has a single node, 60 fibers where theta has two nodes, and 42 fibers where theta is as in Proposition 1.5. Furthermore, we have

**Proposition 2.4.** The total spaces $A$ and $\Theta$ are smooth.

**Proof.** We show that the tangent spaces to $A$ and $\Theta$ have dimension 6 and 5 respectively everywhere. Let $p \in A_t$, resp. $p \in \Theta_t$, be a point of the fiber of $A \rightarrow l$, resp. $\Theta \rightarrow l$, at $t \in l$. If $A_t$ is smooth at $p$, it follows from [ITW, Proposition 3.1] and Lemma 2.3 that, for a generic choice of $l$, both $A$ and $\Theta$ (when $p \in \Theta$) are smooth at $p$. Assume therefore that $A_t$ is singular at $p$. In such a case, it follows from the description of $\Theta_t$ in Proposition 1.5 that, if $p \in \Theta$, $\Theta_t$ is also singular at $p$. By the description of $A_t$ in Section 1.2, resp. $\Theta_t$ in Proposition 1.5, the tangent space to $A_t$ at $p$, resp. $\Theta_t$ at $p$, has dimension 6, resp. 5. We therefore need to show that the tangent space to the total space $A$, resp. $\Theta$, is equal to the tangent space of the fiber. The tangent space to the fiber is the kernel of the differential of the map $A \rightarrow l$, resp. $\Theta \rightarrow l$. Since the map $\Theta \rightarrow l$ is the scheme-theoretic restriction of the map $A \rightarrow l$, we need to show that the differential of the map $A \rightarrow l$ is 0 at $p$ to obtain the smoothness of $A$ at $p$ and also of $\Theta$ at $p$ when $p \in \Theta$.

The total space $A$ is the inverse image of the generic line $l \subset |H|$ in the relative Prym variety $P_H \rightarrow |H|$ constructed in [AFS15]. By [AFS15, Prop. 3.10, Prop. 4.4, Prop. 5.1], the singular locus of $P_H$ lies above a union of lines or points $m_i$ in $|H|$. We can therefore assume that $l$ does not meet any of the $m_i$. Furthermore, since all pull-backs are scheme-theoretic and all fibers reduced, the restriction of the differential of $P_H \rightarrow |H|$ to $A$ is the differential of the projection $A \rightarrow l$. The rank of the differential of $P_H \rightarrow |H|$ is not maximal at $p$ (see loc. cit.), i.e., its image is a proper
subspace of the tangent space of \( |H| \) at \( t \). Since \( l \) is generic, the tangent space of \( l \) at \( t \) intersects this image in 0. Therefore the differential of \( A \to l \) is 0 at \( p \).

\[ \square \]

3. **General facts about the Clemens-Schmid exact sequence**

We briefly review some general facts about the Clemens-Schmid exact sequence. We will apply the general theory in this section to compute the local monodromy representations near the degenerate theta divisors in the pencil.

3.1. **The Clemens-schmid exact sequence.** Let

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{i_t} & Y_t \\
\downarrow & & \downarrow \\
\{0\} & \xleftarrow{} & V \\
\end{array}
\]

be a one-parameter semistable degeneration (i.e., the total space \( Y \) is smooth and the central fiber \( Y_0 \) is reduced with simple normal crossing support) over a small disk \( V \), and \( 0 \neq t \in \partial V \) a general point. The total space \( Y \) deformation retracts to \( Y_0 \). For such a family, the image of the monodromy representation

\[
\rho : \pi_1(V \setminus \{0\}, t) \to GL(H^\bullet(Y_t))
\]

is generated by a unipotent operator \( T : H^\bullet(Y_t) \to H^\bullet(Y_t) \), i.e. \( (T - Id)^k = 0 \) for some integer \( k \) [Lan73]. Thus

\[
N := \log T := (T - Id) - \frac{1}{2}(T - Id)^2 + \frac{1}{3}(T - Id)^3 + ...
\]

is nilpotent.

It follows from the work of Clemens-Schmid [Cle77], [Sch73] and Steenbrink [Ste76] that one can define mixed Hodge structures on \( H^\bullet(Y_t) \), \( H^\bullet(Y) \) and \( H_\bullet(Y) \) such that we have an exact sequence of mixed Hodge structures (with suitable weight shifts)

\[
(3.1) \quad H_{2n+2-m}(Y) \xrightarrow{\alpha} H^m(Y) \xrightarrow{i_t^*} H^m(Y_t)_{\text{lim}} \xrightarrow{N} H^m(Y_t)_{\text{lim}} \xrightarrow{\beta} H_{2n-m}(Y)
\]

where \( n \) is the relative dimension of the fibration, \( \alpha \) is the composition

\[
(3.2) \quad H_{2n+2-m}(Y) \xrightarrow{\text{PD}} H^m(Y, \partial Y) \xrightarrow{} H^m(Y),
\]

and \( \beta \) is the composition

\[
(3.3) \quad H^m(Y_t) \xrightarrow{\text{PD}} H_{2n-m}(Y_t) \xrightarrow{i_t^*} H_{2n-m}(Y).
\]

Here ‘PD’ stands for Poincaré duality. The mixed Hodge structure on \( H^\bullet(Y_t) \) is not the usual pure Hodge structure but rather the ‘limit mixed Hodge structure’ (c.f. Section 3.3). We use the notation \( H^\bullet(Y_t)_{\text{lim}} \) to distinguish it from the pure Hodge structure.
3.2. The weight filtrations on \( H^m(Y) \) and \( H_m(Y) \). Put

\[
H^m := H^m(Y) \cong H^m(Y_0), \\
H_m := H_m(Y) \cong H_m(Y_0).
\]

Recall from [Mor84, p. 103] that there is a Mayer-Vietoris type spectral sequence abutting to \( H^\bullet(Y_0) \) with \( E_1 \) term

\[
E_1^{p,q} = H^q(Y_0^{[p]}).
\]

Here \( Y_0^{[p]} \) is the disjoint union of the codimension \( p \) strata of \( Y_0 \), i.e.,

\[
Y_0^{[p]} := \bigsqcap_{i_0, \ldots, i_p} Z_{i_0} \cap \ldots \cap Z_{i_p}
\]

where the \( Z_{i_j} \) are distinct irreducible components of \( Y_0 \).

The differential \( d_1 \)

\[
\begin{array}{ccc}
E_1^{p,q} & \xrightarrow{d_1} & E_1^{p+1,q} \\
\cong & \Downarrow & \cong \\
H^q(Y_0^{[p]}) & \xrightarrow{d_1} & H^q(Y_0^{[p+1]})
\end{array}
\]

is the alternating sum of the restriction maps on all the irreducible components. By [Mor84, p. 103] this sequence degenerates at \( E_2 \).

The weight filtration is given by

\[
W_k H^m := \bigoplus_{p+q=m, \ q \leq k} E_\infty^{p,q} = \bigoplus_{p+q=m, \ q \leq k} E_2^{p,q}.
\]

Therefore the weights on \( H^m \) go from 0 to \( m \) and

\[
Gr_k H^m \cong E_2^{m-k,k} = \frac{\ker(d_1 : H^k(Y_0^{[m-k]}) \to H^k(Y_0^{[m-k+1]}))}{\operatorname{Im}(d_1 : H^k(Y_0^{[m-k-1]}) \to H^k(Y_0^{[m-k]}))}.
\]

There is also a weight filtration on \( H_m \):

\[
W_{-k} H_m := (W_{k-1} H^m)^\perp
\]

under the perfect pairing between \( H^m \) and \( H_m \). With this definition,

\[
Gr_{-k} H_m \cong (Gr_k H^m)^\vee.
\]

3.3. The limit mixed Hodge structure \( H^m(Y_t)_{\text{lim}} \). The weight filtration associated to the nilpotent operator \( N \) has the following form,

\[
0 \subset W_0 \subset W_1 \subset \ldots \subset W_{2m} = H^m(Y_t).
\]

We refer to [Mor84, pp. 106-109] for the precise definition of the monodromy weight filtration and only summarize the properties we need here.

In the applications in this paper, the nilpotent operator \( N \) satisfies

\[
N^2 = 0.
\]
Thus the monodromy weight filtration satisfies the following
\[ W_k = 0 \text{ for } k \leq m - 2, \]
\[ W_{m-1} = \text{Im}(N), \]
\[ W_m = \text{Ker}(N), \]
\[ W_k = H^m(Y_t) \text{ for } k \geq m + 1. \]

Let \( K_t^m := \text{Ker}(N) \subset H^m(Y_t) \) be the monodromy invariant subspace. It inherits an induced weight filtration from \( H^m(Y_t) \). The graded pieces of \( H^m(Y_t) \) thus satisfy
\[ \text{Gr}_m H^m(Y_t) \mid_{\text{lim}} \sim \frac{\text{Ker}(N)}{\text{Im}(N)} \]
\[ \text{Gr}_{m+1} H^m(Y_t) \mid_{\text{lim}} \sim \frac{\text{Gr}_{m-1} H^m(Y_t) \mid_{\text{lim}}}{\text{Gr}_{m-1} K_t^m \mid_{\text{lim}}} = \text{Im}(N). \]

The weight filtrations on \( H^m \) and \( K_t^m \) are related by the Clemens-Schmid exact sequence. Below are the basic facts we will use (see [Mor84, pp. 107-109])

1. \( i_t^* \) induces an isomorphism
\[ \text{Gr}_k H^m \xrightarrow{\cong} \text{Gr}_k K_t^m \text{ for } k \leq m - 1. \]

2. There is an exact sequence
\[ 0 \longrightarrow \text{Gr}_{m-2} K_t^{m-2} \longrightarrow \text{Gr}_{m-2n-2} H_{2n+2-m} \xrightarrow{\alpha} \text{Gr}_m H^m \longrightarrow \text{Gr}_m K_t^m \longrightarrow 0. \]

The limit Hodge filtration on \( H^m(Y_t) \mid_{\text{lim}} \) is given by ([Mor84], [Sch73])
\[ F^p = \lim_{z \to \infty} \exp(-zN)F^p(z) \]
where \( f : U' \to U \setminus \{0\}, f(z) = e^{2\pi iz} \) is the universal cover of the punctured disk and \( F^p \) is the usual Hodge filtration on \( H^m(Y_{f(z)}) \) on the fixed underlying space \( H^m(Y_t) \).

4. Local monodromy representations near \( N_0 \)

4.1. Local monodromy near \( \theta_{null} \). The local monodromy representation on the cohomology of the theta divisor near a general point \((A_0, \Theta_0) \in \theta_{null}\) is given by the classic Picard-Lefschetz formula. Fix a point \( p_0 \in l \cap \theta_{null} \) and pick a small disk \( U \subset l \) containing \( p_0 \). We have a family of theta divisors with smooth total space \( \Theta_U \) (see Proposition 2.4):
\[
\begin{array}{ccc}
\Theta_0 & \longrightarrow & \Theta_U \\
\downarrow & & \downarrow \\
p_0 & \longrightarrow & U.
\end{array}
\]

The local monodromy representation on the cohomology of a general fiber \( \Theta_t \) for \( t \in U \setminus \{p_0\} \)
\[ \rho : \pi_1(U \setminus \{p_0\}, t) \longrightarrow GL(H^k(\Theta_t)) \]
is trivial when \( k \neq 4 \). When \( k = 4 \), the Picard-Lefschetz formula (see, for instance, [Voi03, p. 78]) shows that \( \rho(\pi_1(U \setminus \{p_0\}, t)) \) is generated by

\[
T_U : H^4(\Theta_t) \to H^4(\Theta_t)
\]

\[
\alpha \mapsto \alpha - \langle \alpha, \gamma \rangle \gamma
\]

where \( \langle , \rangle \) is the intersection product on \( H^4(\Theta_t) \), and \( \gamma \in H^4(\Theta_t) \) is the class of the vanishing 4-sphere with \( \langle \gamma, \gamma \rangle = 2 \).

One checks immediately that

\[
T^2_U = \text{Id}.
\]

4.2. **Local monodromy near** \( N'_0 \). Next we fix a point \( p_0 \in l \cap N'_0 \) and a small disk \( U \subset l \) containing \( p_0 \). The central fiber \( \Theta_0 \) of the family \( \Theta_U \) has two ordinary double points \( x \) and \( -x \).

If we make a degree two base change \( V \to U \) ramified at \( p_0 \):

\[
\begin{array}{ccc}
\Theta_V & \longrightarrow & \Theta_U \\
\downarrow & & \downarrow \\
V & \longrightarrow & U,
\end{array}
\]

then blow up the two singular points of \( \Theta_V \), we obtain a family

\[
\begin{array}{ccc}
\tilde{\Theta}_0 & \longrightarrow & \tilde{\Theta}_V \\
\downarrow & & \downarrow \\
p_0 & \longrightarrow & V,
\end{array}
\]

where the central fiber \( \tilde{\Theta}_0 = \Theta'_0 \cup Q_1 \cup Q_2 \) is reduced with simple normal crossing support. Here \( \Theta'_0 \) is the blow-up of \( \Theta_0 \) at the two singular points and \( Q_1 \cong Q_2 \) are smooth quadric 4-folds. The double loci \( \Theta'_0 \cap Q_1 \) and \( \Theta'_0 \cap Q_2 \) are smooth quadric 3-folds.

Since \( V \to U \) is a degree 2 ramified cover, the local monodromy operator \( T_V \) for the family \( \tilde{\Theta}_V \to V \) is equal to \( T^2_U \in GL(H^4(\Theta_t)) \).

**Proposition 4.1.** Notation as above, \( T_V = T^2_U = Id \in GL(H^4(\Theta_t)) \).

**Proof.** Since the central fiber \( \tilde{\Theta}_0 = \Theta'_0 \cup Q_1 \cup Q_2 \) only has a double locus, we have

\[
Gr_k H^4(\Theta_t) = 0
\]

for \( k \neq 3, 4, 5 \). Since \( H^3(\Theta'_0 \cap Q_1) \oplus H^3(\Theta'_0 \cap Q_2) = 0 \), we conclude

\[
Gr_5 H^4(\Theta_t) \cong Gr_3 H^4(\Theta_t) \cong Gr_3 H^4(\tilde{\Theta}_0) = \text{Im}(N_V) = 0,
\]

where \( N_V := \log T_V = 0 \). Therefore \( T_V = Id \). \( \square \)
Near the boundary $\Delta$, the family of Prym varieties $A_U \to U$ parametrized by a small disk $U \subset l$ has smooth general fiber $(A_t, \Theta_t)$ and central fiber $(P, \Upsilon)$ as in Proposition 1.5. We use the Clemens-Schmid exact sequence to compute the monodromy action.

5.1. The semi-stable reduction. Making a ramified base change $V \to U$ of order 2 of the family

$\begin{align*}
A_V & \longrightarrow A_U \\
V & \longrightarrow U,
\end{align*}$

and then blowing up the singular locus $P \setminus G_{pq}$ of $A_V$, we obtain a family $\widetilde{A}_V \to V$.

Proposition 5.1. The central fiber $\widetilde{A}_0$ of the family $(\widetilde{A}_V, \widetilde{\Theta}_V) \to V$ is the union of two copies $P'_1$ and $P''_2$ of $P''$, with $B_0 \subset P'_1$ identified with $B_\infty \subset P''_2$ via the identity map and $B_\infty \subset P'_1$ identified with $B_0 \subset P''_2$ via translation by $b$. The intersection $P'_1 \cap P''_2 = B_{0\infty} \sqcup B_{\infty0}$ is the disjoint union of two copies of $B$.

Proof. Clearly the main component $P'_1 \cong P''$. We will show the exceptional divisor $P''_2$ is also isomorphic to $P''$. In the semistable family $\widetilde{A}_V \to V$, we have

$N_{B_{0\infty}/P'_1} \cong N_{B_{0\infty}/P''}.$

Therefore $P''_2$ contains the total space of $\mathcal{O}_B(\Xi_b - \Xi) \cong \mathcal{O}_{B_0}(-B_0) \cong N_{B_{0\infty}/P''} = P''_2 \setminus B_{\infty0}$ as a Zariski open subset. Applying the same argument to $B_{0\infty}$, we see that $P''_2$ also contains the total space of $\mathcal{O}_B(\Xi - \Xi_b) \cong N_{B_{\infty0}/P''} = P''_2 \setminus B_{0\infty}$ as an open subset. We conclude that $P''_2 \cong \mathbb{P}_B(\mathcal{O}_B(\Xi - \Xi_b) \oplus \mathcal{O}_B(\Xi_b - \Xi)) \cong P''$. The statement about the gluing follows from the fact that after contracting $P''_2$, the infinity and zero sections of $P''_2$ are identified via translation by $b$. \qed

Corollary 5.2. The central fiber $\tilde{\Theta}_0$ of the family $(\tilde{A}_V, \tilde{\Theta}_V) \to V$ is the union $\Upsilon'' \cup Q_\Xi$, where $\Upsilon'' = Bl_{\Xi \cap \Xi_b} B$ and the conic bundle $Q_\Xi$ is the restriction of $P''_2 \to B$ to $\Xi$. The intersection $\Upsilon'' \cap Q_\Xi = \Xi_{0\infty} \sqcup \Xi_{\infty0}$ is the disjoint union of two copies of $\Xi$.

Proof. Immediate. \qed

5.2. The weight filtration on $H^m(\widetilde{A}_0)$. By Section 3.2 and Proposition 5.1, the weight filtration on $H^m(\widetilde{A}_0)$ only has the following possibly nontrivial graded pieces

$Gr_m H^m(\widetilde{A}_0) = \text{Ker}(d_1 : H^m(P'_1) \oplus H^m(P''_2) \longrightarrow H^m(B_{0\infty}) \oplus H^m(B_{\infty0}))$

and

$Gr_{m-1} H^m(\widetilde{A}_0) = \text{Coker}(d_1 : H^{m-1}(P'_1) \oplus H^{m-1}(P''_2) \longrightarrow H^{m-1}(B_{0\infty}) \oplus H^{m-1}(B_{\infty0}))$
Proposition 5.3. We have
\[ Gr_m H^m(\widetilde{A}_0) \cong H^{m-2}(B) \oplus H^m(P^\nu), \]
and
\[ Gr_{m-1} H^m(\widetilde{A}_0) \cong H^{m-1}(B). \]

Proof. By Remark 1.4, \( P^\nu \to B \) is a topologically trivial \( \mathbb{P}^1 \) bundle. The statements then follow easily from Proposition 5.1 and the Künneth formula. \( \square \)

Corollary 5.4. The monodromy weight filtration on \( H^m(A_t)_{\lim} \) satisfies
\[ Gr_{m+1} H^m(A_t)_{\lim} \cong Gr_{m-1} H^m(A_t)_{\lim} \cong H^{m-1}(B). \]
Furthermore, \( \dim_{\mathbb{C}} Gr_m H^m(A_t)_{\lim} = \binom{10}{m} - 2 \binom{8}{m-1} \).

Proof. By (3.5) and (3.6), \( Gr_{m+1} H^m(A_t)_{\lim} \cong Gr_{m-1} H^m(A_t)_{\lim} \cong Gr_{m-1} H^m(\widetilde{A}_0) \) which is isomorphic to \( H^{m-1}(B) \) by Proposition 5.3. The second part follows from Sequence (3.7). \( \square \)

5.3. The weight filtration on \( H^m(\widetilde{\Theta}_0) \). By Section 3.2 and Proposition 5.2, the weight filtration on \( H^m(\widetilde{\Theta}_0) \) only has the following possibly nontrivial graded pieces
\[ Gr_m H^m(\widetilde{\Theta}_0) = \ker(d_1 : H^m(\gamma^\nu) \oplus H^m(Q_\Xi) \longrightarrow H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty0})) \]
and
\[ Gr_{m-1} H^m(\widetilde{\Theta}_0) = \text{coker}(d_1 : H^{m-1}(\gamma^\nu) \oplus H^{m-1}(Q_\Xi) \longrightarrow H^{m-1}(\Xi_{0\infty}) \oplus H^{m-1}(\Xi_{\infty0})). \]

Proposition 5.5. For \( m \leq 4 \),
\[ Gr_m H^m(\widetilde{\Theta}_0) \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \oplus H^{m-2}(\Xi), \]
and for all \( m \),
\[ Gr_{m-1} H^m(\widetilde{\Theta}_0) \cong H^{m-1}(\Xi). \]

Proof. By Corollary 5.2, \( H^m(\gamma^\nu) \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \) and the restriction map \( H^m(\gamma^\nu) \to H^m(\Xi_{0\infty}) \) can be identified with the map
\[ H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \overset{(\gamma^\nu, \iota^*_i)}{\longrightarrow} H^m(\Xi). \]
Thus the image of
\[ H^m(\gamma^\nu) \longrightarrow H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty0}) \]
is contained in the image of
\[ H^m(Q_\Xi) \cong H^m(\Xi) \oplus H^{m-2}(\Xi) \longrightarrow H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty0}), \]
which is equal to the diagonal of \( H^m(\Xi_{0\infty}) \oplus H^m(\Xi_{\infty0}) \). Thus
\[ Gr_{m-1} H^m(\widetilde{\Theta}_0) \cong H^{m-1}(\Xi). \]
Next we compute \( Gr_m H^m(\tilde{\Theta}_0) \subset H^m(\Upsilon') \oplus H^m(Q_\Xi) \). By the previous discussion, for any \( x \in H^m(\Upsilon') \), we can find \( y \in H^m(Q_\Xi) \) such that \((x, y) \in Gr_m H^m(\tilde{\Theta}_0)\). Thus we have an exact sequence

\[
0 \rightarrow H^{m-2}(\Xi) \rightarrow Gr_m H^m(\tilde{\Theta}_0) \rightarrow H^m(\Upsilon') \rightarrow 0
\]

Therefore, we have a noncanonical isomorphism

\[
Gr_m H^m(\tilde{\Theta}_0) \cong H^{m-2}(\Xi) \oplus H^m(\Upsilon') \cong H^m(B) \oplus H^{m-2}(\Xi \cap \Xi_b) \oplus H^{m-2}(\Xi)
\]

\[\blacksquare\]

**Corollary 5.6.** The monodromy weight filtration on \( H^m(\Theta_t)_{\text{lim}} \) satisfies

\[
Gr_{m+1} H^m(\Theta_t)_{\text{lim}} \cong Gr_{m-1} H^m(\Theta_t)_{\text{lim}} \cong H^{m-1}(\Xi).
\]

Furthermore, \( \dim \mathbb{C} Gr_m H^m(\Theta_t)_{\text{lim}} = h^m(\Theta_t) - 2h^{m-1}(\Xi) \).

**Proof.** Analogous to the proof of Corollary 5.4. \[\blacksquare\]

### 5.4. The vanishing cocycles near the boundary.

Let \( Z \to |H| \cong \mathbb{P}^5 \) be the 2-to-1 cover ramified exactly along \( \Gamma := D + \mathbb{P}^{-1}_H(N_0) \) and set \( X := \nu^{-1}l, \ U := Z \setminus \Gamma \). Note that \( Z \) exists since \( \Gamma \) has even degree by Proposition 1.1 and Corollary 2.2. The curve \( X \) is a 2-to-1 cover of \( l \) ramified along \( X \cap \Gamma \). After base change to \( X \) and blowing up the singular locus of each singular theta divisor, we obtain a family \((\tilde{A}, \tilde{\Theta})\) with general fiber \((A_t, \Theta_t)\).

\[
\begin{array}{ccc}
\Theta_t & \xrightarrow{i_t} & \tilde{\Theta} \\
\downarrow j_t & & \downarrow j \\
A_t & \xrightarrow{h_t} & \tilde{A} \\
\downarrow p & & \downarrow p \\
\{t\} & \longrightarrow & X.
\end{array}
\]

The total spaces of \( \tilde{A} \) and \( \tilde{\Theta} \) are smooth and the local pictures are described in Sections 4.1, 4.2 and 5.1.

For each \( s_i, i = 1, \ldots, 42 \), corresponding to the degeneration in Section 1 (also see Section 5.1), choose a small disk \( V_i \ni s_i \) and pick a general point \( t_i \in V_i \). Let \( \gamma_i \subset X \) be a general path connecting \( t \) with \( t_i \). The family \( \tilde{\Theta}|_{\bigcup_{s_i}} \) deformation retracts to \( \Theta_t \). Thus we have induced **diffeomorphisms**

\[
\psi_i : \Theta_t \longrightarrow \Theta_{t_i}.
\]
Over each $V_i$ we have the Clemens-Schmid exact sequences \((3.1)\) for the degenerations of the abelian varieties and their theta divisors

\[
\begin{align*}
(5.1) \quad & H^m(\tilde{\Theta}_{V_i}) \xrightarrow{i^*_i} H^m(\Theta_{t_i})_{\text{lim}} \xrightarrow{N_i} H^m(\Theta_{t_i})_{\text{lim}} \xrightarrow{\beta_i} H_{10-m}(\tilde{\Theta}_{V_i}) \\
\downarrow j_* & \downarrow j^{-1}_* \downarrow j^{-1}_* \downarrow j_* \\
& H^{m+2}(\tilde{A}_{V_i}) \xrightarrow{i^*_i} H^{m+2}(A_{t_i})_{\text{lim}} \xrightarrow{j^{-1}_*} H^{m+2}(A_{t_i})_{\text{lim}} \xrightarrow{j_*} H_{10-m}(\tilde{A}_{V_i}) .
\end{align*}
\]

Here $j_* : H^m(\tilde{\Theta}_{V_i}) \to H^{m+2}(\tilde{A}_{V_i})$ is defined to be the transpose of $j^* : H^{10-m}(\tilde{A}_{V_i}) \to H^{10-m}(\tilde{\Theta}_{V_i})$ under Poincaré duality and is a morphism of mixed Hodge structures [ITW, Section 8].

Put $V^m_i := \psi^*_i \text{Ker} \beta_i = \psi^*_i \text{Im}(N_i) = \psi^*_i \text{Gr}_{m-1}H^m(\Theta_{t_i})_{\text{lim}} \subset H^m(\Theta_{t_i})_{\text{lim}}$.

**Proposition 5.7.** The space $V_i$ is the space of ‘local vanishing $m$-cocycles’, i.e., cohomology classes whose Poincaré dual vanishes in $\Theta_{V_i}$.

**Proof.** This follows immediately from the definition of $\beta_i$ in \((3.3)\). \qed

By Corollary 5.6, we have

\[
\text{Im}(N_i) = \text{Gr}_{m-1}H^m(\Theta_{t_i})_{\text{lim}} \cong \text{Gr}_{m-1}H^m(\tilde{\Theta}_{V_i}) \cong H^{m-1}(\Xi).
\]

When $m = 4$, we can further rewrite the above isomorphisms as

\[
(5.2) \quad \text{Gr}_3H^4(\Theta_{t_i})_{\text{lim}} \cong H^3(\Xi) \cong H^3(B) \oplus \mathbb{H}^3_i \cong \psi^*_i \text{Gr}_3H^4(A_{t_i})_{\text{lim}} \oplus \mathbb{H}^3_i,
\]

where $\mathbb{H}^3_i \subset H^3(\Xi)$ is the primal cohomology of $\Xi$ in $B$, which is 10-dimensional. Let $\mathbb{H}^3_i \subset V^4_i \subset H^4(\Theta_{t_i})$ be the image of $\mathbb{H}^3_i$ under the composition

\[
H^3(B) \oplus \mathbb{H}^3_i \cong \text{Gr}_3H^4(\Theta_{t_i})_{\text{lim}} \subset H^4(\Theta_{t_i})_{\text{lim}} \xrightarrow{\psi^*_i} H^4(\Theta_{t_i}).
\]

### 6. Global monodromy

Let $H^m(\Theta_{t_i})_{\text{var}} := \text{Ker}(i_{t*} : H^m(\Theta_{t_i}) \to H^{m+2}(\tilde{\Theta}))$ and $H^m(A_{t_i})_{\text{var}} := \text{Ker}(h_{t*} : H^m(A_{t_i}) \to H^{m+2}(\tilde{A}))$ be the variable cohomology of $\Theta_{t_i}$ in $\tilde{\Theta}$ and $A_{t_i}$ in $\tilde{A}$, respectively.

#### 6.1. The primal cohomology and the variable cohomology.

The next four propositions describe the variable middle cohomology $H^4(\Theta_{t_i})_{\text{var}}$ and its relation with the primal cohomology $\mathbb{K}_t$.

**Proposition 6.1.** The variable cohomology $H^m(\Theta_{t_i})_{\text{var}}$ is equal to $\sum_{i=1}^{42} V^m_i$.

**Proof.** By Equation \((4.1)\) and Proposition 4.1, when the theta divisor has one or two nodes, the local monodromy representation is trivial after we make a base change of order 2. Thus from the Clemens–Schmid sequence, there are no ‘local vanishing cocycles’ near these singular theta divisors. Therefore the space of vanishing cocycles is generated by the ‘local vanishing cocycles’ near $\Theta_{s_i}$, $i = 1, ..., 42$. \qed
**Proposition 6.2.** The pull-back maps $i^*_t : H^4(\tilde{\Theta}) \to H^4(\Theta_t)$ and $(j \circ i)_t^* : H^4(\tilde{A}) \to H^4(\Theta_t)$ have the same image. As a consequence, $H^4(\Theta_t)_{\text{var}} = (\text{Ker}(j \circ i)_t^* : H^4(\Theta_t) \to H^8(\tilde{A}))$.

**Proof.** Choose another general point $u \neq t$ in $X$. Write $W := X \setminus \{u\}$, and $(\tilde{A}_W, \tilde{\Theta}_W) := (p^{-1}(W), (p \circ j)^{-1}(W))$.

Consider the Gysin sequence

$$
\begin{array}{ccccccc}
H^4(\tilde{A}) & \longrightarrow & H^4(\tilde{A}_W) & \stackrel{\text{Res}}{\longrightarrow} & H^3(\Theta_u) & \stackrel{h_u^*}{\longrightarrow} & H^5(\tilde{A}) \\
\downarrow j^* & & \downarrow j_W^* & & \cong & \downarrow j_u^* & \downarrow j^* \\
H^4(\tilde{\Theta}) & \longrightarrow & H^4(\tilde{\Theta}_W) & \stackrel{\text{Res}}{\longrightarrow} & H^3(\Theta_u) & \longrightarrow & H^5(\tilde{\Theta})
\end{array}
$$

where $\text{Res}$ denotes Griffiths’ residue map. We claim that $j_W^*: H^k(\tilde{A}_W) \to H^k(\tilde{\Theta}_W)$ is an isomorphism for $k \leq 4$ and injective for $k = 5$ (this is the Lefschetz hyperplane theorem in a slightly modified setting). To this end, apply the long exact sequence of singular cohomology of the pair $(\tilde{A}_W, \tilde{\Theta}_W)$. The relative cohomology $H^k(\tilde{A}_W, \tilde{\Theta}_W)$ is isomorphic to $H_{12-k}(\tilde{A}_W \setminus \tilde{\Theta}_W)$ [Voi03, p. 33].

Note that $\tilde{\Theta}$ is $p$-ample, and therefore $\tilde{\Theta} + kA_u$ is ample in $\tilde{A}$ for some $k > 0$. We conclude that the open set $\tilde{A}_W \setminus \tilde{\Theta}_W = \tilde{A} \setminus (\tilde{\Theta} \cup A_u)$ is affine, thus has the homotopy type of a CW-complex of real dimension 6. Therefore $H_{12-k}(\tilde{A}_W \setminus \tilde{\Theta}_W) = 0$ for $k \leq 6$, which implies the claim.

By Proposition 6.1 and Corollaries 5.4 and 5.6, $H^3(\Theta_u)_{\text{var}} := \text{Ker}(h_u^*) \cong H^3(\Theta_u)_{\text{var}}$, thus by the Gysin sequence and the fact that $j_W^*$ is an isomorphism when $k = 4$, the restriction map $H^4(\tilde{\Theta}) \to H^4(\tilde{\Theta}_W)$ has the same image as the composition $H^4(\tilde{A}) \to H^4(\tilde{A}_W) \to H^4(\tilde{\Theta}_W)$. Taking the restriction map from $H^4(\tilde{\Theta}_W)$ to $H^4(\Theta_t)$, the first statement follows immediately.

The second statement follows from the fact that Gysin push-forward is the transpose of the pull-back map. \hfill \Box

**Proposition 6.3.** The primal cohomology $\mathbb{K}_t := \text{Ker}(j_t^* : H^4(\Theta_t) \to H^6(\tilde{A}_t))$ is contained in the variable cohomology $H^4(\Theta_t)_{\text{var}}$.

**Proof.** By Proposition 6.2, we have $H^4(\Theta_t)_{\text{var}} = (\text{Ker}(j \circ i)_t^* : H^4(\Theta_t) \to H^8(\tilde{A}))$, which implies $\mathbb{K}_t \subset H^4(\Theta_t)_{\text{var}}$. \hfill \Box

**Proposition 6.4.** The primal cohomology $\mathbb{K}_t$ is equal to $\sum_{i=1}^{42} \mathbb{H}_i$.

**Proof.** The morphism $j_* : H^4(\tilde{\Theta}_V) \to H^6(\tilde{A}_V)$ in (5.1) is a morphism of mixed Hodge structures. The induced morphism

$$
\begin{array}{ccc}
\text{Gr}_3 H^4(\tilde{\Theta}_V) & \longrightarrow & \text{Gr}_3 H^6(\tilde{A}_V) \\
\downarrow \cong & & \downarrow \cong \\
H^3(\Xi) & \longrightarrow & H^5(B)
\end{array}
$$

is Gysin pushforward. By construction, $\mathbb{H}' \subset \text{Gr}_3 H^4(\tilde{\Theta}_V) \subset H^4(\tilde{\Theta}_V)$ is contained in $\text{Ker}(j_*)$. Thus by sequence (5.1), $i_t^* \mathbb{H}' \subset \text{Ker}(j_{t, *} : H^4(\Theta_{t, t}) \to H^6(A_{t, t}))$, or equivalently, $\mathbb{H}_t \subset \mathbb{K}_t$. It remains
to show $\mathbb{K}_t \subset \sum_{i=1}^{42} \mathbb{H}_i$. To this end, pick any $\alpha \in \mathbb{K}_t$, by Proposition 6.1 and Equation (5.2), we can write $\alpha = \sum_{i=1}^{42} (x_i + y_i)$, where $x_i \in j_t^*H^4(A_t)$ and $y_i \in \mathbb{H}_i \subset \mathbb{K}_t$. From the direct sum decomposition

$$H^4(\Theta_t) = j_t^*H^4(A_t) \oplus \mathbb{K}_t,$$

we conclude $\sum_{i=1}^{42} x_i = 0$ and $\alpha \in \sum_{i=1}^{42} \mathbb{H}_i$. $\square$

6.2. The proof of the main theorem. From now on we will abuse notation by considering $N_i$ in (5.1) as an endomorphism on $H^4(\Theta_t)$ via $\psi_i^*$ and then restricting it to $\mathbb{K}_t$. With the new notation, $N_i : \mathbb{K}_t \to \mathbb{K}_t$ satisfies

(6.1) $N_i^2 = 0,$

(6.2) $N_i(\mathbb{K}_t) = \mathbb{H}_i.$

Since the monodromy operator preserves the intersection product $\langle \cdot, \cdot \rangle$ on $\mathbb{K}_t$, $N_i$ also satisfies the equality

(6.3) $\langle N_i(x), y \rangle + \langle x, N_i(y) \rangle = 0$

for any $x, y \in \mathbb{K}_t$.

Each $N_i$ induces a ‘limit mixed Hodge structure’ $\mathbb{K}_i^{\text{lim}}$ on $\mathbb{K}_t$ as in Section 3.3.

**Lemma 6.5.** We have $\cap_{i=1}^{42} \text{Ker}(N_i) = 0$.

*Proof.* Equation (6.3) implies that $\langle N_i(x), y \rangle = 0$ for any $x \in \mathbb{K}_t$ and $y \in \text{Ker}(N_i)$. Thus $\text{Ker}(N_i) \perp \mathbb{H}_i$. Any element in $\cap_{i=1}^{42} \text{Ker}(N_i)$ is therefore perpendicular to all $\mathbb{H}_i$, $i = 1, ..., 42$. The statement now follows immediately from Proposition 6.4 and the fact that the intersection product is nondegenerate. $\square$

**Lemma 6.6.** With the notation of Section 5.4, all $\mathbb{H}_i$, $i = 1, ..., 42$ are conjugate under the monodromy representation

$$\rho : \pi_1(\mathcal{V}, t) \to \text{Aut}(\mathbb{K}_t, \langle \cdot, \cdot \rangle).$$

*Proof.* For any $i \neq j$, choose a path $\delta'$ in $l$ connecting $t_i$ and $t_j$. By perturbing $\delta'$, we can assume $\delta'$ does not intersect the inverse image of $N_0$. We can lift $\delta'$ to a path $\delta \subset X \cap \mathcal{V}$ as a smooth section over $\delta'$ in the tubular neighborhood of the smooth locus $\mathcal{D}^0$ of $\mathcal{D}$ in $\mathcal{V}$. A $C^\infty$-trivialization of the total space of the theta divisors over $\delta$ induces a map on cohomology, which sends $\mathbb{H}_i \subset H^4(\Theta_{t_i})$ to $\mathbb{H}_j \subset H^4(\Theta_{t_j})$. This precisely means that under the monodromy action, $\rho(\gamma_i \cdot \delta \cdot \gamma_j^{-1})$ sends $\mathbb{H}_i$ to $\mathbb{H}_j$. $\square$

*Proof.* of **Theorem 0.1.** It suffices to show that for very general $t \in X \cap \mathcal{V}$, $\mathbb{K}_t$ is an irreducible Hodge structure. Suppose $0 \subsetneq F_t \subset \mathbb{K}_t$ is a rational Hodge substructure, then $F_t$ is an invariant subspace under the action of the Mumford-Tate group $MT(\mathbb{K}_t)$. For very general $t$, $MT(\mathbb{K}_t)$ contains the identity component $I_{\mathcal{V}}$ of the algebraic monodromy group $G_{\mathcal{V}}$, i.e., the Zariski
The irreducibility of the primal cohomology closure in $GL(\mathbb{K}_t)$ of the monodromy group $\rho(\pi_1(V))$, (c.f. [Sch11, Prop. 6]), thus by further passing to a finite étale cover $V'$ of $V$, we can assume $\mathbb{F}_t$ is invariant under $\rho(\pi_1(V'))$. Therefore, we obtain a local subsystem $\mathbb{F}_{V'} \subset \mathbb{K}_{V'}$ over $V'$.

Note that $I_{V'} = I_V$, since $I_{V'} \subset I_V$ is of finite index and $I_V$ is connected. Moreover, $T_i = \exp(N_i) \in I_V = I_{V'}$. (Because $T_i$ is in the image of the exponential map $\exp : gl(\mathbb{K}_t) \to GL(\mathbb{K}_t)$.) We conclude that $\mathbb{F}_t$ is invariant under $T_i$ and therefore $N_i$. Each $N_i$ then induces a ‘limit mixed Hodge structure’ $\mathbb{F}_i^\lim$ on $\mathbb{F}_t$.

By Lemma 6.5, for any $0 \neq x \in \mathbb{F}_t$, $x \notin \text{Ker}(N_i)$ for some $i$, thus $0 \neq N_i(x) \in \mathbb{F}_t \cap \mathbb{H}_i = \mathbb{F}_t \cap W_3 \mathbb{K}_t^\lim = W_3 \mathbb{F}_t^\lim$. Since $\mathbb{H}_i = W_3 \mathbb{K}_t^\lim$ is an irreducible pure Hodge structure (follows from the main result of [IS95]), we conclude $\mathbb{H}_i \subset \mathbb{F}_t$. By Lemma 6.6, the $\mathbb{H}_i$ are conjugate under the monodromy group $\pi_1(V)$, thus $\mathbb{H}_i \subset \mathbb{F}_t$ for all $i$ and, by Proposition 6.4, $\mathbb{F}_t = \mathbb{K}_t$. □

References


