A signature for some subgroups of the permutation group of [0,1[

UC San Diego Group Actions Seminar

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ICJ (Lyon)

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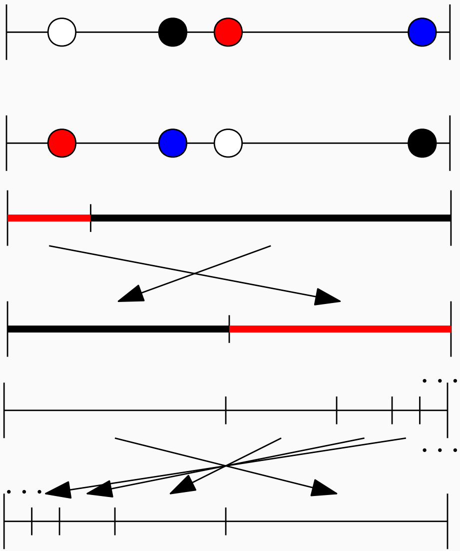
Context

Permutation group

Let X be the interval $[0, 1[$.

We denote by $\mathfrak{S}(X)$ the permutation group of X.

Permutation group



Permutation group

Let $\mathfrak{S}_{\text{fin}}$ be the normal subgroup of $\mathfrak{S}(X)$ consisting of all finitely supported permutations.

Proposition
There exists a unique non-trivial group homomorphism :

$$\varepsilon_{\text{fin}} : \mathfrak{S}_{\text{fin}} \to \mathbb{Z}/2\mathbb{Z}$$

We define $\mathfrak{A}_{\text{fin}}$ as the kernel of ε_{fin}.

Proposition (Consequence of a remark of Vitali in 1915)
There does not exist any group homomorphism from $\mathfrak{S}(X)$ to $\mathbb{Z}/2\mathbb{Z}$ that extends ε_{fin}.

Vitali's remark

Proof.
Every element of $\mathfrak{S}(\mathbb{Z})$ is a product of squares. We explicit here how the transposition of two distinct points of $\mathbb{Z}$ is the product of squares.

$$\begin{array}{ccccc}
\sigma & = & (1\ 2)(3\ 4)(5\ 6)\dots & & \\
\tau & = & (\dots\ -1\ 1\ 3\ 5\ \dots)(\dots\ 0\ 2\ 4\ 6\ \dots) & & \\
\sigma' & = & (3\ 4)(5\ 6)\dots & = & \tau\sigma\tau^{-1} \\
(1\ 2) & = & \sigma\sigma'^{-1} & = & [\sigma,\tau]
\end{array}$$

$$\begin{array}{ccl}
[a,b] & = & aba^{-1}b^{-1} \\
 & = & (ab)^2b^{-1}a^{-2}b^{-1} \\
 & = & (ab)^2b^{-2}ba^{-2}b^{-1} \\
 & = & (ab)^2b^{-2}(ba^{-1}b^{-1})^2
\end{array}$$

□

Kapoudjian class

$$1 \to \mathfrak{S}_{\text{fin}} \to \mathfrak{S}(X) \to \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}} \to 1$$

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathfrak{S}(X)/\mathfrak{A}_{\text{fin}} \to \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}} \to 1$$

This exact sequence gives us a central extension of $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$ by $\mathbb{Z}/2\mathbb{Z}$ thus an element of the cohomology group $H^2(\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}, \mathbb{Z}/2\mathbb{Z})$. We called this element the Kapoudjian class of $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$.

This element is not trivial because the exact sequence does not split.

Kapoudjian class

Let G be a subgroup of $\mathfrak{S}(X)/\mathfrak{S}_{\mathrm{fin}}$ and we denote by $\widehat{G}$ its preimage in $\mathfrak{S}(X)$. We have :

$$1 \to \mathfrak{S}_{\mathrm{fin}} \to \widehat{G} \to G \to 1$$

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widehat{G}/\mathfrak{A}_{\mathrm{fin}} \to G \to 1$$

Similarly we define the Kapoudjian class of $G \in H^2(G, \mathbb{Z}/2\mathbb{Z})$.

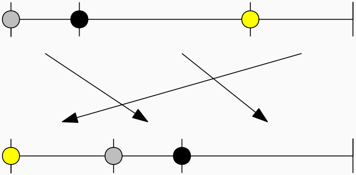
Proposition
The Kapoudjian class of G is trivial if and only if there exists a group homomorphism from $\widehat{G}$ onto $\mathbb{Z}/2\mathbb{Z}$ that extends $\varepsilon_{\mathrm{fin}}$.

Subgroups of $\mathfrak{S}([0,1[)$

IET

The group IET (Interval Exchange Transformations) is defined as the subgroup of $\mathfrak{S}(X)$ consisting of elements that are :

1. continuous outside a finite set;
2. right continuous;
3. piecewise a translation.

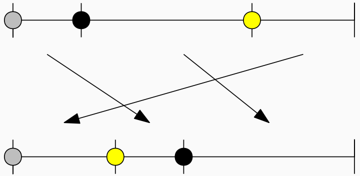


We notice that IET does not contain $\mathfrak{S}_{\text{fin}}$.

IET^+

The group $\widehat{\mathrm{IET}^+}$ is defined as the subgroup of $\mathfrak{S}(X)$ consisting of elements that are :

1. continuous outside a finite set;
2. piecewise a translation.



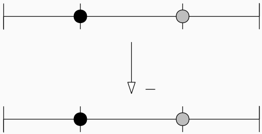
We can notice that $\mathrm{IET}^+ = \widehat{\mathrm{IET}^+}/\mathfrak{S}_{\mathrm{fin}}$ is isomorphic to IET.

$\mathrm{IET}^{\bowtie}$

The group $\widehat{\mathrm{IET}^{\bowtie}}$ is defined as the subgroup of $\mathfrak{S}(X)$ consisting of elements that are :

1. continuous outside a finite set;
2. piecewise isometric.

reflection :

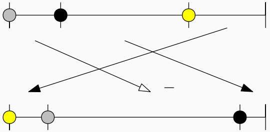


We notice that $\widehat{\mathrm{IET}^{\bowtie}}$ is generated by $\widehat{\mathrm{IET}^{+}}$ and reflections.

$\mathrm{IET}^{\bowtie}$ is the group of all Intervals Exchange Transformations with flips.

$\mathrm{PC}^{+}, \mathrm{PC}^{\bowtie}$

$\widehat{\mathrm{PC}^{\bowtie}}$ is the subgroup of $\mathfrak{S}(X)$ consisting of all piecewise continuous elements and $\widehat{\mathrm{PC}^{+}}$ is the subgroup of $\widehat{\mathrm{PC}^{\bowtie}}$ consisting of all elements that are piecewise increasing function.



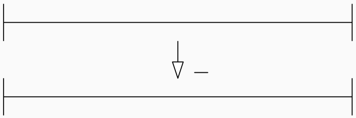
Proposition

We have the following equalities :

1. $\widehat{\mathrm{PC}^{+}} = \mathrm{Homeo}^{+}(\mathbb{S}^1)\widehat{\mathrm{IET}^{+}}$;
2. $\widehat{\mathrm{PC}^{\bowtie}} = \mathrm{Homeo}^{+}(\mathbb{S}^1)\widehat{\mathrm{IET}^{\bowtie}}$

A signature for $\widehat{\mathsf{IET}}^{\bowtie}$

Intuition for the value on reflections



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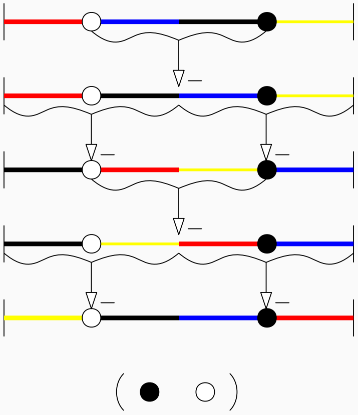


Associated partitions

Let $f \in \widehat{\mathrm{IET}^{\bowtie}}$ and $\mathcal{P}$ be a finite partition into intervals of $[0, 1[$. We said that $\mathcal{P}$ is a partition associated with f if f is continuous on the interior of every interval of $\mathcal{P}$. We denote by $f(\mathcal{P})$ the arrival partition.

We denote by Π_f the set of all partition associated with f.

There exists a unique partition associated with f that has a minimal number of interval denoted by $\mathcal{P}_f^{\min}$. This partition is also minimal for the refinement.

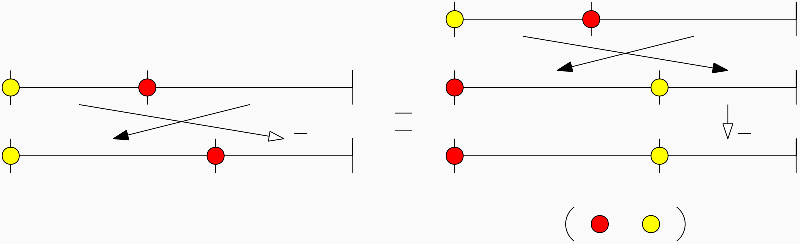
Associated partitions

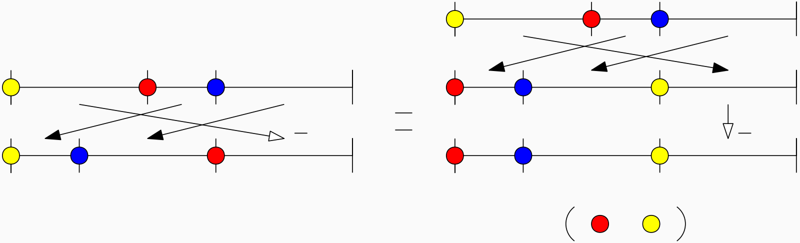
For every subinterval I of $[0,1[$, we denote by r_I is the reflection of the interval I.

Proposition
For every $f \in \widehat{\mathrm{IET}^{\bowtie}}$ and $\mathcal{P} \in \Pi_f$, there exist a unique $\sigma_{(f,\mathcal{P})} \in \mathfrak{S}_{\mathrm{fin}}$ and a unique subset $A(f,\mathcal{P})$ of $f(\mathcal{P})$ such that $\sigma_{(f,\mathcal{P})}(\prod_{I \in A(f,\mathcal{P})} r_I)f$ belongs to IET *(right continuous and piecewise a translation).*

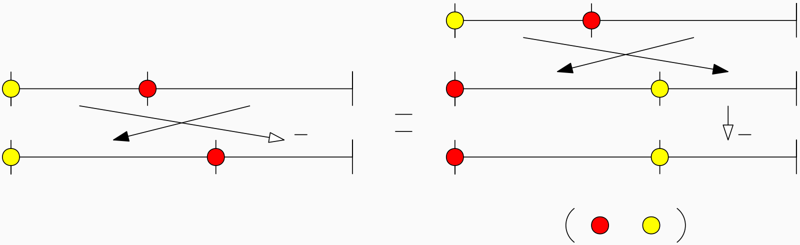
The permutation $\sigma_{(f,\mathcal{P})}$ is called the default of pseudo-right continuity of f in regards to $\mathcal{P}$. We denote by $R(f,\mathcal{P})$ the cardinal of $A(f,\mathcal{P})$.

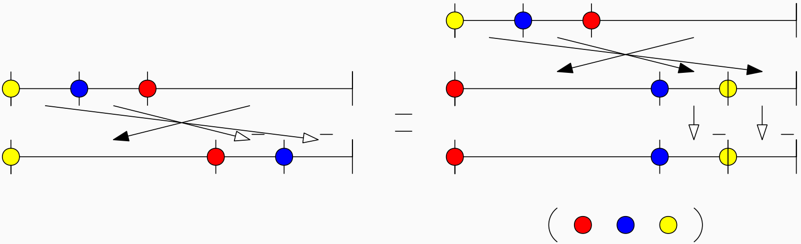
Associated partitions





Associated partitions





The signature

Lemma
Let $f \in \widehat{\mathsf{IET}}^{\bowtie}$ *and* $\mathcal{P} \in \Pi_f$*. The value* $\varepsilon_{\mathrm{fin}}(\sigma_{(f,\mathcal{P})}) + R(f,\mathcal{P})$ [mod 2] *does not depend on* $\mathcal{P}$*.*

We define $\varepsilon(f) = \varepsilon(\sigma_{(f,\mathcal{P}_f^{\min})}) + R(f,\mathcal{P}_f^{\min})$ [mod 2].

Group homomorphism

Proposition

Let $f, g \in \widehat{\mathrm{IET}^{\bowtie}}$. There exists $\mathcal{P} \in \Pi_g$ such that $g(\mathcal{P}) \in \Pi_f$.

Proof.
Let A be the set of discontinuities of f. We can always refine $\mathcal{P}_g^{\min}$ in a partition $\mathcal{P}$ in order to have $g^{-1}(A)$ inside the set of endpoints of intervals in $\mathcal{P}$. Hence $g(\mathcal{P})$ is a refinement of $\mathcal{P}_f^{\min}$, thus it is inside Π_f. □

Then for this partition we obtain :

1. $R(f \circ g, \mathcal{P}) = R(f, g(\mathcal{P})) + R(g, \mathcal{P})$ [mod 2] ;
2. $\sigma_{(f \circ g, \mathcal{P})} = \sigma_{(f, g(\mathcal{P}))} \circ f \circ \sigma_{(g, \mathcal{P})} \circ f^{-1}$;
3. $\varepsilon(f \circ g) = \varepsilon(f) + \varepsilon(g)$.

Theorem (L. 2020)
The map ε is a group homomorphism that extends $\varepsilon_{\mathrm{fin}}$.

Normal subgroups

Normal subgroups of $\widehat{\mathsf{IET}^{\bowtie}}$

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widehat{\mathsf{IET}^{\bowtie}}/\mathfrak{A}_{\text{fin}} \to \mathsf{IET}^{\bowtie} \to 1$$

$$\widehat{\mathsf{IET}^{\bowtie}}/\mathfrak{A}_{\text{fin}} \simeq \mathsf{IET}^{\bowtie} \times \mathbb{Z}/2\mathbb{Z}$$

Theorem (Arnoux 1981)
The group $\mathsf{IET}^{\bowtie}$ *is simple.*

Theorem (L. 2020)
The group $\widehat{\mathsf{IET}^{\bowtie}}$ *has five normal subgroups given by the list :*

$$\{\{1\}, \mathfrak{A}_{\text{fin}}, \mathfrak{S}_{\text{fin}}, \mathsf{Ker}(\varepsilon), \widehat{\mathsf{IET}^{\bowtie}}\}$$

More generally

Theorem (L. 2020)
There exists a group homomorphism ε *from* $\widehat{\mathsf{PC}^{\bowtie}}$ *onto* $\mathbb{Z}/2\mathbb{Z}$ *that extends* ε_{fin}.

Theorem (L. 2020)
Let G *be a subgroup of* $\mathsf{PC}^{\bowtie}$ *that is simple nonabelian. Then* $\widehat{G}$ *has exactly five normal subgroups given by the list :*

$$\{\{1\}, \mathfrak{A}_{\text{fin}}, \mathfrak{S}_{\text{fin}}, \mathsf{Ker}(\varepsilon|_{\widehat{G}}), \widehat{G}\}$$

Thanks for your attention!

Precisions about Vitali's remark i

Let $\sigma \in \mathfrak{S}(\mathbb{Z})$. Its decomposition into disjoint cycles may contain finite cycle, infinite cycle with finite/infinite number of fixed points.

The product of two cycles of the same length $\ell \in \mathbb{N} \cup \{\infty\}$, is a square.

$$(\dots\ a_{-1}\ a_0\ a_1\ \dots)(\dots\ b_{-1}\ b_0\ b_1\ \dots) = (\dots\ a_{-1}\ b_{-1}\ a_0\ b_0\ a_1\ b_1\ \dots)^2$$

Let $n \in 2\mathbb{N} \cup \{\infty\}$ we have that any product of n disjoint cycles of same length (length in $\mathbb{N} \cup \{\infty\}$) is a square.

An infinite cycle with finite number of fixed points is the product of an infinite cycle with infinite fixed points with an infinite product of disjoint transpositions.

$$(\dots\ p_0\ q_0\ p_1\ q_1\ \dots) = (\dots\ (p_0\ q_0)\ (p_1\ q_1)\ \dots)\ (\dots\ q_0\ q_1\ \dots)$$

Precisions about Vitali's remark ii

An infinite product of finite cycles is the product of an infinite cycle with an infinite product of disjoint transpositions.

$$((a_1\ a_2\ \ldots\ a_{r_1})\ (a_{r_1+1}\ a_{r_1+2}\ \ldots\ a_{r_2})\ldots) =$$
$$((a_{r_1}\ a_{r_1+1})\ (a_{r_2}\ a_{r_2+1})\ \ldots)(\ldots\ a_{r_2}\ a_{r_1}\ a_1\ a_2\ \ldots\ a_{r_1-1}\ a_{r_1+1}\ \ldots a_{r_2-1}\ldots)$$

Any cycle with infinite fixed points is a commutator thus the product of three squares. There exists $d_1, d_2, \ldots$ disjoint cycles whose length are the length of c. Let $\Sigma = d_1 d_2 \ldots$ and $\Sigma' = c d_1 d_2 \ldots$. Then Σ and Σ' are conjugated and $c = \Sigma' \Sigma^{-1}$ is a commutator.

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