Invariant measures for horospherical actions and Anosov groups.

(Joint work with Hee Oh)

Introduction.

e.g.

G : connected, semisimple linear Lie group. $G = SL_3(\mathbb{R})$

Γ : Zariski dense discrete subgroup of G.

N : maximal horospherical subgroup of G. $N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

$\exists a \in G$ s.t.

$$N = \{ g \in G : a^k g a^{-k} \to e \text{ as } k \to \infty \}$$

We are interested in the "measure rigidity" problem for

$\Gamma \backslash G \curvearrowleft N$

measure = Radon measure
(locally finite Borel measure)

Previously known

1) when Γ is a lattice,
$\mathrm{Vol}(\Gamma \backslash G) < \infty$.

All N-invariant, ergodic measures are completely classified.
(Dani, Ratner)

2) When $\begin{cases} G \text{ is of rank } 1 \\ \Gamma \text{ is geometrically finite} \end{cases}$

$G = PSL(2,\mathbb{R})$
$PSL(2,\mathbb{C})$
$SO^\circ(n,1)$

$\exists!$ nontrivial N-invariant ergodic measure, called Burger Roblin measure on $\Gamma \backslash G$.
(Burger, Roblin, Winter)

3) When $\begin{cases} G \text{ is of rank } 1 \\ \Gamma \text{ is geometrically infinite} \end{cases}$

Babillot, Ledrappier :
$\exists$ continuous family of N-invariant ergodic measure on $\Gamma\backslash G$.

$\exists$ complete classification for certain classes.
(Sarig, Oh-Pan, Landesberg-Lindenstrauss)

Less is known

when G is of rank ≥ 2

and Γ is <u>not</u> a lattice.

(If μ : N-inv, ergodic meas on $\Gamma\backslash G$

& $|\mu| < \infty$ by Ratner).

In the rest of the talk,
I will focus on class of discrete subgroups of ∞-covolume called <u>Anosov groups</u>.
(Generalization of convex cocompact subgroups in rank 1 case).

<u>Anosov groups</u>

$G = SL(3, \mathbb{R})$

P minimal parabolic subgroup of G

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

G/P Furstenberg boundary.

(Benoist) $\exists!$ <u>Γ-minimal set</u> inside G/P.
called limit set $\Lambda (= \Lambda_\Gamma)$

$\exists!$ open G-orbit inside $G/P \times G/P$ $\circlearrowleft G$

$G.(P^+, P^-)$ $\quad P^+ = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$ $\quad P^- = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$

We say

$(\xi_1, \xi_2) \in G/P \times G/P$ is in a general position if

(ξ_1, ξ_2) lies in the unique open G-orbit.

$G = SL(2, \mathbb{R})$

$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\quad G/P \simeq \partial \mathbb{H}^2$

$\{ (x,y) \in \underline{\partial \mathbb{H}^2 \times \partial \mathbb{H}^2} : x \neq y \}$

We say $\Gamma < G$ <u>Anosov subgroup</u> if

1) Γ is a <u>word hyperbolic group</u>
   $\Gamma = \pi_1(S_g)$

2) $\exists$ Γ-equivariant embedding

$$\xi : \overset{\partial \mathbb{H}^2}{\partial \Gamma} \longrightarrow G/P$$

s.t. $(\xi(x), \xi(y))$ is in general position whenever $x \neq y \in \partial \Gamma$.

Examples Hitchin representation.

1) $S_g =$ 

genus g surface.

$\Gamma = \pi_1(S_g)$

$$\begin{tikzcd} \Gamma \arrow[r, "\text{Holonomy representation}"] \arrow[dr, dashed, "\rho"'] & PSL(2,\mathbb{R})/\sim \arrow[d, "\text{irreducible representation}"] \\ & PSL(n,\mathbb{R})/\sim \end{tikzcd}$$

$\rho \in \underline{Hom(\Gamma, PSL(n,\mathbb{R}))/\sim}$

The connected component of containing ρ is called <u>Hitchin component</u>.

2) Schottky group

We can choose $g_1, \cdots, g_k \in G$ "loxodromic" element

s.t.

$$\Gamma := \langle g_1^{\pm 1}, \cdots, g_k^{\pm 1} \rangle \simeq \mathbb{F}_k$$

(Benoist)

<u>Main theorem</u>

$G = SL_3(\mathbb{R})$

$$P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

$P = N\,A\,M$ (Langlands decomposition)

N : unipotent radical of P

$$N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

A : maximal $\mathbb{R}$-split torus

$$A = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

M : maximal cpt subgroup that commutes with A

$$M = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

$$\mathcal{E} := \{ [g] \in \Gamma\backslash G : \underline{\underline{gP}} \in \Lambda \}$$

$(g \in G)$ $\quad$ $\Lambda \subseteq G/P$

$\exists!$ (Γ-minimal subset)

$\mathcal{E}$ is the unique P-minimal subset of $\Gamma\backslash G$.

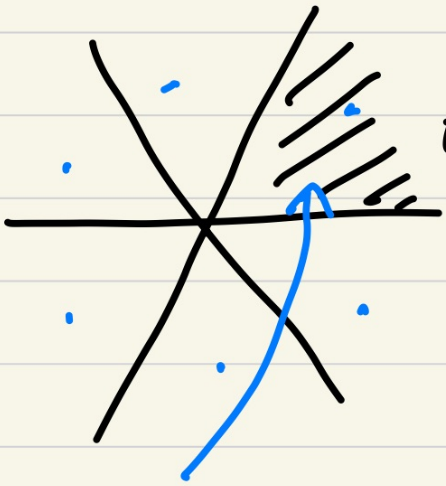
$\boxed{\Gamma\backslash G \longleftrightarrow G/P \quad \text{duality}}$

An N-invariant measure μ on $\Gamma\backslash G$ is called nontrivial if μ is supported on $\mathcal{E}$.

$$\operatorname{supp} \mu \subset \mathcal{E}.$$

Patterson Sullivan theory

$\mathfrak{a} = \mathrm{Lie}(A)$ $\quad A = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$



$\mathfrak{a}^+$

$$\mathfrak{a} = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_1+t_2+t_3=0 \right\} \simeq \mathbb{R}^2$$

$t_1 \geq t_2 \geq t_3$.

ψ : linear form on $\mathfrak{a}$ $(\psi \in \mathfrak{a}^*)$

A measure ν on G/P is called (Γ, ψ)-PS measure if

① ν is supported on Λ

② $$\boxed{\frac{d(\gamma.\nu)}{d\nu}(\xi) = e^{\underline{\psi(\beta_\xi(o,\gamma o))}}}$$

$e^{\boxed{\text{cocycle}}}$

$$\left(\begin{array}{l}\text{for all } \gamma \in \Gamma \\ \text{and all } \xi \in \Lambda\end{array}\right)$$

- ν is Γ-quasi-inv
- $\beta_\xi(o, \gamma o)$ is an analogue of Busemann function in rank 1 case.

  $\cap$

  $\mathfrak{a}$

Cartan projection

$$\underline{G = K A^+ K}$$

$G = SL(3, \mathbb{R})$

$K = SO(3, \mathbb{R})$

$$A^+ = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

$a_1 \geq a_2 \geq a_3$.

For ${}^\forall g \in G$,

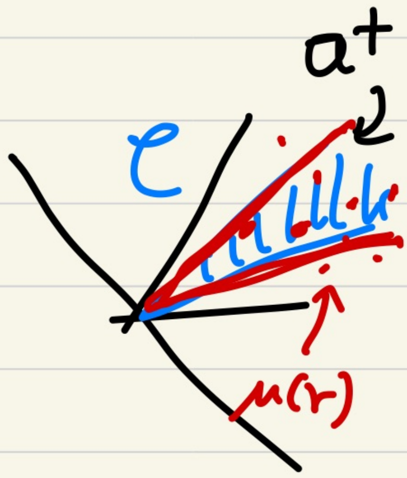
$\exists! \boxed{\underline{\mu(g)}} \in \mathfrak{a}^+$ s.t.

$$g \in \underline{K}\, \underline{e^{\underline{\mu(g)}}} \cdot \underline{K}$$

~ critical exponent δ_Γ

Growth indicator $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R}$

is defined by



① For $\forall \mathcal{C} \subseteq \mathfrak{a}^+$

$$h_{\mathcal{C}} = \limsup_{N\to\infty} \frac{1}{N} \log \#\{\gamma\in\Gamma : \mu(\gamma)\in\mathcal{C}, \|\mu(\gamma)\| \le N\}$$

exp. growth rate of $\{\mu(\gamma) : \gamma\in\Gamma\}$

② For $x \in \mathfrak{a}$,

$$\psi_\Gamma(x) := \|x\| \cdot \inf_{x\in\mathcal{C}} h_{\mathcal{C}}$$

$$\psi_\Gamma(tx) = t\,\psi_\Gamma(x)$$

for $t\in\mathbb{R}$.

Parametrization

$$D_\Gamma^\star := \left\{ \psi \in \mathfrak{a}^* : \begin{array}{l} \underline{\psi \geq \psi_\Gamma} \\ \underline{\psi(x) = \psi_\Gamma(x)} \\ \text{for some } x \in \operatorname{int} \mathcal{L}_\Gamma \end{array} \right\}$$

where ($\mathcal{L}_\Gamma$: limit cone) is defined to be the asymptotic cone of $\{\mu(\gamma) : \gamma \in \Gamma\}$

$\Gamma < G$ is Anosov,

① $D_\Gamma^\star \underset{\text{homeo}}{\simeq} \mathbb{P}(\operatorname{int} \mathcal{L}_\Gamma)$

② (Quint) For $\forall \psi \in D_\Gamma^\star$, there exists a unique

(Γ, ψ) – PS measure ν_ψ on G/P

<u>Thm</u> (L.–Oh) The map

$$\begin{aligned} D_\Gamma^\star &\to \left\{ \begin{array}{c} \text{PS measures} \\ \text{on } G/P \end{array} \right\} \\ \psi &\mapsto \nu_\psi \end{aligned}$$

is a homeomorphism.

For $\psi \in D_\Gamma^\star$

Define Burger Roblin measure by (m_ψ^{BR})

$$\begin{array}{ccl} G & \simeq & G/P \times P \ni nam \\ \pi \downarrow & & e^{\psi(\log a)}\, dn\, da\, dm \\ G/P & & \end{array}$$

$G/P \leftarrow \nu_\psi$

$$\boxed{dm^{BR}(g) = d\nu_\psi(gP)\, e^{\psi(\log a)}\, dn\, da\, dm}$$

This is left Γ-invariant,

($\because$ ν_ψ is (Γ,ψ)-PS)

right NM-invar, A-quasi-inv.

<u>Thm</u>. (L.–Oh) The map

$$\begin{array}{ccc} D_\Gamma^{\star} & \longrightarrow & \left(\begin{array}{c}\text{Space of all}\\ NM\text{-invariant}\\ \text{ergodic, } A\text{-q.i.}\\ \text{measure}\end{array}\right) \\ \Psi & & \\ \psi & \longmapsto & \underline{[m_\psi^{BR}]} \end{array}$$

N

is a homeomorphism

AMN – q.i. ergodic

NM-inv A-q.i

AMN-ergodic.