**Exercise 24.11. (a)** Notice \( f_n(x) \equiv f(x) \) and thus \( f_n \to f \) uniformly.

To show \( g_n \to g \) uniformly, notice for any \( \epsilon > 0 \), pick \( N > 1/\epsilon \), we have for any \( n > N \)

\[
|g_n(x) - g(x)| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \epsilon
\]

which proves uniform continuity.

**(b)** Since \( f_n(x)g_n(x) = \frac{x}{n^2} \), and \( f(x)g(x) \equiv 0 \), therefore, for any \( \epsilon < 1 \) and any \( n \), pick \( x = n^2 \), and \( f_n(x)g_n(x) = n > 1 \), which means the convergence is not uniform.

This is just the same as Exercise 24.2.

**Exercise 24.13.** Pick and fix any \( \epsilon > 0 \). Since \( f_n \to f \) uniformly, there exists \( N = N(\epsilon) \), such that, for any \( n \geq N \), we have

\[
|f_n(x) - f(x)| < \frac{\epsilon}{3}
\]

Particularly, this is true when \( n = N \). Also, since \( f_N \) is uniformly continuous over the interval \((a,b)\), we have \( \delta = \delta(\epsilon, N) \), such that for any \( x, y \in (a,b) \), \( |x - y| < \delta \), we have

\[
|f_N(x) - f_N(y)| < \frac{\epsilon}{3}
\]

Now, for any \( x, y \in (a,b), |x - y| < \delta \), we have

\[
|f(x) - f(y)| = |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)|
\]

use triangle inequality

\[
\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|
\]

use uniform convergence for the first and the third terms, and use uniform continuity of \( f_N \) for the second term

\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\]

which proves uniform continuity of \( f \).
Exercise 24.14. (a) If $x = 0$, we have

$$f_n(0) = 0 \to 0$$

If $x > 0$, fix $x$. We have

$$0 \leq \lim_{n \to \infty} \frac{nx}{1 + n^2x^2} \leq \lim_{n \to \infty} \frac{nx}{n^2x^2} \leq \lim_{n \to \infty} \frac{1}{nx} = 0$$

The last equality depends on the fact that $x > 0$. Hence, by squeezing theorem, we have, for $x > 0$

$$\lim_{n \to \infty} \frac{nx}{1 + n^2x^2} = 0$$

Combine both cases, we have for any $x \in [0, \infty)$

$$f(x) = \lim_{n \to \infty} f_n(x) = 0$$

This convergence is pointwise.

(b) For any $\epsilon < \frac{1}{2}$, and any $n \in \mathbb{N}$, pick $x = \frac{1}{n}$, we have

$$f_n\left(\frac{1}{n}\right) = \frac{n \cdot \frac{1}{n}}{1 + n^2 \cdot \frac{1}{n^2}} = \frac{1}{2} > \epsilon$$

which shows the convergence is not uniform.

(c) For any $\epsilon > 0$, pick $N > 1/\epsilon$, we have for any $n > N$

$$f_n(x) = \frac{nx}{1 + n^2x^2} \leq \frac{nx}{n^2x^2} = \frac{1}{n} \cdot \frac{1}{x}$$

use the fact that $x \geq 1$

$$\leq \frac{1}{n} < \epsilon$$

which proves uniform convergence.

Exercise 25.2. I am going to claim the limit function is $f(x) \equiv 0$. For any $\epsilon > 0$, pick $N > 1/\epsilon$. We have for any $n > N$

$$|f_n(x) - f(x)| = \left| \frac{x^n}{n} \right| = \frac{|x^n|}{n}$$

notice $x \in [-1, 1]$, and hence $x^n \in [-1, 1]$

$$\leq \frac{1}{n} < \epsilon$$

which proves uniform convergence.
**Exercise 25.4.** For any $\epsilon > 0$, due to uniform convergence, there exists $N > 0$, such that for any $n > N$, we have, for any $x \in S$

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

Thus, for any $n, m > N$, we have for any $x \in S$

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$$

use triangle inequality

$$\leq |f_n(x) - f(x)| + |f_m(x) - f(x)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which proves the uniform Cauchy property.

**Exercise 25.5.** Pick $\epsilon = 1$. Due to uniform convergence, there exists $N$, such that, for any $n \geq N$, and $x \in S$, we have

$$|f_n(x) - f(x)| \leq \epsilon = 1$$

In particular, this holds for $n = N$. Since $f_N$ is bounded, there exists $M = M(N)$, such that $|f_N(x)| \leq M, \forall x \in S$. Hence, for all $x \in S$

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)|$$

$$\leq 1 + M$$

which proves boundedness.

**Exercise 25.6. (a)** Since $x \in [-1, 1]$, we have $|a_kx^k| \leq |a_k|$. According to Weierstrass M-test, the convergence is uniform. Since each term $a_kx^k$ is a continuous function of $x$, by Theorem 25.5, the uniformly convergent limit of continuous functions is again a continuous function, which completes the proof.

**(b)** Notice $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, and the original series a continuous function by part(b).

**Exercise 25.8.** First, let’s consider the radius of convergence.

$$\lim_{n \to \infty} \sqrt[n]{n^22^n} = \lim_{n \to \infty} \sqrt[n]{n^2} \cdot \lim_{n \to \infty} \sqrt[n]{2^n}$$

notice $\sqrt[n]{n^2} = (\sqrt[n]{n})^2 \to 1^2 = 1$

$$= 1 \cdot 2$$

Therefore, the radius is 2. Notice, here we did not use lim sup, since the limit itself exists.
Now we are left to show the convergence is uniform and the limit is a continuous function on \([-2, 2]\). We have
\[
\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n} = \frac{1}{n^2} \cdot \left(\frac{x}{2}\right)^n
\]
Since \(x \in [-2, 2]\), we have \(\left(\frac{x}{2}\right)^n \in [-1, 1]\), and \(|\frac{x^n}{n^2 2^n}| \leq \frac{1}{n^2}\). Now the conclusion follows from Exercise 25.6.