At this point, we will try to avoid using Riemann integrability, since it’s not yet covered in the lectures, and it is actually equivalent to Darboux integrability by Theorem 32.9.

Exercise 32.1. For a partition $P$, where

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Now, let’s first consider the upper Darboux sum

$$U(x^3, P) = \sum_{i=0}^{n-1} x_i^3(x_{i+1} - x_i) > \frac{1}{4}(x_i^3 + x_i^2x_{i+1} + x_i^2x_{i+1} + x_i^3)(x_{i+1} - x_i)$$

$$= \frac{1}{4} \sum_{i=0}^{n-1} x_{i+1}^4 - x_i^4 = \frac{1}{4} (x_n^4 - x_0^4) = \frac{b^4}{4}$$

which shows for all partition $P$, we have

$$U(x^3, P) > \frac{b^4}{4}$$

This gives the fact, after taking infimum over all possible partions, that

$$U(x^3) \geq \frac{b^4}{4}$$

Similarly, for the lower Darboux sum

$$L(x^3, P) = \sum_{i=0}^{n-1} x_i^3(x_{i+1} - x_i) < \frac{1}{4}(x_i^3 + x_i^2x_{i+1} + x_i^2x_{i+1} + x_i^3)(x_{i+1} - x_i)$$

$$= \frac{1}{4} \sum_{i=0}^{n-1} x_{i+1}^4 - x_i^4 = \frac{1}{4} (x_n^4 - x_0^4) = \frac{b^4}{4}$$

which shows for all partition $P$, we have

$$L(x^3, P) < \frac{b^4}{4}$$
This gives the fact, after taking supremum over all possible partitions, that

\[ L(x^3) \leq \frac{b^4}{4} \]

I shall give the other directions of inequalities for \( U(x^3) \) and \( L(x^3) \).

Suppose we have a sequence of "even partition" \( P_n \), where the partition points are

\[ x_i = \frac{i}{n} \cdot b, \quad i = 0, 1, \ldots, n \]

Now

\[ U(x^3, P_n) = \sum_{i=0}^{n-1} \left( \frac{i+1}{n} \cdot b \right)^3 \cdot \frac{b}{n} = \frac{b^4}{n^4} \sum_{i=0}^{n-1} (i + 1)^3 \]

we shall use the result of Exercise 1.3

\[ = \frac{b^4}{n^4} \sum_{i=1}^{n} i^3 = \frac{b^4}{n^4} \cdot \left( \frac{n(n+1)}{2} \right)^2 \]

\[ = \frac{b^4}{4} \cdot \frac{n^2(n+1)^2}{n^4} \rightarrow \frac{b^4}{4} \]

Therefore, as \( n \rightarrow \infty \), we have \( U(x^3, P_n) \rightarrow \frac{b^4}{4} \), which means, after taking infimum, \( U(x^3) \leq \frac{b^4}{4} \). Considering the other direction of the inequality, we can conclude that

\[ U(x^3) = \frac{b^4}{4} \]

Similarly

\[ L(x^3, P_n) = \sum_{i=0}^{n-1} \left( \frac{i}{n} \cdot b \right)^3 \cdot \frac{b}{n} = \frac{b^4}{n^4} \sum_{i=0}^{n-1} (i + 1)^3 \]

we shall use the result of Exercise 1.3

\[ = \frac{b^4}{n^4} \sum_{i=1}^{n} i^3 = \frac{b^4}{n^4} \cdot \left( \frac{n(n-1)}{2} \right)^2 \]

\[ = \frac{b^4}{4} \cdot \frac{n^2(n-1)^2}{n^4} \rightarrow \frac{b^4}{4} \]

Therefore, as \( n \rightarrow \infty \), we have \( L(x^3, P_n) \rightarrow \frac{b^4}{4} \), which means, after taking supremum, \( L(x^3) \geq \frac{b^4}{4} \). Considering the other direction of the inequality, we can conclude that

\[ L(x^3) = \frac{b^4}{4} \]

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In conclusion

\[ U(x^3) = \frac{b^4}{4} \quad L(x^3) = \frac{b^4}{4} \]

Actually, the justifications before the red words are not needed, because when we have

\[ U(x^3) \leq \frac{b^4}{4} \quad L(x^3) \geq \frac{b^4}{4} \]

we can immediately conclude that

\[ U(x^3) \leq \frac{b^4}{4} \leq L(x^3) \]

but since

\[ U(x^3) \geq L(x^3) \]

This actually gives the result that

\[ U(x^3) = L(x^3) = \frac{b^4}{4} \]

**Exercise 30.2.** (a) Let’s first consider the lower Darboux integral. Let the partition \( P \) consisting of the partition points

\[ 0 = x_0 < x_1 < x_2 < \cdots < x_n = b \]

Denote by \( M_i, m_i \), the supremum and infimum of the function value on the \( i \)-th interval, respectively.

\[ L(f, P) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i) = \sum_{i=0}^{n-1} 0 \cdot (x_{i+1} - x_i) = 0 \]

Therefore, the lower Darboux sum is always zero, regardless of the partition, hence we have

\[ L(f) = 0 \]

Now let’s consider the upper Darboux sum.

\[ U(f, P) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i) \]

due to the denseness of the irrationals, we have the fact that \( M_i = x_{i+1} \)

\[ = \sum_{i=0}^{n-1} x_{i+1}(x_{i+1} - x_i) > \frac{1}{2} \sum_{i=0}^{n} (x_i + x_{i+1})(x_{i+1} - x_i) \]

\[ = \frac{1}{2} \sum_{i=0}^{n-1} x_{i+1}^2 - x_i^2 \]

\[ = \frac{b^2}{2} \]
which shows

\[ U(f) \geq \frac{b^2}{2} \]

Now, find a sequence of “even partitions” \( P_n \) constituting of the partition points

\[ x_i = \frac{i}{b}, \quad i = 0, 1, \ldots, n \]

\[
U(f, P_n) = \sum_{i=0}^{n-1} \frac{b(i+1)}{n} \cdot \frac{b}{n} = \frac{b^2}{n^2} \sum_{i=0}^{n-1} (i + 1)
\]

\[
= \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2} \rightarrow \frac{b^2}{2}
\]

Taking the infimum, we know that

\[ U(f) \leq \frac{b^2}{2} \]

In conclusion

\[ U(f) = \frac{b^2}{2} \]

(b) It’s not integrable since \( U(f) \neq L(f) \).

**Exercise 32.5.** Omitted.

This is just a rephrase of the proof using Exercise 4.8. In essence, you still probably want to use Lemma 32.2 and Lemma 32.3.

**Exercise 32.6.** We always have the fact that

\[ L(f) \leq U(f) \quad U(f) \leq U_n \quad L(f) \geq L_n \]

Hence we have

\[ 0 \leq U(f) - L(f) \leq \lim(U_n - L_n) = 0 \]

This gives the fact of integrability since

\[ U(f) = L(f) \]

Actually, there is one thing we need to clarify; that is the value \( U(f) = L(f) \) is finite. Since

\[ L_1 \leq L(f) \leq U(f) \leq U_1 \]

which indeed proves the value is finite.

Notice, for every \( n \in \mathbb{N} \), we have

\[ L_n \leq L(f) \leq U(f) \leq U_n \]
By the definition of limit, \( \forall \epsilon > 0, \exists N \in \mathbb{N}_+ \), such that \( \forall n > N \), we have the fact that
\[
0 \leq U_n - L_n \leq \epsilon
\]
and hence
\[
0 \leq U_n - U(f) \leq \epsilon, \quad 0 \leq L(f) - L_n \leq \epsilon
\]
By the definition again, is shows
\[
\lim_{n \to \infty} U_n = U(f) = L(f) = \lim_{n \to \infty} L_n
\]
which by definition equals
\[
\int_a^b f
\]

**Exercise 32.7.** Notice, the hint is right, but it’s very strange that linearity is not introduced in this textbook until next section, and we will try to avoid using the hint. Interested students can consider the integral and integrability of \( f - g \), but as I said, I try to avoid linearity of integrals.

Assume, \( f \) and \( g \) take different values at \( y_1, y_2, \cdots y_k \in [a, b] \), and \(|g| < M\), and \(|f| < M\), for some \( M > 0 \) on \([a, b]\). That is \( M \) is an upper bound for both \(|f|\) and \(|g|\).

By Theorem 32.5 and the integrability of \( f \), we have \( \forall \epsilon > 0 \), there exists partition \( P \), such that
\[
U(f, P) - L(f, P) < \frac{\epsilon}{2}
\]
We can add more partition points, so that
\[
\text{mesh}(P) < \frac{\epsilon}{4kM}
\]
Notice, adding more partition points, by Lemma 32.2, preserves the inequality
\[
U(f, P) - L(f, P) < \frac{\epsilon}{2}
\]
Now, we stick to this particular partition, and suppose the partition points are
\[
a = x_0 < x_1 < \cdots < x_n = b
\]
Denote by \( M^f_i, M^g_i, m^f_i, m^g_i \), the suprema of \( f \) and \( g \) and the infimums of \( f \) and \( g \) on the \( i \)-th interval, respectively. Recall that
\[
U(f, P) - L(f, P) = \sum_{i=0}^{n-1} (M^f_i - m^f_i)(x_{i+1} - x_i)
\]
Now
\[
U(g, P) - L(g, P) = \sum_{i=0}^{n-1} (M^g_i - m^g_i)(x_{i+1} - x_i)
\]
\[
= \sum_{y_j \notin [x_i, x_{i+1}]} (M^g_i - m^g_i)(x_{i+1} - x_i) + \sum_{\exists y_j \in [x_i, x_{i+1}]} (M^g_i - m^g_i)(x_{i+1} - x_i)
\]
notice if \( y_j \notin [x_i, x_{i+1}] \), then \( f \) and \( g \) coincide on that interval

\[
< (U(f, P) - L(f, P)) + \sum_{\exists y_j \in [x_i, x_{i+1}]} (M^q_i - m^q_i)(x_{i+1} - x_i)
\]

notice there are at most \( k \) summands in the second sum, and the recall the boundedness of \( g \) and the mesh

\[
< \frac{\epsilon}{2} + k \cdot 2M \cdot \frac{\epsilon}{4kM} \\
= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
	his, again by Theorem 32.5, proves the integrability of \( g \).

Now we shall show that

\[
\int f = \int g
\]

Consider

\[
U(g, P) - U(f) = \left( U(g, P) - U(f, P) \right) + \left( U(f, P) - U(f) \right) \\
\leq \left( U(g, P) - U(f, P) \right) + \left( U(f, P) - L(f, P) \right) \\
< \sum_{\exists y_j \in [x_i, x_{i+1}]} (M^q_i - m^q_i)(x_{i+1} - x_i) + \frac{\epsilon}{2} \\
< k \cdot 2M \cdot \frac{\epsilon}{4kM} + \frac{\epsilon}{2} = \epsilon
\]

which means

\[
U(g, P) < U(f) + \epsilon
\]

Similarly

\[
L(g, P) - L(f) = \left( L(g, P) - L(f, P) \right) + \left( L(f, P) - L(f) \right) \\
\geq \left( L(g, P) - L(f, P) \right) + \left( L(f, P) - U(f, P) \right) \\
> \sum_{\exists y_j \in [x_i, x_{i+1}]} (m^q_i - m^f_i)(x_{i+1} - x_i) - \frac{\epsilon}{2} \\
> -k \cdot 2M \cdot \frac{\epsilon}{4kM} - \frac{\epsilon}{2} = -\epsilon
\]

which means

\[
L(g, P) > L(f) - \epsilon
\]

Comining these two together, we have

\[
L(f) - \epsilon < L(g, P) \leq L(g) = U(g) \leq U(g, P) < U(f) + \epsilon
\]

that is to say

\[
L(f) - \epsilon < L(g) = U(g) < U(f) + \epsilon
\]
Let \( \epsilon \to 0 \), and by the fact that \( U(f) = L(f) \), we have

\[
L(f) = L(g) = U(g) = U(f)
\]

This concludes the proof.

**Exercise 32.8.** Since \( f \) is integrable on \([c, d]\), by Theorem 32.5, for any \( \epsilon > 0 \), we will have a partition \( P \) constituting of partition points

\[
a = x_0 < x_1 < x_2 < \cdots < x_n = b
\]

such that

\[
U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon
\]

By Lemma 32.2, the above inequality is preserved if we add more partition points to \( P \), so we can without loss of generality, assume \( c \) and \( d \) are two of the partition points, say, for some \( 0 < k < l < n \), \( x_k = c, x_l = d \). Now, let \( Q \) be the partition of the interval \([c, d]\) with partition points

\[
c = x_k < x_{k+1} < \cdots < x_l = d
\]

and

\[
U(f, Q, [c, d]) - L(f, Q, [c, d]) = \sum_{i=k}^{l} (M_i - m_i)(x_{i+1} - x_i) \leq \sum_{i=0}^{n-1} (M_i - m_i)(x_{i+1} - x_i)
\]

\[
= U(f, P, [a, b]) - L(f, P, [a, b]) < \epsilon
\]

which, again by Theorem 32.5, proves the integrability of \( f \) over the interval \([c, d]\).