Math 142B Lecture, Wednesday, March 11, 2020

Theorem (Fundamental Theorem of Calculus, II)
Let \( f: [a, b] \rightarrow \mathbb{R} \) be an integrable function. Define \( F: [a, b] \rightarrow \mathbb{R} \) by \( F(x) = \int_a^x f(t) \, dt \). Then

1. \( F \) is continuous on \([a, b]\).
2. If \( f \) is continuous at some \( x_0 \in (a, b) \), then \( F \) is differentiable at \( x_0 \) and \( F'(x_0) = f(x_0) \).

Proof
1. Since \( f \) is bounded, \( \exists M > 0 \) such that \( |f(x)| \leq M \) for all \( x \in [a, b] \).

Let \( x_0 \in [a, b] \) and \( \epsilon > 0 \). Let \( x \in [a, b] \) with \( |x - x_0| < \frac{\epsilon}{M} \).

Assume \( x \geq x_0 \). Then we have

\[
|F(x) - F(x_0)| = \left| \int_{x_0}^x f(t) \, dt \right| \leq \int_{x_0}^x |f(t)| \, dt \\
\leq \int_{x_0}^x M \, dt = M \cdot (x - x_0) < \epsilon.
\]

Similarly, if \( x \leq x_0 \), then

\[
|F(x) - F(x_0)| < \epsilon.
\]

Thus, \( |F(x) - F(x_0)| < \epsilon \), \( \forall x \in [a, b] \) with \( |x - x_0| < \frac{\epsilon}{M} \).

This shows that \( F \) is continuous at \( x_0 \).

2. Assume \( f \) is continuous at \( x_0 \). Let \( \epsilon > 0 \).

Then \( \exists \delta > 0 \) such that if \( x \in [a, b] \) and \( |x - x_0| < \delta \), then \( |f(x) - f(x_0)| < \epsilon \).

Let \( x \in [a, b] \) with \( |x - x_0| < \delta \).

Assume \( x \geq x_0 \) (the case \( x \leq x_0 \) is similar).
Then \( \forall t \in [x_0, x] \), we have \( |t - x_0| \leq |x - x_0| < \delta \), hence \( |f(t) - f(x_0)| < \varepsilon \).

Since \( \frac{F(x) - F(x_0)}{x - x_0} = \int_{x_0}^{x} \frac{f(t)\, dt}{x - x_0} \), we get that

\[
\left| \frac{F(x) - F(x_0) - f(x_0)}{x - x_0} \right| = \left| \int_{x_0}^{x} \frac{f(t)\, dt}{x - x_0} - \frac{\int_{x_0}^{x} f(t)\, dt - \int_{x_0}^{x_0} f(x_0)\, dt}{x - x_0} \right|
\leq \int_{x_0}^{x} |f(t) - f(x_0)|\, dt \frac{1}{x - x_0}
\leq \int_{x_0}^{x} \varepsilon \, dt \frac{1}{x - x_0} = \varepsilon.
\]

In conclusion, \( \forall \varepsilon > 0, \exists \delta > 0 \) such that if \( x \in [a, b] \) and \( |x - x_0| < \delta \), then \( \left| \frac{F(x) - F(x_0) - f(x_0)}{x - x_0} \right| < \varepsilon \).

This shows that \( \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0) \), that is, \( F \) is differentiable at \( x_0 \) and \( F'(x_0) = f(x_0) \). \( \boxdot \)
Theorem 2 (change of variables)
Let \( u : J \to \mathbb{R} \) be a differentiable function, where \( J \) is an open interval, such that \( u' \) is continuous on \( J \).
Let \( f : I \to \mathbb{R} \) be a continuous function, where \( I \) is an open interval such that \( u(x) \in I \), \( \forall x \in J \).
Then
1. \( f \circ u : J \to \mathbb{R} \) is continuous and
2. \( \int_{u(a)}^{u(b)} f(u) \, du = \int_{a}^{b} f(u(x)) \cdot u'(x) \, dx \), \( \forall a < b \), \( u \in \mathcal{C}(J) \).

Proof: We already proved (1). For part (2), define
\[
F(u) = \int_{u(a)}^{u(b)} f(t) \, dt, \quad \forall u \in I.
\]
Since \( F \) is continuous \( \implies \) \( F \) is differentiable on \( I \) and
\[
F'(u) = f(u), \quad \forall u \in I.
\]
Thus, we have
\[
\int_{u(a)}^{u(b)} f(u) \, du \quad \overset{\text{FTC, I}}{=} \quad F(u(b)) - F(u(a))
\]
\[
\overset{\text{FTC, I}}{=} \int_{a}^{b} (F \circ u)' \, du
\]
\[
\overset{\text{chain rule}}{=} \int_{a}^{b} F'(u) \cdot u' \, du
\]
\[
= \int_{a}^{b} f(u(x)) \cdot u'(x) \, dx \quad \blacklozenge
\]
EXAMPLE Calculate \( \int_{0}^{1} \sqrt{1-u^2} \, du \).

Let \( u(x) = \sin x \), then \( u'(x) = \cos x \), so applying the change of variables formula gives that

\[
\int_{0}^{1} \sqrt{1-u^2} \, du = \int_{0}^{\frac{\pi}{2}} \sqrt{1-\sin^2 x \cdot \cos x} \, dx
\]

\[
= \int_{0}^{\frac{\pi}{2}} \sqrt{1-\sin^2 x} \cdot \cos x \, dx
\]

\[
= \int_{0}^{\frac{\pi}{2}} \cos^2 x \, dx
\]

\[
= \int_{0}^{\frac{\pi}{2}} \frac{\cos 2x + 1}{2} \, dx
\]

\[
= \left( \frac{\sin 2x}{4} + \frac{x}{2} \right) \bigg|_{0}^{\frac{\pi}{2}}
\]

\[
= \frac{\pi}{4} \cdot \Box
\]