Solutions to practice midterm 1

1. (1) F, (2) T, (3) F, (4) T, (5) T

2. see Theorem 23.1, Definition 24.1, Definition 24.2, Definition 25.3, top of page 205 and Theorem 25.6 in the textbook.

3. \[ \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x + \frac{1}{n}} = \frac{0}{x+0} = 0, \text{ for all } x \in (0, \infty) \]

3.6 Let \( \delta > 0 \). Let \( \epsilon > 0 \).

Then for \( \forall n > \frac{1}{\epsilon \delta} \) and \( \forall x \in (\delta, \infty) \) we have

\[ |f_n(x) - f(x)| = \left| \frac{1}{nx+1} \right| = \frac{1}{nx+1} \]
\[ \leq \frac{1}{nx} \]
\[ \leq \frac{1}{\delta \epsilon} \quad (\text{since } x > \delta) \]
\[ < \frac{1}{\epsilon \delta} \quad (\text{since } n > \frac{1}{\epsilon \delta}) \]

This shows that \( f_n \to f \) uniformly on \((\delta, \infty)\).
(3) Assume by contradiction that $f_n \to f$ uniformly on $(0, +\infty)$.

Then $\exists N \in \mathbb{N}$ such that

$$\frac{1}{nx+1} = |f_n(x) - f(x)| < \frac{1}{2}, \forall x \in (0, +\infty).$$

However, if $x = \frac{1}{n}$, then

$$\frac{1}{nx+1} = \frac{1}{n \cdot \frac{1}{n} + 1} = \frac{1}{2}$$

which gives a contradiction.

(4) Since $f$ is bounded, $\exists M$ such that

Let $\varepsilon > 0$. $|f(x)| \leq M \ \forall x \in S$.

Since $f_n \to f$ uniformly, $\exists N$ such that

$$|f_n(x) - f(x)| < \min \left\{ 1, \frac{\varepsilon}{1 + 2M} \right\} \ \forall n > N \text{ and } x \in S.$$

Thus, $\forall n > N$ we have

$$|f_n(x) + f(x)| = |f_n(x) - f(x) + 2f(x)|$$

$$\leq |f_n(x) - f(x)| + 2|f(x)|$$

$$< 1 + 2M, \forall x \in S.$$ 

Hence, $\forall n > N$ we get that

$$|f_n(x)^2 - f(x)^2| = |f_n(x) + f(x)| \cdot |f_n(x) - f(x)|$$

$$\leq (1 + 2M) \cdot |f_n(x) - f(x)|$$

$$< (1 + 2M) \cdot \frac{\varepsilon}{1 + 2M} = \varepsilon, \forall x \in S.$$ 

This shows that $f_n^2 \to f^2$ uniformly on $S$. 

4.(b) \( \beta = \limsup \left( \frac{1}{2^n} \right)^{\frac{1}{n^2}} = \limsup \frac{1}{2^{\frac{1}{n}}} = \frac{1}{1} = 1 \), so \( R = \frac{1}{\beta} = 1 \).

Moreover, for all \( n \in \mathbb{N} \) we have \( |\frac{x^{n^2}}{2^n}| \leq \frac{1}{2^n} \) for every \( x \in [-1,1] \).

Since \( \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \), the Weierstrass M-test implies that \( \sum_{n=1}^{\infty} \frac{x^{n^2}}{2^n} \) converges uniformly on \([-1,1]\).

5.(a) \( \beta = \limsup \left( \frac{1}{2^n n!} \right)^{\frac{1}{2n}} \)

\[
= \limsup \left( \frac{1}{\sqrt{2^n n!}} \right)^{\frac{1}{n}}
\]

\[
= \lim \frac{1}{\sqrt[2n]{2^n n!}} = \lim \frac{1}{\sqrt{2 (n+1)}} = 0
\]

Thus, \( R = \frac{1}{\beta} = +\infty \).
5.6 Let $n \in \mathbb{N}$. Then \( \sup_{x \in \mathbb{R}} \left\{ \frac{x^{2n}}{(2^nn!)} \right\} = +\infty \).

Indeed, if $M > 0$, then $\exists x \in \mathbb{R}$ such that $\frac{x^{2n}}{2^nn!} > M$.

Take any $x \geq \left( \frac{M}{2^nn!} \right)^{\frac{1}{2n}}$.

Hence, the set \( \left\{ \frac{x^{2n}}{2^nn!} \mid x \in \mathbb{R} \right\} \) is unbounded above.

This implies that \( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2^nn!)} \) does not converge uniformly on $\mathbb{R}$.

(Recall: if \( \sum_{n=0}^{\infty} f_n(x) \) converges uniformly on $\mathbb{R}$, then \( \lim_{n \to \infty} \left( \sup_{x \in \mathbb{R}} \left\{ |f_n(x)| \right\} \right) = 0 \))

See Example 5 on page 206.

5.6 Since $R = +\infty$, Theorem 26.5 implies that $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2^nn!)}$ is differentiable on $\mathbb{R}$ and

\( A x \in \mathbb{R} \), \( f'(x) = \sum_{n=0}^{\infty} \frac{X^{2n-1}}{2^nn!(n-1)!} \)