# First isomorphism theorem for rings

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**Theorem 1.** *Let $f : R \to S$ be a surjective ring homomorphism. Let $I$ be an ideal of $R$ such that $\ker f \subset I$. Then*

1. *$f(I)$ is an ideal in $S$.*
2. *$R/I \simeq S/f(I)$ as rings.*

*Proof.* First we show that $f(I)$ is an ideal in $S$. We already know that it is a subgroup of the abelian group $(S, +)$. Fix $a \in I$ and $s \in S$. We need to show that $sf(a) \in f(I)$ and $f(a)s \in f(I)$.

Since $f$ is surjective, there exists $r \in R$ such that $f(r) = s$. Since $I$ is an ideal of $R$ and $a \in I$, we know that $ar \in I$ and $ra \in I$. Hence $f(ar) \in f(I)$ and $f(ra) \in f(I)$.

Therefore

$$sf(a) = f(r)f(a) = f(ra) \in f(I)$$

and

$$f(a)s = f(a)f(r) = f(ar) \in f(I).$$

To prove the first part, let $g : S \to S/f(I)$ be the ring homomorphism given by $g(s) = s + f(I)$. Then $g$ is clearly surjective, which implies that $h = g \circ f : R \to S/f(I)$ is a surjective ring homomorphism.

Key observation: it is now enough to prove that $\ker h = I$, because once we have this equality the fundamental isomorphism theorem will imply the desired result.

We have

$$a \in \ker h \iff h(a) = 0 + f(I) \iff f(a) + f(I) = 0 + f(I) \iff f(a) \in f(I).$$

This proves that $f(\ker h) = f(I)$. On the other hand, it is clear that if $a \in \ker f$ then $f(a) = 0$ and therefore $h(a) = 0 + f(I) \implies a \in \ker h$. That is to say, $\ker f \subset \ker h$.

Since $\ker h$ and $I$ are both ideals of $R$ that contain $\ker f$ and $f(\ker h) = f(I)$, problem 8b in the quiz practice problems implies that $\ker h = I$.

$\square$