

**HOMEWORK 2**

DUE WEDNESDAY, OCTOBER 23, 2019 IN CLASS

## PART I: FROM THE TEXTBOOK

Read Propositions 2.12 and 4.2 and their proofs.

Chapter I, Section 3: **6**

Chapter I, Section 4: **2**

Chapter I, Section 6: **2, 6, 7**

## PART II

1. (a) Let  $K/F$  be a finite separable field extension. We define the relative trace and relative norm

$$\mathrm{Tr}_{K/F}, \mathrm{N}_{K/F} : K \rightarrow F, \quad \mathrm{Tr}_{K/F} = \sum \sigma(\alpha) \quad \text{and} \quad \mathrm{N}_{K/F} = \prod \sigma(\alpha)$$

where  $\sigma$  ranges over the embeddings of  $K$  into an algebraic closure  $\bar{F}$  of  $F$  that fix  $F$ . (It is clear that the relative trace is additive and the relative norm is multiplicative.)

Prove that if  $F \subset K \subset L$  are finite separable extensions, then

$$\mathrm{Tr}_{K/F} \circ \mathrm{Tr}_{L/K} = \mathrm{Tr}_{L/F} \quad \text{and} \quad \mathrm{N}_{K/F} \circ \mathrm{N}_{L/K} = \mathrm{N}_{L/F}.$$

- (b) Assume  $K/F$  is a finite extension of fields, not necessarily separable. What happens if we define the relative trace and the relative norm of an element  $\alpha \in K$  as the trace and determinant, respectively, of  $F$ -linear endomorphism of  $K$  given by the multiplication by  $\alpha$ ? Do the same properties hold?
2. Let  $K = \mathbb{Q}(\alpha)$  where  $\alpha$  is a root of the polynomial  $\alpha^3 + 2\alpha - 1 = 0$ . Show that  $K$  has 1 real embedding and 2 complex embeddings into  $\mathbb{C}$ . Find the ring of integers  $\mathcal{O}_K$  and a fundamental unit of  $K$ .
3. Find a system of fundamental units in  $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ .

## PART III: OPTIONAL

These problems are optional. However they illustrate important facts and you should try them if you have not seen them before. They will not be graded, but you are welcome to come and discuss them with me.

1. Give an example of a Dedekind domain with infinite ideal class group.

*Hint: elliptic curves might be useful.*

2. For any positive real numbers  $a_1, \dots, a_m$  and  $t$  define

$$I(a_1, \dots, a_m; t) = \int_{\substack{x_1, \dots, x_m \geq 0 \\ x_1 + \dots + x_m \leq t}} x_1^{a_1} \dots x_m^{a_m} dx_1 \dots dx_m.$$

Prove that

(a)

$$I(a_1, \dots, a_m; t) = t^{m+\sum a_i} I(a_1, \dots, a_m; 1)$$

(b)

$$\int_0^1 x^{a-1} (1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

(c)

$$I(a_1, \dots, a_m; 1) = I(a_1, \dots, a_{m-1}; 1) \frac{\Gamma(1+a_m)\Gamma(m+a_1+a_2+\dots+a_{m-1})}{\Gamma(1+m+a_1+\dots+a_m)}$$

(d)

$$I(a_1; 1) = \frac{1}{1+a_1} = \frac{\Gamma(1+a_1)}{\Gamma(2+a_1)}$$

(e)

$$I(a_1, \dots, a_m; t) = t^{m+\sum a_i} \frac{\Gamma(1+a_1) \dots \Gamma(1+a_m)}{\Gamma(1+m+a_1+\dots+a_m)}.$$