HOMEWORK 3

DUE WEDNESDAY, OCTOBER 30, 2019 IN CLASS

Part I: from the textbook

Chapter I, Section 8: **4, 10, 11**

Part II

1. Let d be a square free integer and set $K = \mathbb{Q}(\sqrt{d})$ and denote by D the discriminant of K.

(a) Show that 2 splits completely in K if and only if $D \equiv 1 \pmod 8$.

(b) Show that 2 is inert in K if and only if $D \equiv 5 \pmod 8$.

(c) Show that 2 ramifies completely in K if and only if $D \equiv 0, 4 \pmod 8$.

2. Let $S = \mathbb{Z}[\alpha]$, where $\alpha^3 = \alpha + 1$.

(a) Prove that

$$23S = (23, \alpha - 10)^2 (23, \alpha - 3).$$

(b) Show that $(23, \alpha - 10, \alpha - 3) = S$.

(c) $(23, \alpha - 10)$ and $(23, \alpha - 3)$ are relatively prime ideals.

3. Let $\omega = \frac{-1+\sqrt{-3}}{2}$ one of the primitive third roots of unity. Set $K = \mathbb{Q}(\omega)$.

(a) Compute the $N_{K/\mathbb{Q}}(a + b\omega)$ where $a, b \in \mathbb{Q}$.

(b) Prove that $\mathcal{O}_K = \mathbb{Z}[\omega]$.
Hint: K is a quadratic extension of $\mathbb{Q}$.

(c) Show that $\mathcal{O}_K$ is a PID.

(d) Show that $\alpha = 1 - \omega$ is prime in $\mathcal{O}_K$ and that $3 = u(1 - \omega)^2$ for some unit $u \in \mathcal{O}_K^\times$.

4. Let $p > 2$ be a prime number and $\zeta = \zeta_p = e^{\frac{2\pi i}{p}}$ one of the primitive pth roots of unity. Set $K = \mathbb{Q}(\zeta_p)$. Note that $1, \zeta, \ldots, \zeta^{p-2}$ are algebraic integers and they form of a $\mathbb{Q}$-basis of K.

(a) Compute $\mathrm{Tr}_{K/\mathbb{Q}}(\zeta^j)$ and $\mathrm{Tr}_{K/\mathbb{Q}}(1 - \zeta^j)$ for $0 \le j \le p - 1$.
Hint: Use the minimal polynomial of ζ.

(b) Compute $\mathrm{N}_{K/\mathbb{Q}}(1 - \zeta)$.
Hint: Consider $f(t - 1)$ where f is the minimal polynomial of ζ.

(c) Show that $(1 - \zeta)\mathcal{O}_K \cap \mathbb{Z} = p\mathbb{Z}$.
Hint: Use the previous part to show that $p = (1 - \zeta)(1 - \zeta^2) \ldots (1 - \zeta^{p-1})$.

(d) Show that $\mathrm{Tr}_{K/\mathbb{Q}}(y(1-\zeta)) \in p\mathbb{Z}$ for all $y \in \mathcal{O}_K$.

(e) Prove that $\mathcal{O}_K = \mathbb{Z}[\zeta]$.

Hint: Show that if $\alpha = a_0 + a_1\zeta + \cdots + a_{p-2}\zeta^{p-2}$ with $a_j \in \mathbb{Q}$, then

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha(1-\zeta)) = pa_0.$$

(f) Compute the discriminant of K.

Hint: Use the proof of exercise 4, Part II, HW1.