HOMEWORK 7

DUE WEDNESDAY, DECEMBER 4, 2019 IN CLASS

Part I: from the textbook

Chapter I, Section 12: **4, 5**

Read about primitive Dirichlet characters in Chapter VII, Section 2, page 434.

Part II

1. Find the primes above 2 in $\mathbb{Z}\left[\sqrt{5}\right]$ and in $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

2. Let $K = \mathbb{F}_q$ be a finite field with q elements. Define the *zeta function* of K to be

$$Z(t) = (1-t)^{-1} \prod_{P \in K[t] \text{ monic, irred}} \left(1 - t^{\deg P}\right)^{-1}$$

(a) Prove that $Z(t)$ is a rational function and determine this rational function. Note that in this case the Riemann Hypothesis says that all zeros of $Z(t)$ have absolute value equal to $q^{-1/2}$. If you have found the correct rational expression for $Z(t)$, this is trivial.

(b) Show that

$$\pi_q(m) \sim \frac{q}{q-1}\frac{q^m}{m} \quad \text{as } m \to \infty$$

where $\pi_q(m)$ denotes the number of monic irreducible polynomials with degree up to m in $K[t]$. This is the equivalent of the prime number theorem for the polynomial ring $K[t]$.

Note: The first term in the product that defines $Z(t)$ is the equivalent of the Gamma factor in this context.

3. Let K be a number field. Define the *Dedekind zeta function* of K to be

$$\zeta_K(s) = \sum_{\substack{0 \neq I \subset \mathcal{O}_K \\ \text{ideal}}} \frac{1}{\mathbb{N}(I)^s}.$$

(a) Prove that $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

(b) Prove that the series that defines $\zeta_K(s)$ is absolutely convergent for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. (Note that the convergence is uniform on compacts, which implies that $\zeta_K(s)$ is an analytic function on the right half-plane $\mathrm{Re}(s) > 1$. You don't have to prove this.)

(c) Prove that for $\mathrm{Re}(s) > 1$ we have

$$\zeta_K(s) = \prod_{\substack{0 \neq P \subset \mathcal{O}_K \\ \text{prime ideal}}} \left(1 - \frac{1}{\mathbb{N}(P)^s}\right)^{-1}.$$

4. Let χ be a primitive Dirichlet character modulo D. Define the *L-function* of χ to be

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

(a) Prove that if $D = 1$, $L(s, \chi)$ is the Riemann zeta function.

(b) Prove that the series that defines $L(s, \chi)$ is absolutely convergent for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. (Note that the convergence is uniform on compacts, which implies that $L(s, \chi)$ is an analytic function on the right half-plane $\mathrm{Re}(s) > 1$. You don't have to prove this.)

(c) Prove that for $\mathrm{Re}(s) > 1$ we have

$$L(s, \chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

5. Let $K = \mathbb{Q}(\sqrt{-1})$. Prove that

$$\zeta_K(s) = \zeta(s) L(s, \chi)$$

where

$$\chi(n) = \begin{cases} 1 & n \equiv 1 \pmod 4 \\ -1 & n \equiv 3 \pmod 4 \\ 0 & n \text{ even.} \end{cases}$$

What is the modulus of χ? Is χ primitive?

Hint: Group the prime ideals of K according the rational prime below.

6. Let $d \equiv 2, 3 \pmod 4$ be a square free integer. Set $K = \mathbb{Q}(\sqrt{d})$ and $D = \mathrm{disc}(K)$. Prove that

$$\zeta_K(s) = \zeta(s) L(s, \chi_D)$$

where

$$\chi_D(n) = \left(\frac{D}{n}\right)$$

is the Jacobi symbol. (See for instance wikipedia for the definition of the Jacobi symbol). What is the modulus of χ_D? Is χ_D primitive?

Note: The same sort of statement is true for $d \equiv 1 \pmod 4$, in the sense that there exists a quadratic Dirichlet character modulo $d = \mathrm{disc}\, \mathbb{Q}(\sqrt{d})$ such that $\zeta_{\mathbb{Q}(\sqrt{d})}(s) = \zeta(s) L(s, \chi_d)$ and this character χ_d is an extension of the Legendre symbol $\left(\frac{d}{p}\right)$. However, one has to be careful how one extends the Legendre symbol to even integer, namely how one defines $\chi_d(n)$ for n even. This is not an issue in the exercise above as in that case the discriminant is even.