AWS 2022: Modular forms on exceptional groups

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CHAPTER 1

Introduction

This course is about *quaternionic modular forms* (QMFs). A QMF is a very special type of automorphic form, much like holomorphic modular forms (HMFs) are a special type of automorphic form. In fact, quaternionic modular forms appear to behave very similarly to holomorphic modular forms, and this is one reason to be interested in them. In this introduction, I will briefly explain what are QMFs and try to motivate why they are interesting objects to study. I begin by briefly reviewing holomorphic modular forms.

1.1. Holomorphic modular forms

Suppose G is a semisimple $\mathbf{Q}$ group, with symmetric space $X_G = G(\mathbf{R})^0/K^0$, where K^0 is a maximal compact subgroup of $G(\mathbf{R})^0$. For some groups G, the symmetric space X_G has a structure of a complex manifold, for which the $G(\mathbf{R})^0$ action is via biholomorphic maps. This is the case[1] if $G(\mathbf{R})^0$ is, for example, isogenous to one of the following groups:

(1) $\mathrm{Sp}_{2n}(\mathbf{R})$
(2) $\mathrm{SO}(2,n)$
(3) $U(p,q)$

In these cases, one can make a classical definition of holomorphic modular forms. The holomorphic modular forms can be thought of sections of holomorhpic line bundles on $\Gamma\backslash X_G$, where $\Gamma \subseteq G(\mathbf{Q})$ is an arithmetic subgroup. Another way to think of the holomorphic modular forms is as very special automorphic forms $\varphi : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbf{C}$.

Compared to general automorphic forms, holomorhpic modular forms are special for at least a few reasons. One reason is that, in many situations[2], they have a classical Fourier expansion and corresponding Fourier coefficients. General automorphic forms do not have as nice of a notion of Fourier coefficients. These Fourier coefficients can contain very interesting arithmetic information. For example, if $\theta(q) = \sum_{n\in\mathbf{Z}} q^{n^2}$ is the classical θ-function, which is a modular form of weight $1/2$, then $\theta(q)^k = \sum_{n\geq 0} r_k(n)q^n$ where $r_k(n)$ is the number of ways of writing n as the sum of k squares. So the Fourier coefficients of $\theta(q)^k$, which is a special modular form of weight $k/2$, see the arithmetically interesting numbers $r_k(n)$.

Another reason holomorphic modular forms are interesting is that they are frequently, although certainly not always, the objects for which we can say arithmetically interesting things about their L-values. To be more precise, automorphic representations π have associated L-functions, $L(\pi, r, s)$. For certain automorphic representations π, the L-functions $L(\pi, r, s)$ are conjectured to be *motivic*, i.e., equal to the L-functions of certain motives. One can then transfer the conjectures about motivic L-functions (e.g., the Deligne and Bloch-Kato conjectures) to the automorphic L-functions. For example, one can ask if the special L-values

[1] This is not an exhaustive list

[2] If X_G is of tube type and G has an appropriate rational parabolic subgroup

$L(\pi, r, s = s_0)$ are algebraic (after dividing by a certain period) and if they can be p-adically interpolated. Working with holomorphic modular forms enables one to prove statements of this sort. For example, there is the recent work of Eischen-Harris-Li-Skinner [**EHLS20**] who construct p-adic L-functions for holomorphic modular forms on unitary groups.

Finally, because of their familiarity and extra structure, holomorphic modular forms are a great testing ground for potentially new phenomena. For theorems about automorphic forms–whether it be regarding L-values, periods, the trace formula, Galois representations–the first special cases that are proved are in the context of holomorphic modular forms. Trying to find and test new phenomena is yet another reason why holomorphic modular forms deserve special attention.

1.2. Quaternionic modular forms

For most semisimple groups G, the symmetric space X_G is does not have a $G(\mathbf{R})^0$-invariant complex structure. Consequently, there are no holomorhpic modular forms on X_G, and no obvious notion of very special[3] automorphic forms on $G(\mathbf{A})$. Phrased this way, it begs the question: Are there a class of groups G, and a class of very special automorphic on these groups G, that can in some ways take the place of holomorphic modular forms?

Quaternionic modular forms are a potential answer to the above question. Their study was initiated by Gross-Wallach [**GW94**], Wallach [**Wal03**], and Gan-Gross-Savin [**GGS02**]. I have been studying them for the last few years, trying to provide evidence that they behave like holomorphic modular forms. In this course, I will define quaternionic modular forms and give some of this evidence that quaternionic modular forms are *arithmetic* in a way similar to how holomorphic modular forms are arithmetic.

So, what are the groups G above, and what are the quaternionic modular forms? This will be described in detail below, but for now let me give a brief indication. Suppose G is a semisimple group for which X_G has a $G(\mathbf{R})$-invariant complex structure. If φ is an automorphic form on $G(\mathbf{A})$ that corresponds to a holomorphic modular form f_φ of some integer scalar weight w, then φ has two properties:

(1) $\varphi(gk) = z(k)^{-w}\varphi(g)$ for all $k \in K^0$, where $z : K^0 \to U(1)$ is a certain fixed surjection;
(2) $D_{CR,w}\varphi(g) \equiv 0$, where $D_{CR,w}$ is a certain linear differential operator corresponding to the fact f_φ satisfies the Cauchy-Riemann equation.

And conversely, the automorphic forms φ with these properties correspond to holomorphic modular forms f_φ of weight w. One can also make a definition in terms of the representation theory of real group $G(\mathbf{R})$: At least if w is sufficiently large, φ corresponds to a holomorphic modular form of weight w if the $(\mathfrak{g}, K^0)$-module generated by φ is a *holomorphic discrete series* representation π_w and φ spans a one-dimensional minimal K^0-type in this representation. In other words, holomorphic modular forms correspond to special vectors in automorphic representations $\pi = \pi_f \otimes \pi_\infty$, where π_∞ is a special type of representation of the real group $G(\mathbf{R})^0$.

[3]Some notions of special automorphic forms for G, which we do not consider "very special" for this purpose, are as follows: 1) functorial lifts from smaller groups (not wide enough a class of automorphic forms); 2) cohomological automorphic forms (too broad a class of automorphic forms)

Now, the groups G with holomorphic modular forms have K^0 such that K^0 possesses a surjection to the smallest nontrivial connected compact group, $U(1)$. Gross and Wallach had the insight to ask the question, "What if $G(\mathbf{R})^0$ is such that K^0 possesses a surjection to the next smallest compact group, $\mathrm{SU}(2)/\mu_2 = \mathrm{SO}(3)$?" The list of these groups includes (strictly) the following groups:

- split G_2
- split F_4
- $E_{n,4}$, $n = 6, 7, 8$ (groups of type E_6, E_7, E_8 with real rank four)
- $\mathrm{SO}(4, n)^0$ with $n \geq 3$.

The G for which $G(\mathbf{R})^0$ is as above are called quaternionic groups; they possess quaternionic modular forms. Set $\mathbf{V}_w = Sym^{2w}(\mathbf{C}^2)$, a representation of $\mathrm{SU}(2)/\mu_2$, or of K^0 via the surjection $K^0 \to \mathrm{SU}(2)/\mu_2$. A quaternionic modular form of integer weight w is an automorphic form $\varphi : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbf{V}_w$ satisfying

(1) $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $k \in K^0$;
(2) $D_w\varphi(g) \equiv 0$, for a certain specific linear differential operator D_w.

Like with holomorphic modular forms, one can also make a definition of quaternionic modular forms in terms of certain (discrete series) representations of $G(\mathbf{R})^0$. In more detail, if w is sufficiently large, there is a discrete series representation π_w of $G(\mathbf{R})^0$ whose minimal K^0-type is $\mathbf{V}_w^\vee \simeq \mathbf{V}_w$. quaternionic modular forms are automorphic forms that correspond to the (entire) minimal K^0-type of π_w, embedded in the space of automorphic forms on G.

1.3. Why study quaternionic modular forms

From my point of view, the primary purpose of this course is to convince you that quaternionic modular forms are interesting gadgets which deserve further study[4]. Let me briefly indicate some ways in which quaternionic modular forms are interesting, leaving a more detailed description to the rest of the notes.

1.3.1. Quaternionic modular forms possess Fourier coefficients. One of the first things to say about quaternionic modular forms is that they have a Fourier expansion and corresponding Fourier coefficients, very similar to that of holomorphic modular forms. In other words, associated to a quaternionic modular form φ, one can define a list of complex *numbers* $a_\varphi(\lambda)$, where λ varies in some lattice Λ. The proof of the existence of these Fourier coefficients began with work of Wallach [**Wal03**], was used by Gan-Gross-Savin [**GGS02**], and then was made more complete and explicit in [**Pol20a**].

The existence and properties of these Fourier coefficients are somewhat miraculous. Here is, in my mind, an important example: Suppose $G_1 \subseteq G_2$ are two quaternionic groups, embedded appropriately, and φ_2 on G_2 is a quaternionic modular form of weight w. Then the pullback φ_1 of φ_2 to G_1 is again a quaternionic modular form of weight w. Moreover, one can show that the Fourier coefficients of φ_1 are **finite sums** of the Fourier coefficients of φ_2. Note that, for a general automorphic form φ_2, one would not be able to say anything of content about a Fourier expansion of φ_1 from that of φ_2.

[4]The secondary purpose of this course is to convince you that exceptional groups are beautiful, and that you can work with them concretely.

1.3.2. The Fourier coefficients of quaternionic modular forms appear to be arithmetic. The Fourier coefficients of a quaternionic modular form are defined in a very transcendental way. Nevertheless, they appear to be highly arithmetic. Here are examples:

(1) There are degenerate Eisenstein series E_{2k} on G_2 of even weight $2k$. Using work of Jiang-Rallis [**JR97**], Gan-Gross-Savin [**GGS02**] showed that if the non-degenerate Fourier coefficients of the E_{2k} are nonzero[5], then they are essentially values of Zeta functions $\zeta_E(1-2k)$ of totally real cubic fields E.
(2) In the works [**Pol20b**], [**Pol20c**], it is shown that for two very special quaternionic modular forms on $E_{8,4}$, their Fourier coefficients are (nonzero) rational numbers.
(3) In the paper [**Pol20b**], I gave an example of a quaternionic modular forms on $E_{6,4}$ that is *distinguished*. I will define this notion precisely below, but for now let me say that a distinguished modular form φ is one whose non-degenerate Fourier coefficients $a_\varphi(\lambda)$ are 0 unless a certain *arithmetic* condition on λ is satisfied.
(4) In the paper [**Pol19**], I proved that in every even weight $w \geq 16$, there is a nonzero cuspidal modular form on G_2 with all Fourier coefficients algebraic numbers.
(5) In forthcoming joint work with Spencer Leslie, we define the notion of modular forms of half-integral weight on exceptional groups and prove that they have a similar notion of Fourier coefficients. Moreover, we construct a modular form of weight $1/2$ on G_2 whose Fourier coefficients see the 2-torsion in the narrow class groups of totally real cubic fields.

Based on the above-mentioned evidence, I want to take this opportunity to make the following conjecture:

CONJECTURE 1.3.1. *Suppose G is a quaternionic exceptional group, and $w \geq 1$ is an integer. Then there exists a basis $\{\varphi_i\}$ of quaternionic modular forms on G of weight w, such that all the Fourier coefficients $a_{\varphi_i}(\lambda)$ of the φ_i are algebaic numbers.*

Conjecture 1.3.1 says that quaternionic modular forms possess a very surprising *arithmeticity.*

REMARK 1.3.2. For the group GL_2, the Fourier coefficients are essentially the Satake parameters, and so algebraicity of Satake parameters implies that of the Fourier coefficients. For larger groups, the relationship between Satake parameters and Fourier coefficients is much more elaborate. In particular, algebraicity of Satake parameters does not in any clear way imply that of the Fourier coefficients.

1.3.3. The representation theory of quaternionic real representations is particularly nice. I won't have much to say about this in the course, but I did want to take this opportunity to reference work of Wallach [**Wal15**], work of Gross-Wallach [**GW94**], work of Loke [**Lok99**]. Moreover, I also want to mention the work [**Wei06**] of Marty Weissman and the recent work [**Dal21**] of Rahul Dalal, although they are more global in nature and don't exactly fit into this category. The paper of Dalal gives a dimension formula for the space of level one even weight modular forms on G_2.

[5] I believe one still does not know if these Fourier coefficients are nonzero, although of course it is believed that they are nonzero. This would be a good project for someone!

1.3.4. Quaternionic modular forms are potentially a testing ground for new phenomena. Because of their very rich structure, and because they appear to possess surprising arithmeticity, quaternionic modular forms behave very much like holomorphic modular forms. I suspect that they are ripe for study. In particular, like holomorphic modular forms, they may be a fertile testing ground for as-yet-undiscovered phenomena.

The rest of the notes will try to describe quaternionic modular forms in more detail, and explain some of the above-mentioned results.

CHAPTER 2

The group G_2

In this chapter, we describe the group G_2 in ways that generalize to the other exceptional groups. For deeper reading on the group G_2 and modular forms on G_2, we refer the reader to [**Pol19**].

2.1. The group G_2

We begin by defining the group G_2. We will define G_2 in a way that will generalize to all the (quaternionic) exceptional Lie groups. We work over a field k of characteristic 0.

Let $\mathfrak{sl}_3$ be the Lie algebra of SL_3. We may identify $\mathfrak{sl}_3$ with the trace zero 3×3 matrices. Let V_3 denote the standard representation of $\mathfrak{sl}_3$ and $V_3^\vee$ its dual. Then $End(V_3) \simeq V_3 \otimes V_3^\vee$. Note that if $v \in V_3$ and $\delta \in V_3^\vee$, then $v \otimes \delta - \frac{\delta(v)}{3} 1_3$ has trace 0, so is an element of $\mathfrak{sl}_3$.

Fix an identification $\wedge^3 V_3 \simeq k$. This gives rise to an identification $\wedge^2 V_3 \simeq V_3^\vee$ and $\wedge^2 V_3^\vee \simeq V_3$. We can describe this identification in bases. Let v_1, v_2, v_3 be the standard basis of V_3 and $\delta_1, \delta_2, \delta_3$ be the dual basis of $V_3^\vee$. Then we identify $v_i \wedge v_{i+1}$ with δ_{i-1} (indices taken modulo 3) and $\delta_j \wedge \delta_{j+1}$ with v_{j-1}.

Now we set $\mathfrak{g}_2 = \mathfrak{sl}_3 \oplus V_3 \oplus V_3^\vee$, a k vector space of dimension 14. This is a $\mathbf{Z}/3$-grading, with $\mathfrak{sl}_3$ in degree 0, V_3 in degree 1 and $V_3^\vee$ in degree 2. One defines a Lie bracket on $\mathfrak{g}_2$ as follows. First, if $\phi, \phi' \in \mathfrak{sl}_3$, then $[\phi, \phi']$ is the usual Lie bracket on $\mathfrak{sl}_3$: $\phi \circ \phi' - \phi' \circ \phi$. Next, if $\phi \in \mathfrak{sl}_3$, $v \in V_3$ and $\delta \in V_3^\vee$, then $[\phi, v] = \phi(v) \in V_3$ and $[\phi, \delta] = \phi(\delta) \in V_3^\vee$. Here recall that the action on the dual space V_3 is defined as $\phi(\delta)(v) = -\delta(\phi(v))$, so that $\langle \phi(v), \delta \rangle + \langle v, \phi(\delta) \rangle = 0$, where $\langle \, , \rangle$ is the canonical pairing between V_3 and $V_3^\vee$.

If $v, v' \in V_3$, then $[v, v'] = 2v \wedge v' \in \wedge^2 V_3 \simeq V_3^\vee$, and similarly, if $\delta, \delta' \in V_3^\vee$, then $[\delta, \delta'] = 2\delta \wedge \delta' in \wedge^2 V_3 \simeq V_3$. Finally, if $\delta \in V_3^\vee$ and $v \in V_3$, then $[\delta, v] = 3v \otimes \delta - \delta(v) 1_3$. All other Lie brackets are determined by linearity and antisymmetry.

Proposition 2.1.1. *With these definitions, $\mathfrak{g}_2$ is a Lie algebra, i.e., the Jacobi identity is satisfied.*

Proof. This is a fun exercise, which we leave to the reader. □

Proposition 2.1.2. *The Lie algebra $\mathfrak{g}_2$ is simple.*

Proof. Any ideal I of $\mathfrak{g}_2$ will be a representation of $\mathfrak{sl}_3$, and thus will be a direct sum of irreducible $\mathfrak{sl}_3$ pieces. The rest is an exercise. □

The algebraic group G_2 is defined as the (identity component of) the automorphism group of $\mathfrak{g}_2$: For a Lie algebra $\mathfrak{g}$, define

$$Aut(\mathfrak{g}) = \{g \in \mathrm{GL}(\mathfrak{g}) : g[X, Y] = [gX, gY] \; \forall X, Y \in \mathfrak{g}\}$$

and $G(\mathfrak{g}) = Aut(\mathfrak{g})^0$, the connected component of the identity. When $\mathfrak{g}$ is semisimple, $G(\mathfrak{g})$ is a connected algebraic group with Lie algebra $\mathfrak{g}$, and of adjoint type. We define $G_2 = G(\mathfrak{g}_2)$.

As a Cartan subalgebra $\mathfrak{h}$, we may take the usual (diagonal) Cartan of $\mathfrak{sl}_3$. Indeed, it is clear that this $\mathfrak{h}$ acts diagonally on $\mathfrak{g}_2$ with distinct nonzero weights.

2.2. The $\mathbf{Z}/2$-grading and the Heisenberg parabolic

The Lie algebra $\mathfrak{g}_2$ possesses a 5-step $\mathbf{Z}$-grading, and a closely related $\mathbf{Z}/2$-grading. It will be useful to review these gradings on $\mathfrak{g}_2$, which we will do presently.

Let $E_{ij} = v_i \otimes \delta_j \in End(V_3) \simeq V_3 \otimes V_3^\vee$ be the matrix with a 1 in the (ij) place and 0's elsewhere. For the 5-step $\mathbf{Z}$-grading:

- In degree 2, put kE_{13}
- In degree 1, put $kE_{12} + kv_1 + k\delta_3 + kE_{23}$
- In degree 0, put $k\delta_2 + \mathfrak{h} + kv_2$
- In degree -1, put $kE_{32} + kv_3 + k\delta_1 + kE_{21}$
- In degree -2, put kE_{31}.

Exercise 2.2.1. *Write $F_i\mathfrak{g}_2$ for the degree i piece. Find an element $h \in \mathfrak{h}$ so that $[h, F_k\mathfrak{g}_2] = kF_k\mathfrak{g}_2$. Deduce that $[F_j\mathfrak{g}_2, F_k\mathfrak{g}_2] \subseteq F_{j+k}\mathfrak{g}_2$.*

The degree 0 piece is isomorphic to $\mathfrak{gl}_2$. Write W for the degree 1 piece.

Exercise 2.2.2. *Prove that W is isomorphic to the representation $Sym^3 \otimes \det()^{-1}$ of $\mathfrak{gl}_2 \simeq F_0\mathfrak{g}_2$.*

The $\mathbf{Z}/2$-grading is defined as follows: Set $\mathfrak{g}_0 = F_{-2}\mathfrak{g}_2 \oplus F_0\mathfrak{g}_2 \oplus F_2\mathfrak{g}_2$ and $\mathfrak{g}_1 = F_{-1}\mathfrak{g}_2 \oplus F_1\mathfrak{g}_2$. It is clear that this is a $\mathbf{Z}/2$-grading.

Exercise 2.2.3. *Prove that $\mathfrak{g}_0 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{g}_1 \simeq V_2 \otimes W$ as a representation of $\mathfrak{g}_0$.*

Let P the subgroup of G_2 that stabilizes the line kE_{13}. One can show that P is a parabolic subgroup of G_2, with Lie algebra $\bigoplus_{k\geq 0} F_k\mathfrak{g}_2$, the part of $\mathfrak{g}_2$ with non-negative components in the $\mathbf{Z}$-grading. We call P the Heisenberg parabolic of G_2. The group P has a Levi decomposition $P = MN$ with $M \simeq \mathrm{GL}_2$. The Lie algebra of M is $F_0\mathfrak{g}_2$ and the Lie algebra of N is $F_1\mathfrak{g}_2 \oplus F_2\mathfrak{g}_2$. Set $Z = [N, N]$. Then one can identify Z with $F_2\mathfrak{g}_2$ via the exponential map, and one can identify $N^{ab} = N/[N, N] = N/Z$ with $W = F_1\mathfrak{g}_2$ via the exponential map.

2.3. The Cartan involution

In this section the ground field is the field $\mathbf{R}$ of real numbers. We describe the Cartan involution θ on $\mathfrak{g}_2$, and the corresponding decomposition $\mathfrak{g}_2 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where $\mathfrak{k}_0 = \mathfrak{g}_2^{\theta=1}$ and $\mathfrak{p}_0 = \mathfrak{g}_2^{\theta=-1}$.

To describe the involution, we use the $\mathbf{Z}/3$-model of $\mathfrak{g}_2$. Recall that v_1, v_2, v_3 and $\delta_1, \delta_2, \delta_3$ are our fixed bases of V_3 and $V_3^\vee$.

- On $\mathfrak{sl}_3$, which we identify with the trace 0 three-by-three matrices, we define $\theta(X) = -X^t$
- On V_3, θ is given by $\theta(v_j) = \delta_j$
- On $V_3^\vee$, θ is given by $\theta(\delta_j) = v_3$.

Exercise 2.3.1. *Check that θ is a Lie algebra involution on $\mathfrak{g}_2$, i.e., $\theta[X, Y] = [\theta X, \theta Y]$ for $X, Y \in \mathfrak{g}_2$.*

In fact, θ is a Cartan involution, i.e., the bilinear form $B_\theta(X,Y) := -B(X,\theta(Y))$ is positive definite on $\mathfrak{g}_2$, where B is the Killing form.

The group $G_2(\mathbf{R})$ has a corresponding maximal compact subgroup: Set $K_{G_2} = \{k \in G_2(\mathbf{R}) : k\theta = \theta k \text{ on } \mathfrak{g}_2\}$. Equivalently, K_{G_2} is the subgroup of $G_2(\mathbf{R})$ that also preserves B_θ. In fact, $K_{G_2} \simeq (\mathrm{SU}(2) \times \mathrm{SU}(2))/\{\pm 1\}$.

Set $\mathfrak{k} = \mathfrak{k}_0 \otimes \mathbf{C}$ and $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbf{C}$. Then $\mathfrak{k} \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ and $\mathfrak{p} \simeq V_2 \otimes W$. For details, see [**Pol19**, section 4.1]. In fact, there is an explicit $c \in G_2(\mathbf{C})$ such that $c(\mathfrak{k}) = \mathfrak{g}_0 \otimes \mathbf{C}$ and $c(\mathfrak{p}) = \mathfrak{g}_1 \otimes \mathbf{C}$. This is the exceptional Cayley transform, which can be found in [**Pol20a**].

CHAPTER 3

Modular forms on G_2

In this chapter, we describe modular forms on G_2, and what is known about them.

3.1. Warm-up: Holomorphic modular forms

We warm up to the definition by first revisiting the definition of holomorphic modular forms for SL_2.

3.1.1. Classical definition. Let $\mathfrak{h}$ denote the complex upper half-plane. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ and $z \in \mathfrak{h}$ set $j(g,z) = cz + d$. A function $f : \mathfrak{h} \to \mathbf{C}$ is a holomorphic modular form of weight ℓ if

(1) f is holomorphic
(2) $f(\gamma z) = j(\gamma, z)^{\ell} f(z)$ for all $\gamma \in \Gamma$ some congruence subgroup
(3) The function on $\mathrm{SL}_2(\mathbf{R})$ defined by $g \mapsto j(g,i)^{-\ell} f(g \cdot i)$ is of moderate growth. (See [**BJ79**] for the notion of moderate growth.)

Denote the above space of modular forms by $M_\ell(\Gamma)$. Holomorphic modular forms have a classical Fourier expansion: Assume for simplicity that Γ contains the subgroup $\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbf{Z}\}$. Then $f(z) = \sum_{n \geq 0} a_f(n) e^{2\pi i n z}$ with the Fourier coefficients $a_f(n) \in \mathbf{C}$.

3.1.2. Semi-classical definition. We now give a semi-classical definition of holomorphic modular forms. We say a smooth function $\varphi : \mathrm{SL}_2(\mathbf{R}) \to \mathbf{C}$ is a holmorphic modular form of weight ℓ if:

(1) φ is of moderate growth
(2) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$, some congruence subgroup
(3) $\varphi(g k_\theta) = e^{-i\ell\theta}\varphi(g)$ for all $g \in \mathrm{SL}_2(\mathbf{R})$ and $k_\theta = \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in \mathrm{SO}(2)$.
(4) $D_{CR}\varphi \equiv 0$ where D_{CR} is a linear differential operator described below.

To describe the operator D_{CR}, we proceed as follows. Let $X_+ = \left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$ and $X_- = \left(\begin{smallmatrix} 1 & -i \\ -i & -1 \end{smallmatrix}\right)$. Then $\mathfrak{sl}_2 \otimes \mathbf{C} = \mathfrak{k} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$, where $\mathfrak{k}$ is the complexified Lie algebra of $\mathrm{SO}(2)$, and $\mathfrak{p}_\pm$ is spanned by $X_\pm$. One defines $D_{CR}\varphi = X_-\varphi$, the differential of the right regular action.

Let $A_\ell(\Gamma)$ denote the above space of modular forms.

EXERCISE 3.1.1. *Prove that the maps* $f(z) \mapsto \varphi_f(g) := j(g,i)^{-\ell} f(g \cdot i)$ *and* $\varphi(g) \mapsto f_\varphi(z) := j(g_z, i)^{\ell}\varphi(g_z)$ *define inverse bijections between* $M_\ell(\Gamma)$ *and* $A_\ell(\Gamma)$*. Here* g_z *is any* $g \in \mathrm{SL}_2(\mathbf{R})$ *with* $g \cdot i = z$.

Assume for simplicity that Γ contains the subgroup $\left(\begin{smallmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\right) = y^{\ell/2} \sum_{n \geq 0} a_\varphi(n) e^{2\pi i n x} e^{-2\pi n y}.$$

We can re-express this Fourier expansion as follows. Define the function $W_n : \mathrm{SL}_2(\mathbf{R}) \to \mathbf{C}$ as

$$W_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k_\theta\right) = y^{\ell/2} e^{-i\ell\theta} e^{2\pi i n x} e^{-2\pi n y}.$$

Then $\varphi(g) = \sum_{n\geq 0} a_\varphi(n) W_n(g)$ with Fourier coefficients $a_\varphi(n) \in \mathbf{C}$.

3.1.3. Adelic definition. We now give the adelic definition of holomorphic modular forms and their Fourier expansion.

A smooth function $\varphi : \mathrm{SL}_2(\mathbf{A}) \to \mathbf{C}$ is a holomorphic modular form of weight ℓ if:

(1) φ is of moderate growth
(2) φ is right invariant under and open compact subgroup of $\mathrm{SL}_2(\mathbf{A}_f)$
(3) φ is left-invariant under $\mathrm{SL}_2(\mathbf{Q})$
(4) $\varphi(gk_\theta) = e^{-i\ell\theta}\varphi(g)$ for all $g \in \mathrm{SL}_2(\mathbf{A})$ and $k_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SO}(2)$.
(5) $D_{CR}\varphi \equiv 0$ where D_{CR} is the linear differential operator described above.

The Fourier expansion of φ is

$$\varphi(g_f g_\infty) = \sum_{n \in \mathbf{Q}, n \geq 0} a_{\varphi,n}(g_f) W_n(g_\infty)$$

for locally constant functions $a_{\varphi,n}(g_f)$, the Fourier coefficients of φ. Here $g_f \in \mathrm{SL}_2(\mathbf{A}_f)$ and $g_\infty \in \mathrm{SL}_2(\mathbf{R})$.

3.2. The definition

We now give the definition of modular forms on G_2, mimicking the adelic definition of holomorphic modular forms on SL_2.

Let $V_\ell = Sym^{2\ell}(\mathbf{C}^2) \boxtimes 1$ as a representation of $K_{G_2} \simeq (\mathrm{SU}(2) \times \mathrm{SU}(2))/\pm 1$. Here the $\mathrm{SU}(2)$ factors are ordered so that $\mathfrak{p} \simeq \mathbf{C}^2 \boxtimes W$ (as opposed to $W \boxtimes \mathbf{C}^2$). To define modular forms on G_2, we need a certain differential operator D_ℓ that will take the place of D_{CR} above.

Suppose $\varphi : G_2(\mathbf{R}) \to \mathbf{V}_\ell$ is a smooth function, satisfying $\varphi(gk) = k^{-1}\varphi(g)$ for all $g \in G_2(\mathbf{R})$ and $k \in K_{G_2}$. We define $D_\ell\varphi : G_2(\mathbf{R}) \to Sym^{2\ell-1}(\mathbf{C}^2) \boxtimes W$ as follows.

First, we define $\widetilde{D}_\ell\varphi : G_2(\mathbf{R}) \to \mathbf{V}_\ell \otimes \mathfrak{p}^\vee$ as:

$$\widetilde{D}_\ell\varphi = \sum_j X_j\varphi \otimes X_j^\vee$$

where $\{X_j\}$ is a basis of $\mathfrak{p}$ and $\{X_j^\vee\}$ is the dual basis of $\mathfrak{p}^\vee$. Here $X_j\varphi$ is the right regular action of $\mathfrak{p} \subseteq \mathfrak{g}$ on φ. One checks easily that $\widetilde{D}_\ell\varphi$ is still K_{G_2}-equivariant, i.e., if $\varphi' = \widetilde{D}_\ell\varphi$, then $\varphi'(gk) = k^{-1}\varphi(g)$ for all $k \in K$ and $g \in G$.

Now, because $\mathfrak{p} \simeq \mathfrak{p}^\vee$, we have

$$\mathbf{V}_\ell \otimes \mathfrak{p}^\vee \simeq Sym^{2\ell-1}(\mathbf{C}^2) \boxtimes W \oplus Sym^{2\ell+1}(\mathbf{C}^2) \boxtimes W.$$

Let $pr : \mathbf{V}_\ell \otimes \mathfrak{p}^\vee \to Sym^{2\ell-1}(\mathbf{C}^2) \boxtimes W$ be a K_{G_2} equivariant projection (unique up to scalar multiple). We define $D_\ell = pr \circ \widetilde{D}_\ell$.

Definition 3.2.1. Suppose $\ell \geq 1$ is a non-negative integer. A smooth function $\varphi : G_2(\mathbf{A}) \to \mathbf{V}_\ell$ is a **quaternionic modular form** of weight ℓ if

(1) φ is of moderate growth

(2) φ is right-invariant under an open compact subgroup of $G_2(\mathbf{A}_f)$
(3) φ is left $G(\mathbf{Q})$-invariant, i.e., $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbf{Q})$
(4) φ is K_{G_2}-equivariant, i.e., $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in K_{G_2}$ and
(5) $D_\ell \varphi \equiv 0$.

Quaternionic modular forms have a Fourier expansion and corresponding Fourier coefficients. We briefly state the result now, and will discuss it in more detail below. Denote φ_Z and φ_N the constant terms of φ along Z and N.

We identify elements of W with four-tuples (a,b,c,d) so that $(a,b,c,d) = aE_{12} + bv_1 + c\delta_3 + dE_{23}$. Define a symplectic form on W as $\langle (a,b,c,d),(a',b',c',d')\rangle = ad' - 3bc' + 3cb' - da'$. This form is none other than the commutator of two elements of W, i.e., if $w, w' \in W$ then $[w,w'] = \langle w, w'\rangle E_{13}$. If $w = (a,b,c,d) \in W(\mathbf{R})$, write $w \geq 0$ if $az^3 + 3bz^2 + 3cz + d$ is never 0 on the upper half plane in $\mathbf{C}$; we say such elements of $W(\mathbf{R})$ are positive semi-definite.

THEOREM 3.2.2 ([**Pol19**],[**Pol20a**]). *Suppose $w \in W(\mathbf{R})$ is positive semi-definite. There is a completely explicit function $W_w : G_2(\mathbf{R}) \to \mathbf{V}_\ell$ satisfying the following properties:*

(1) *$W_w(ng) = e^{i\langle w, \overline{n}\rangle} W_w(g)$ for all $n \in N(\mathbf{R})$. Here $\overline{n}$ denotes the image of n in $N^{ab} \simeq W$.*
(2) *$W_w(gk) = k^{-1} \cdot W_w(g)$ for all $k \in K_{G_2}$.*
(3) *$D_\ell W_w(g) \equiv 0$.*
(4) *W_w is of moderate growth.*

Moreover, if φ is a modular form on G_2 of weight ℓ, then

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \in 2\pi W(\mathbf{Q}) : w \geq 0} a_{\varphi,w}(g_f) W_w(g_\infty)$$

for locally constant functions $a_{\varphi,w}$ on $G_2(\mathbf{A}_f)$. Additionally, the constant term φ_N is essentially a holomorphic modular form of weight 3ℓ on $M \simeq \mathrm{GL}_2$.

The functions $a_{\varphi,w}$, or sometime their value at $g_f = 1$, are the Fourier coefficients of φ. We say $a_{\varphi,w}(1)$ is the Fourier coefficient of φ associated to w. When w is in the open orbit of $\mathrm{GL}_2(\mathbf{R})$ acting on $W(\mathbf{R})$, these Fourier coefficients were defined by Gan-Gross-Savin [**GGS02**], using a multiplicity one result of Wallach [**Wal03**], even though these authors did not have the explicit functions $W_w(g)$.

Note that in the theorem, the constant terms φ_N are essentially modular forms of weight 3ℓ on GL_2. So, the Ramanujan cusp form Δ can (and does) show up, but the cusp form of weight 16 does not.

3.3. Examples and theorems

We give some examples and theorem about modular forms on G_2.

3.3.1. Eisenstein series. The easiest family of examples is the degenerate Heisenberg Eisenstein series. To define these, let $x^{2\ell}, \ldots, y^{2\ell}$ be a particular fixed basis of $\mathbf{V}_\ell$, that we will specify in more detail later.

Let $P \subseteq G$ be the Heisenberg parabolic, with $\nu : P \to \mathrm{GL}_1$ the character given by $p \cdot E_{13} = \nu(p) E_{13}$. If $\ell > 2$ is even, there is a weight ℓ modular form associated to inducing sections in $Ind_P^G(|\nu|^{\ell+1})$. In more detail, let $f_\ell(g) \in Ind_{P(\mathbf{R})}^{G(\mathbf{R})}(|\nu|^{\ell+1})$ be the unique K-equivariant,

$\mathbf{V}_\ell$-valued section whose value at $g = 1$ is $x^\ell y^\ell$. Let now $f_{fte} \in Ind_{P(\mathbf{A}_f)}^{G(\mathbf{A}_f)}(|\nu|^{\ell+1})$ be an arbitrary element. Set $f(g) = f_{fte}(g_f)f_\ell(g_\infty)$ and $E(g,f) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f(\gamma g)$. The sum converges absolutely using that $\ell > 2$. Then it can be shown that the value $E(g,f)$ is a quaternionic modular form of weight ℓ. More generally, both for even and odd ℓ, one can make quaternionic modular forms by starting with inducing sections in $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}((\chi \circ \nu)|\nu|^{\ell+1})$, where $\chi : \mathrm{GL}_1(\mathbf{Q})\backslash \mathrm{GL}_1(\mathbf{A}) \to \mathbf{C}^\times$ is an automorphic character satisfying $\chi_\infty(t) = \left(\frac{t}{|t|}\right)^\ell = sgn(t)^\ell$.

Another way to create modular forms is to use Heisenberg Eisenstein series with nontrivial inducing data. One can take a classical holomorphic modular form Φ of weight 3ℓ on $M \simeq \mathrm{GL}_2$, and from it produce a weight ℓ modular form $E(g,\Phi)$ on G_2 if ℓ is sufficiently large. This is spelled out in [**Pol20a**].

Here are some interesting open questions about Eisenstein series:

(1) Do the degenerate Eisenstein series $E(g,f)$ have rational or algebraic Fourier coefficients, when f_{fte} is spherical?
(2) Say that $w \in W(\mathbf{R})$ is non-degenerate if w is in the open orbit of $M(\mathbf{C})$ on $W(\mathbf{C})$. Are the non-degenerate Fourier coefficients of the Eisenstein series $E(g,f)$ nonzero?
(3) If the modular form Φ on GL_2 has algebraic Fourier coefficients, does the same occur for $E(g,\Phi)$? In the setting of holomorphic modular forms on tube domains, an analogous result is due to Harris [**Har81, Har84**]. Nothing is known in the quaternionic case.

Recall (see [**GGS02**]) that there is a correspondence between $\mathrm{GL}_2(\mathbf{Z})$ orbits of integral binary cubic forms and cubic rings. If φ is a level one modular form on G_2, and $w = (a, b/3, c/3, d)$ corresponds to the integral binary cubic form $au^3 + bu^2v + cuv^2 + dv^3$, then we can consider the Fourier coefficient $a_{\varphi,w}(1)$. If φ is of even weight, and $\gamma \in \mathrm{GL}_2(\mathbf{Z})$, it is easy to check that $a_{\varphi,w}(1) = a_{\varphi,\gamma\cdot w}(1)$. Let $A(w)$ be the cubic ring corresponding to the orbit $\mathrm{GL}_2(\mathbf{Z}) \cdot w$. Thus, following [**GGS02**], we can define $a_\varphi(A(w)) = a_{\varphi,w}(1)$ to be the Fourier coefficient of φ associated to the cubic ring $A(w)$. We use this definition in the statement of the following theorem, and also in the subsection below on L-functions.

THEOREM 3.3.1 (Gan-Gross-Savin [**GGS02**], Jiang-Rallis [**JR97**]). *For $\ell \geq 4$ even, let $E_\ell(g)$ be the spherical weight ℓ Eisenstein series on G_2. Assume the non-degenerate Fourier coefficients of $E_\ell(g)$ are nonzero*[1]*. Let $\omega = (a, b/3, c/3, d) \in W$ correspond to the totally real cubic ring $A(\omega)$, via the correspondence between binary cubic forms and cubic rings. If $A(\omega)$ is maximal, then the Fourier coefficient of $E_\ell(g)$ corresponding to ω is $\zeta_{A(\omega)}(1-\ell)$, up to a nonzero constant independent of ω.*

We will have more to say later about some examples of modular forms constructed in [**GGS02**].

3.3.2. Cusp forms. We now explain what is known about cusp forms.

The following is the main theorem of [**Pol21**].

[1]For each ℓ, one only needs to assume that a certain purely archimedean integral (that depends on ℓ) is nonzero

THEOREM 3.3.2. [**Pol21**] *Let $w \geq 16$ be even. Then there are nonzero cuspidal modular forms on G_2 of weight w, all of whose Fourier coefficients are algebraic integers.*

In fact, in weight 20, there is nonzero cuspidal level one modular form on G_2, all of whose Fourier coefficients are integers.

Recently, Dalal has given a dimension formula for the level one cuspidal quaternionic modular forms on G_2.

THEOREM 3.3.3 (Dalal [**Dal21**]). *There is an explicit formula for the dimension of level one cuspidal modular forms on G_2. In particular, the smallest nonzero level one cuspidal modular form on G_2 is in weight 6.*

Here are some questions:

(1) Can the cuspidal weight 6, level one quaternionic modular form of Dalal be constructed explicitly?
(2) Let $w = (0, 1/3, -1/3, 0)$ so that $A(w) \simeq \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$. Do level one quaternionic modular forms with nonzero w Fourier coefficient exist? Do they exist in abundance?

3.3.3. *L*-functions. Nothing is known about L-functions of quaternionic modular forms on groups bigger than G_2. However, on G_2, one can say a bit about the standard L-function of quaternionic modular forms.

Thus let $\pi = \pi_f \otimes \pi_{\ell,\infty}$ be a cuspidal automorphic representation of $G_2(\mathbf{A})$ for which $\pi_{\ell,\infty}$ is the quaternionic discrete series having minimal K_{G_2}-type $\mathbf{V}_\ell$. Let φ be the level one cuspidal modular form associated to π. Then one can write the standard L-function of π as a Dirichlet series in the Fourier coefficients of φ.

THEOREM 3.3.4. [**iDD$^+$21**] *Let $a_\varphi(T)$ denote the Fourier coefficient of φ corresponding to the cubic ring T. Then*

$$\sum_{T \subseteq \mathbf{Z}^3, n \geq 1} \frac{a_\varphi(\mathbf{Z} + nT)}{[\mathbf{Z}^3 : T]^{s-\ell+1} n^s} = a_\varphi(\mathbf{Z}^3) \frac{L(\pi, Std, s - 2\ell + 1)}{\zeta(s - 2\ell + 2)^2 \zeta(2s - 4\ell + 2)}.$$

Here the sum is over the subrings T of $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ and integers $n \geq 1$.

It is also known that the completed L-function has a functional equation, if $a_\varphi(\mathbf{Z}^3) \neq 0$. To state the result, define the archimedean L-factor as

$$L_\infty(\pi_{\ell,\infty}, s) = \Gamma_{\mathbf{C}}(s + \ell - 1)\Gamma_{\mathbf{C}}(s + \ell)\Gamma_{\mathbf{C}}(s + 2\ell - 1)\Gamma_{\mathbf{R}}(s + 1).$$

Here

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2) \quad \text{and} \quad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s),$$

where Γ is the usual gamma function.

The completed L-function is given by

$$\Lambda(\pi, Std, s) = L_\infty(\pi_{\ell,\infty}, s) L(\pi, Std, s).$$

THEOREM 3.3.5. [**iDD$^+$21**] *Suppose that φ is a level one cuspidal modular form on G_2 of positive even weight ℓ that generates the cuspidal automorphic representation π. Further, assume that the Fourier coefficient of φ corresponding to the split cubic ring $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ is nonzero. Then*

$$\Lambda(\pi, Std, s) = \Lambda(\pi, Std, 1 - s)$$

for all $s \in \mathbb{C}$.

The proof of the theorems above is based on a refined analysis of the Rankin-Selberg integral in [**GS15**].

CHAPTER 4

Exceptional algebra

In this chapter we will work our way up to defining the quaternionic exceptional group of type E_8. We begin with some exceptional algebra: composition algebras and cubic norm structures. Our aim is just to give the necessary definitions. For many of the omitted proofs in this chapter, the reader is encouraged to see my course notes [**Pol**, Chapters 1 and 2].

4.1. Composition algebras

Suppose k is a field, not of characteristic 2. In this section we discuss composition algebras, which we are about to define. For more on composition algebras, one can see [**SV00**] in addition to [**Pol**].

Definition 4.1.1. Suppose C is a not-neccessarily-associative k algebra with unit 1, and that C comes equipped with non-degenerate quadratic form $n_C : C \to k$. Then C is said to be a composition algebra if n_C is multiplicative, i.e., $n_C(xy) = n_C(x)n_C(y)$ for all $x, y \in C$.

Composition algebras can be classified, and in fact are always dimension $1, 2, 4$ or 8 over the ground field. Every dimension four composition algebra is a quaternion algebra.

Example 4.1.2. $C = k$ with $n_C(x) = x^2$ is a composition algebra.

Example 4.1.3. $C = E$, an etale quadratic extension of k, with $n_C(x) = N_{E/k}(x)$ the norm.

Example 4.1.4. $C = B$, a quaternion k-algebra, with $n_C(x)$ the reduced norm.

There is a way of defining an involution $*$ on a composition algebra, as follows. Let $(x, y) = n_C(x + y) - n_C(x) - n_C(y)$ be the non-degenerate bilinear form associated to n_C. Note that 1 satisfies $(1, 1) = 2 \neq 0$. Let C^0 be the perpendicular space to 1 under the bilinear form. Define $*$ on C as $(x1 + y)^* = x - y$ if $x \in k$ and $y \in C^0$. In other words,

$$z^* = (z, 1)1 - z$$

for $z \in C$.

Note that $z + z^* \in k \cdot 1$ for all $z \in C$. Also note that $n_C(z) = n_C(z^*)$ for all $z \in C$.

Theorem 4.1.5. *The map $*$ satisfies*

(1) *$z^* z = n_C(z)$ for all $z \in C$.*
(2) *Moreover, $*$ is an algebra involution, i.e., $(xy)^* = y^* x^*$ for all $x, y \in C$.*

Definition 4.1.6. An octonion algebra Θ is an eight-dimensional composition algebra.

Octonion algebras exist. We give two different constructions, called the Zorn model and the Cayley-Dickson construction.

Definition 4.1.7 (The Zorn model). Denote by V_3 the three-dimensional defining representation of SL_3 and by $V_3^\vee$ its dual representation. Recall that we identify $\wedge^2 V_3 \simeq V_3^\vee$ and $\wedge^2 V_3^\vee \simeq V_3$. Denote by Θ the set of two-by-two matrices $\left(\begin{smallmatrix} a & v \\ \phi & d \end{smallmatrix}\right)$ with $a, d \in k$, $v \in V_3$ and $\phi \in V_3^\vee$ with multiplication

$$\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}\begin{pmatrix} a' & v' \\ \phi' & d' \end{pmatrix} = \begin{pmatrix} aa' + \phi'(v) & av' + d'v - \phi \wedge \phi' \\ a'\phi + d\phi' + v \wedge v' & \phi(v') + dd' \end{pmatrix}.$$

The involution $*$ is $\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}^* = \begin{pmatrix} d & -v \\ -\phi & a \end{pmatrix}$ and the norm is $n_\Theta(\begin{pmatrix} a & v \\ \phi & d \end{pmatrix}) = ad - \phi(v)$.

The Cayley-Dickson construction starts with a quaternion algebra B and an element $\gamma \in k^\times$, and defines $\Theta = B \oplus B$ with multiplication as follows.

Definition 4.1.8. Let $*$ denote the involution on the quaternion algebra B. Then the multiplicaton on $\Theta = B \oplus B$ is

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 + \gamma y_2^* y_1, y_2 x_1 + y_1 x_2^*).$$

The involution $*$ on Θ is $(x, y)^* = (x^*, -y)$ and the norm is $n_\Theta((x, y)) = n_B(x) - \gamma n_B(y)$.

The Cayley-Dickson construction and the Zorn model produces composition algebras.

Proposition 4.1.9. *The Zorn model is a special case of the Cayley-Dickson construction, with $B = M_2(k)$ and $\gamma = 1$.*

Proposition 4.1.10. *The Zorn model and the Cayley-Dickson construction define octonion algebras, i.e., the norms are multiplicative.*

4.2. Cubic norm structures

In this section we define another algebraic gadget, which is called a cubic norm structure. For more on cubic norm structures, one can see [**McC04**] in addition to [**Pol**]. You can think of cubic norm structures as generalizations the pair (3×3 matrices, the determinant map). We will jump straight into the definition, and then give the examples.

Thus suppose k is a field of characteristic 0 and J is a finite dimensional k vector space. That J is a cubic norm structure means that it comes equipped with a cubic polynomial map $N : J \to k$, a quadratic polynomial map $\# : J \to J$, an element $1_J \in J$, and a non-degenerate symmetric bilinear pairing $(\, , \,) : J \otimes J \to k$, called the trace pairing, that satisfy the following properties. For $x, y \in J$, set $x \times y = (x+y)^\# - x^\# - y^\#$ and denote $(\, , \, , \,) : J \otimes J \otimes J \to k$ the unique symmetric trilinear form satisfying $(x, x, x) = 6N(x)$ for all $x \in J$. Then

(1) $N(1_J) = 1$, $1_J^\# = 1_J$, and $1_J \times x = (1_J, x) - x$ for all $x \in J$.
(2) $(x^\#)^\# = N(x)x$ for all $x \in J$.
(3) The pairing $(x, y) = \frac{1}{4}(1_J, 1_J, x)(1_J, 1_J, y) - (1_J, x, y)$.
(4) One has $N(x+y) = N(x) + (x^\#, y) + (x, y^\#) + N(y)$ for all $x, y \in J$.

There is a weaker notion of a cubic norm pair. In this case, the pairing $(\, , \,)$ is between J and $J^\vee$, the linear dual of J, the adjoint map $\#$ takes $J \to J^\vee$ and $J^\vee \to J$, and each $J, J^\vee$ have a norm map $N_J : J \to F$ and $N_{J^\vee} : J^\vee \to F$. The adjoints and norms on J and $J^\vee$ satisfy the same compatibilities as above in items (2) and (4).

EXAMPLE 4.2.1. Let $J = k$ with $N(x) = x^3$, $x^\# = x^2$, $1_J = 1$, and $(x, y) = 3xy$.

EXAMPLE 4.2.2. Let $J = M_3(k)$ with $N(x) = \det(x)$, $x^\#$ the adjoint matrix, (i.e., $x^\# = \det(x)x^{-1}$ for invertible x), $(x, y) = \mathrm{tr}(xy)$, $1_J = 1_3$.

EXAMPLE 4.2.3. $J = k \times S$ with S a pointed quadratic space. In more detail, take $1_S \in S$ with $q(1_S) = 1$. Define an involution ι on S fixing 1_S and acting as minus the identity on $(k \cdot 1_S)^\perp$. The norm on J is $N_J(\beta, s) = \beta q_S(s)$, one has $1_J = (1, 1_S)$, and the adjoint map is $(\beta, s)^\# = (q_S(s), \beta\iota(s))$. Finally, the pairing is $((\beta, t), (\beta', t')) = \beta\beta' + (t, \iota(t'))$.

PROPOSITION 4.2.4. *With data as defined above $J = k \times S$ is a cubic norm structure.*

Set $J = H_3(C)$ the Hermitian 3×3 matrices with coefficients in the composition algebra C. We make $J = H_3(C)$ into a cubic norm structure, with the following choice of data:

(1) $N_J(X) = N_J \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} = c_1c_2c_3 - c_1 n_C(x_1) - c_2 n_C(x_2) - c_3 n_C(x_3) + \mathrm{tr}_C(x_1x_2x_3)$.

(2) $X^\# = \begin{pmatrix} c_2c_3 - n_C(x_1) & x_2^*x_1^* - c_3x_3 & x_3x_1 - c_2x_2^* \\ x_1x_2 - c_3x_3^* & c_1c_3 - n_C(x_2) & x_3^*x_2^* - c_1x_1 \\ x_1^*x_3^* - c_2x_2 & x_2x_3 - c_1x_1^* & c_1c_2 - n_C(x_3) \end{pmatrix}$

(3) The pairing (X, X'), in obvious notation, is

$$(X, X') = c_1c_1' + c_2c_2' + c_3c_3' + (x_1, x_1') + (x_2, x_2') + (x_3, x_3').$$

THEOREM 4.2.5. *With data described above, $J = H_3(C)$ is a cubic norm structure.*

4.3. The group M_J

If J is a cubic norm structure, we define the algebraic k-group M_J to be the group of linear automorphisms of J that preserve the norm form N up to scaling. On k-points, one has

$$M_J(k) = \{(\lambda, g) \in \mathrm{GL}_1(k) \times \mathrm{GL}(J) : N(gX) = \lambda N(X) \text{ for all } X \in J\}.$$

We set M_J^1 the subgroup of M_J consisting of those g with $\lambda(g) = 1$ and we set A_J the subgroup of M_J^1 that also stabilizes the element $1_J \in J$. It follows that A_J preserves the bilinear pairing $(\ ,\ )$: if $a \in A_J$, then $(ax, ay) = (x, y)$ for all $x, y \in J$. The group A_J is the automorphism group of J. If $a \in A_J$, then one also has $(ax) \times (ay) = a(x \times y)$ for all $x, y \in J$.

Let

$$\mathfrak{m}(J) = \{(\mu, \Phi) \in k \times End(J) : (\Phi(z_1), z_2, z_3) + (z_1, \Phi(z_2), z_3) + (z_1, z_2, \Phi(z_3)) = \mu(z_1, z_2, z_3)\}.$$

Then $\mathfrak{m}(J)$ is the Lie algebra of M_J. We define an M_J-equivariant map $J \otimes J^\vee \to \mathfrak{m}(J)$. See [**Spr62**] and [**Rum97**].

For $\gamma \in J^\vee$ and $x \in J$, define the element $\Phi_{\gamma,x} \in End(J)$ as

$$\Phi_{\gamma,x}(z) = -\gamma \times (x \times z) + (\gamma, z)x + (\gamma, x)z.$$

PROPOSITION 4.3.1. [**Rum97**, Equation (9)] *One has*

$$(\Phi_{\gamma,x}(z_1), z_2, z_3) + (z_1, \Phi_{\gamma,x}(z_2), z_3) + (z_1, z_2, \Phi_{\gamma,x}(z_3)) = 2(\gamma, x)(z_1, z_2, z_3)$$

for all z_1, z_2, z_3 in J. In particular, $\Phi_{\gamma,x} \in \mathfrak{m}(J)$.

Note that $\Phi_{\gamma,x}(z) = \Phi_{\gamma,z}(x)$. One sets $\Phi'_{\gamma,x} = \Phi_{\gamma,x} - \frac{2}{3}(\gamma, x)$. Then $\Phi'_{\gamma,x} \in \mathfrak{m}(J)^0$, the Lie algebra of M_J^1.

PROPOSITION 4.3.2. *The map $\Phi : J \otimes J^\vee \to \mathfrak{m}(J)$ is equivariant, i.e., if $g \in M_J$ then $Ad(g)\Phi_{\gamma,x} = \Phi_{\widetilde{g}(\gamma),g(x)}$.*

4.4. The $\mathbf{Z}/3$ grading on $\mathfrak{g}(J)$

In this subsection we recall elements from the paper [**Rum97**] (in different notation). Rumelhart constructed a Lie algebra $\mathfrak{g}(J)$ out of a cubic norm structure J, as follows.

Denote by V_3 the defining representation of $\mathfrak{sl}_3$, and by $V_3^\vee$ the dual representation. One defines

$$\mathfrak{g}(J) = \mathfrak{sl}_3 \oplus \mathfrak{m}(J)^0 \oplus V_3 \otimes J \oplus V_3^\vee \oplus J^\vee.$$

We consider $V_3, V_3^\vee$ as left modules for $\mathfrak{sl}_3$, and $J, J^\vee$ as left modules for $\mathfrak{m}(J)^0$. This is a $\mathbf{Z}/3$-grading, with $\mathfrak{sl}_3 \oplus \mathfrak{m}(J)^0$ in degree 0, $V_3 \otimes J$ in degree 1 and $V_3^\vee \oplus J^\vee$ in degree 2. Note that when $J = k$, $\mathfrak{m}(J)^0 = 0$ and $g(J)$ becomes $\mathfrak{g}_2$.

4.4.1. The bracket. Following [**Rum97**], a Lie bracket on $\mathfrak{g}(J)$ is given as follows. First, because V_3 is considered as a representation of $\mathfrak{sl}_3$, there is an identification $\wedge^2 V_3 \simeq V_3^\vee$, and similarly $\wedge^3 V_3^\vee \simeq V_3$. If v_1, v_2, v_3 denotes the standard basis of V_3, and $\delta_1, \delta_2, \delta_3$ the dual basis of $V_3^\vee$, then $v_1 \wedge v_2 = \delta_3$, $\delta_1 \wedge \delta_2 = v_3$, and cyclic permutations of these two identifications.

Take $\phi_3 \in \mathfrak{sl}_3$, $\phi_J \in \mathfrak{m}(J)^0$, $v, v' \in V_3$, $\delta, \delta' \in V_3^\vee$, $X, X' \in J$ and $\gamma, \gamma' \in J^\vee$. The Lie bracket on $\mathfrak{g}(J)$ is defined as

$$\begin{aligned}
[\phi_3, v \otimes X + \delta \otimes \gamma] &= \phi_3(v) \otimes X + \phi_3(\delta) \otimes \gamma. \\
[\phi_J, v \otimes X + \delta \otimes \gamma] &= v \otimes \phi_J(X) + \delta \otimes \phi_J(\gamma) \\
[v \otimes X, v' \otimes X'] &= (v \wedge v') \otimes (X \times X') \\
[\delta \otimes \gamma, \delta' \otimes \gamma'] &= (\delta \wedge \delta') \otimes (\gamma \times \gamma') \\
[\delta \otimes \gamma, v \otimes X] &= (X, \gamma) v \otimes \delta + \delta(v)\Phi_{\gamma,X} - \delta(v)(X, \gamma) \\
&= (X, \gamma)\left(v \otimes \delta - \frac{1}{3}\delta(v)\right) + \delta(v)\left(\Phi_{\gamma,X} - \frac{2}{3}(X, \gamma)\right).
\end{aligned}$$

Note that $v \otimes \delta - \frac{1}{3}\delta(v) \in \mathfrak{sl}_3$ and $\Phi_{\gamma,X} - \frac{2}{3}(X, \gamma) = \Phi'_{\gamma,X} \in \mathfrak{m}(J)^0$.

THEOREM 4.4.1. [**Rum97**] *The vector space $\mathfrak{g}(J)$ is a Lie algebra, i.e., the Jacobi identity is satisfied.*

EXAMPLE 4.4.2. When $J = k$, $\mathfrak{g}(J) = \mathfrak{g}_2$.

EXAMPLE 4.4.3. When $J = H_3(k)$, $\mathfrak{g}(J)$ is of type $\mathfrak{f}_4$.

EXAMPLE 4.4.4. When $J = H_3(E)$ with E a quadratic etale extension of k, $\mathfrak{g}(J)$ is of type $\mathfrak{e}_6$.

EXAMPLE 4.4.5. When $J = H_3(B)$ with B a quaternion algebra, $\mathfrak{g}(J)$ is of type $\mathfrak{e}_7$.

EXAMPLE 4.4.6. When $J = H_3(\Theta)$ with Θ an octonion algebra, $\mathfrak{g}(J)$ is of type $\mathfrak{e}_8$.

Let $G_J = \mathrm{Aut}(\mathfrak{g}(J))^0$ be the identity component of the automorphisms of the Lie algebra $\mathfrak{g}(J)$. Then G_J is a connected adjoint algebraic group of the above types. When $k = \mathbf{R}$, the norm form on the composition algebra C is positive definite, and $J = H_3(C)$, $G_J(\mathbf{R})$ is the adjoint quaternionic exceptional group of the above types. You can take this to be the definition of the quaternionic exceptional groups.

CHAPTER 5

The Freudenthal construction

In the previous section, we gave a construction of the exceptional Lie algebras $\mathfrak{g}(J)$, via a $\mathbf{Z}/3\mathbf{Z}$-grading. In order to discuss quaternionic modular forms on $G_J = \mathrm{Aut}^0(\mathfrak{g}(J))$, we will require a 5-step $\mathbf{Z}$-grading on $\mathfrak{g}(J)$. The purpose of this chapter is to give some preliminaries needed for the description of this 5-step $\mathbf{Z}$-grading.

Throughout this chapter, the omitted proofs can be found in [**Pol**, Chapter 3].

5.1. The Freudenthal construction

Suppose that J is a cubic norm structure, or that $J, J^\vee$ is a cubic norm pair, over a field k of characteristic 0. Define a vector space $W_J = k \oplus J \oplus J^\vee \oplus k$. The space W_J comes equipped with a symplectic pairing $\langle\, , \,\rangle$ and a quartic form q, which are defined as follows. We write a typical element in W_J as $v = (a, b, c, d)$, so that $a, d \in k$, $b \in J$ and $c \in J^\vee$. Then

$$\langle (a, b, c, d), (a', b', c', d') \rangle = ad' - (b, c') + (c, b') - da'$$

and

$$q((a, b, c, d)) = (ad - (b, c))^2 + 4aN(c) + 4dN(b) - 4(b^\#, c^\#).$$

The definition of this algebraic data goes back to Freudenthal.

We now define a group

$$H_J(k) = \{(g, \nu) \in \mathrm{GL}(W_J) \times \mathrm{GL}_1(k) : \langle gv, gv' \rangle = \nu \langle v, v' \rangle \, \forall v, v' \in W_J \text{ and } q(gv) = \nu^2 q(v) \, \forall v \in W_J\}.$$

The element ν is called the similitude. More generally, we let H_J be the algebraic group of linear automorphisms of W_J that preserve the symplectic form $\langle\, , , \,\rangle$ and the quartic form q up to appropriate similitude. We set $H_J^1 = \ker \nu : H_J \to \mathrm{GL}_1$.

In general, the identity component H_J^0 of H_J will be a Levi subgroup of a maximal parabolic subgroup of G_J.

Example 5.1.1. When $J = k$, $H_J \simeq \mathrm{GL}_2$.

Example 5.1.2. When $J = H_3(k)$, $H_J \simeq \mathrm{GSp}_6$.

Example 5.1.3. When $J = H_3(E)$ with E a quadratic etale extension of k, H_J is of type A_5.

Example 5.1.4. When $J = H_3(B)$ with B a quaternion algebra, H_J is of type D_6.

Example 5.1.5. When $J = H_3(\Theta)$ with Θ an octonion algebra, H_J is of type E_7.

Denote by $(\, , \, , \, , \,)_{W_J}$ the unique symmetric four-linear form normalized so that $(v, v, v, v)_{W_J} = 2q(v)$. Define $t : W_J \times W_J \times W_J \to W_J$ as $(w, x, y, z) = \langle w, t(x, y, z) \rangle$ and set $v^\flat = t(v, v, v)$.

5.2. Rank one elements

If J is a cubic norm structure, there is a notion of rank of elements of J, which we now define.

DEFINITION 5.2.1. All elements of J are of rank at most 3. If $N(x) = 0$, then x has rank at most 2. If $x^{\#} = 0$, then x has rank at most one. If $x = 0$, then x has rank 0.

If $J = H_3(k)$, the above definition reduces to the usual notion of rank of a 3×3 symmetric matrix. There is a related notion of rank of elements of W_J.

DEFINITION 5.2.2. All elements of W_J are of rank at most 4. If $q(v) = 0$, then v has rank at most 3. If $v^\flat = 0$, then v has rank at most two. If $(v, v, w', w) = 0$ for all $w' \in (kv)^\perp$, then v has rank at most one. If $v = 0$, then v has rank 0.

EXAMPLE 5.2.3. If $a, d \neq 0$, then $(a, 0, 0, d)$ has rank 4. If $x \in J$ has rank $j = 0, 1, 2, 3$, then $(0, x, 0, 0)$ has rank j in in W_J.

EXAMPLE 5.2.4. The element $(1, 0, 0, 0)$ of W_J has rank one. If $v = (1, 0, c, d)$ has rank one, then $c = d = 0$.

We will need a definition of certain elements of $End(W_J)$, constructed from two elements of W_J.

DEFINITION 5.2.5. For $w, w' \in W_J$ define $\Phi_{w,w'} \in \mathrm{End}(W_J)$ as follows:

$$\Phi_{w,w'}(x) = 6t(w, w', x) + \langle w', x\rangle w + \langle w, x\rangle w'.$$

One can show that v has rank at most one if and only if $\frac{1}{2}\Phi_{v,v}(x) := 3t(v, v, x)+\langle v, x\rangle v = 0$ for all $x \in W_J$. (Note that this condition implies the rank one condition of the definition, but the converse is not at all obvious.)

It is clear that the set of rank one elements is an H_J-set. In fact,

PROPOSITION 5.2.6. *There is one H_J^1-orbit of rank one lines.*

Let $\mathfrak{h}(J)^0$ denote the Lie algebra of H_J^1.

PROPOSITION 5.2.7. *For $w, w' \in W_J$, the endomorphism $\Phi_{w,w'}$ is in $\mathfrak{h}(J)^0$, i.e., it preserves the symplectic and quartic form on W_J. Furthermore, if $\phi \in \mathfrak{h}(J)^0$, then $[\phi, \Phi_{w,w'}] = \Phi_{\phi(w),w'} + \Phi_{w,\phi(w')}$.*

5.3. The exceptional upper half-space

When the ground field $k = \mathbf{R}$ and the pairing $(\; , \;)$ on J is positive definite, the group H_J has an associated Hermitian symmetric space. This space is $\mathcal{H}_J = \{Z = X + iY : Y > 0\}$. Here $Y > 0$ means that $Y = y^2$ for some $y \in J$ with $N(y) \neq 0$.

5.3.1. The positive definite cone.

THEOREM 5.3.1. *The following statement are equivalent:*

1. *$Y \in J$ is positive-definite, i.e., $Y = y^2$ for some $y \in J$ with $N(y) \neq 0$.*
2. *There exists $a \in A_J$ with aY diagonal with positive entries*
3. *$\mathrm{tr}(Y), \mathrm{tr}(Y^{\#})$ and $N(Y)$ are all positive.*

THEOREM 5.3.2. *Let C denote the set of Y in J with $Y > 0$. Then C is connected and convex.*

5.3.2. The upper half space. We now define how $H_J(\mathbf{R})^0$ acts on $\mathcal{H}_J$.

To do this, suppose $Z \in J_{\mathbf{C}}$. Define $r_0(Z) = (1, -Z, Z^{\#}, -N(Z))$. Then one has the following proposition.

PROPOSITION 5.3.3. *Suppose $Z \in \mathcal{H}_J$, so that $Im(Z)$ is positive definite. Suppose moreover that $g \in H_J(\mathbf{R})^0$. Then there is $j(g,Z) \in \mathbf{C}^\times$ and $gZ \in \mathcal{H}_J$ so that $gr_0(Z) = j(g,Z)r_0(gZ)$. This equality defines the factor of automorphy $j(g,Z)$ and the action of $H_J(\mathbf{R})^0$ simultaneously.*

Suppose J is a cubic norm structure. Let $\iota : J \leftrightarrow J^\vee$ be the identification given by the symmetric pairing on J. Define $J_2 : W_J \to W_J$ as $J_2(a,b,c,d) = (d, -\iota(c), \iota(b), -a)$. One checks that $J_2 \in H_J^1$.

We will now explain the stabilizer of $i1_J$ inside of H_J^1.

PROPOSITION 5.3.4. *Suppose $g \in H_J^1(\mathbf{R})$ stabilizers $i1_J \in \mathfrak{H}_J$. Then g commutes with J_2.*

5.3.3. Modular forms. As an aside, we now define holomorphic modular forms on $E_{7,3}$. To do so, let $J = H_3(\Theta)$ be as above, where Θ is an octonion algebra with positive definite norm form. Define $G = H_J^1(\mathbf{R})$. It turns out that the group G is connected, so it acts (transitively) on $\mathcal{H}_J$.

Following Baily [**Bai70**], a discrete subgroup $\Gamma \subseteq G$ is defined as follows. Let $\Theta_0 \subseteq \Theta$ be Coxeter's ring of integral octonions; see, e.g., *loc cit.* Define $J_0 \subseteq J$ to be the integral lattice consisting of matrices $X = \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix}$ with $c_1, c_2, c_3 \in \mathbf{Z}$ and $x_1, x_2, x_3 \in \Theta_0$. Define $W_{J_0} \subseteq W_J$ to be the lattice $W_{J_0} = \mathbf{Z} \oplus J_0 \oplus J_0^\vee \oplus \mathbf{Z}$. Then Γ is defined to be the subgroup of $H_J^1(\mathbf{Q})$ that preserves W_{J_0}.

A modular form for Γ of weight $\ell > 0$ is a holomorphic function $f : \mathcal{H}_J \to \mathbf{C}$ satisfying

(1) $f(\gamma Z) = j(\gamma, Z)^\ell f(Z)$ for all $\gamma \in \Gamma$ and
(2) the function $\phi_f : \Gamma \backslash G \to \mathbf{C}$ defined by $\phi_f(g) = j(g,i)^{-\ell} f(g \cdot i)$ is of moderate growth.

Some results about modular forms on G can be found in [**Bai70**], [**Kim93**], [**GL97**], [**KY16**].

CHAPTER 6

The quaternionic groups

In this chapter we discuss more about the quaternionic exceptional groups. A reference for this material is [**Pol**, Chapter 4] and [**Pol20a**].

6.1. The $\mathbf{Z}/2$-grading

Previously, we defined the Lie algebra $\mathfrak{g}(J)$ via a $\mathbf{Z}/3$-grading. In this section, we redefine $\mathfrak{g}(J)$ via a $\mathbf{Z}/2$-grading. I believe the construction of the Lie algebra $\mathfrak{g}(J)$ in this section essentially goes back to Freudenthal.

Denote by V_2 the defining two-dimensional representation of $\mathfrak{sl}_2$. We have an identification $Sym^2(V_2) \simeq \mathfrak{sl}_2$ as $(v \cdot v')(x) = \langle v', x\rangle v + \langle v, x\rangle v'$. Here $\langle\ ,\ \rangle$ is the standard symplectic pairing on V_2:

$$\langle (a,b)^t, (c,d)^t \rangle = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = ad - bc.$$

We define

$$\mathfrak{g}(J) = \mathfrak{g}(J)_0 \oplus \mathfrak{g}(J)_1 := \left(\mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0\right) \oplus (V_2 \otimes W_J).$$

Here $\mathfrak{g}(J)_0 = \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0$ is the zeroth graded piece of $\mathfrak{g}(J)$, and $\mathfrak{g}(J)_1 = V_2 \otimes W_J$ is the first graded piece of $\mathfrak{g}(J)$.

6.1.0.1. *The bracket.* We define a map $[\ ,\ ] : \mathfrak{g}(J) \otimes \mathfrak{g}(J) \to \mathfrak{g}(J)$ as follows: If $\phi, \phi' \in \mathfrak{g}(J)_0 = \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0$, $v, v' \in V_2$, and $w, w' \in W_J$, then

$$[(\phi, v\otimes w), (\phi', v'\otimes w')] = \left([\phi,\phi'] + \frac{1}{2}\langle w, w'\rangle (v \cdot v') + \frac{1}{2}\langle v, v'\rangle \Phi_{w,w'}, \phi(v' \otimes w') - \phi'(v \otimes w)\right).$$

With this definition, we have the following fact.

Proposition 6.1.1. *The bracket $[\ ,\ ,]$ on $\mathfrak{g}(J)$ satisfies the Jacobi identity.*

Proof. To check the Jacobi identity $\sum_{cyc}[X,[Y,Z]] = 0$, by linearity it suffices to check it on the various $\mathbf{Z}/2$-graded pieces. Then there are four types identities that must be checked. Namely, if $0, 1, 2$ or 3 of the elements X, Y, Z are in $\mathfrak{g}(J)_1 = V_2 \otimes W_J$. If all three of X, Y, Z are in $\mathfrak{g}(J)_0 = \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0$, then the Jacobi identity is of course satisfied. If two of X, Y, Z are in $\mathfrak{g}(J)_0$, then the Jacobi identity is satisfied. This fact is equivalent to the fact that the bracket $[\ ,\ ]_\alpha$ defines a Lie algebra action of $\mathfrak{g}(J)_0$ on $\mathfrak{g}(J)_1$: $[\phi,\phi'](x) = \phi(\phi'(x)) - \phi'(\phi(x))$ for $x \in \mathfrak{g}(J)_1$ and $\phi, \phi' \in \mathfrak{g}(J)_0$. If one of X, Y, Z is in $\mathfrak{g}(J)_0$, then the Jacobi identity is satisfied by the equivariance of the map $\mathfrak{g}(J)_1 \otimes \mathfrak{g}(J)_1 \to \mathfrak{g}(J)_0$. Finally, when X, Y, Z are all in $\mathfrak{g}(J)_1$, a simple direct computation shows that $\sum_{cyc}[X,[Y,Z]] = 0$.

In more detail, suppose $X_1 = v_1 \otimes w_1$, $X_2 = v_2 \otimes w_2$ and $X_3 = v_3 \otimes w_3$. We must evaluate:

$$-2\sum_{cyc}[X_1,[X_2,X_3]] = 2\sum_{cyc}[v_2 \otimes w_2, v_3 \otimes w_3](v_1 \otimes w_1)$$

$$= \sum_{cyc} (\langle v_2, v_3\rangle \Phi_{w_2,w_3} + \langle w_2, w_3\rangle v_2 \cdot v_3)(v_1 \otimes w_1)$$
$$= \sum_{cyc} \langle v_2, v_3\rangle v_1 \otimes (6t(w_1, w_2, w_3) + \langle w_3, w_1\rangle w_2 + \langle w_2, w_1\rangle w_3)$$
$$+ \sum_{cyc} \langle w_2, w_3\rangle(\langle v_3, v_1\rangle v_2 + \langle v_2, v_1\rangle v_3) \otimes w_1.$$

The term $t(w, w', w'')$ drops out right away because it is symmetric by applying the identity $\sum_{cyc} \langle v_2, v_3\rangle v_1 = 0$ for $v_1, v_2, v_3 \in V_2$. The other cyclic sums cancel in pairs. $\square$

One can give an explicit identification between the Lie algebra $\mathfrak{g}(J)$ defined in this section and the one defined via the $\mathbf{Z}/3$-grading, which is why we have given both Lie algebras the same name. See [**Pol20a**, Proposition 4.2.1].

6.2. The Heisenberg parabolic

We assume in this subsection that the Lie algebra $\mathfrak{g}(J)$ is defined over a ground field F of characteristic 0. Recall that the group G_J is defined to be the connected component of the identity of the automorhpism group of $\mathfrak{g}(J)$. This is a connected reductive adjoint group.

For notation, we write e_0, h_0, f_0 for the usual $\mathfrak{sl}_2$-triple inside $\mathfrak{sl}_2 \subseteq \mathfrak{g}(J)_0$, so that $e_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $h_0 = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$, and $f_0 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Set $e = (1, 0)^t$ and $f = (0, 1)^t$ the standard basis of V_2.

6.2.0.1. *The 5-grading.* We now define the 5-grading on $\mathfrak{g}(J)$. Namely, the components of $\mathfrak{g}(J)$ in each graded piece are

- In degree -2: spanned by f_0
- In degree -1: $f \otimes W_J$
- In degree 0: $Fh_0 \oplus \mathfrak{h}(J)^0$
- In degree 1: $e \otimes W_J$
- In degree 2: spanned by e_0.

Note that this is the grading associated to the eigenvalues of h_0 on $\mathfrak{g}(J)$. The degree 0 piece $Fh_0 \oplus \mathfrak{h}(J)^0$ is the Lie algebra $\mathfrak{h}(J)$, via the map $\alpha h_0 + \phi_0 \mapsto \alpha \mathrm{Id}_{W_J} + \phi_0$, where $\phi_0 \in \mathfrak{h}(J)^0$ and Id_{W_J} denotes the identity on W_J.

6.2.0.2. *The Heisenberg parabolic.* We now define the Heisenberg parabolic of G_J. Define $P \subseteq G_J$ to be the $g \in G_J$ stabilizing the line Fe_0 generated by e_0. Then P is parabolic. For instance, the variety G/P is the subset of the projective space $\mathbf{P}(\mathfrak{g}(J))$ consisting of those X with $[X, [X, y]] + 2B_{\mathfrak{g}}(X, y)X = 0$ for all $y \in \mathfrak{g}(J)$. Thus G/P is cut out by closed conditions, so is projective.

Equivalently, define $\mathfrak{p}_{Heis} \subseteq \mathfrak{g}(J)$ to consist of the elements X so that $[X, e_0] \in Fe_0$. Then the Heisenberg parabolic is equivalently defined to be the $g \in G_J$ satisfying $g\mathfrak{p}_{Heis} = \mathfrak{p}_{Heis}$. Furthermore, $\mathfrak{p}_{Heis}$ consists exactly of the element of $\mathfrak{g}(J)$ non-negative degree in the 5-grading.

The Lie algebra of P is $\mathfrak{p}_{Heis}$. Define a Levi subgroup M of P to be the subgroup of P that preserves the 5-grading. Equivalently, M is the subgroup of P that also fixes the line spanned by f_0. The Levi subgroup M is exactly the group H_J^0, as we now prove.

LEMMA 6.2.1. *The map $M \to \mathrm{GL}_1 \times \mathrm{GL}(W_J)$ defined by the conjugation action of M on the degree 2 and degree 1 pieces of the 5-grading defines an isomorphism $M \simeq H_J$. The GL_1-projection is the similitude.*

PROOF. See [**Pol20a**, Lemma 4.3.1]. □

Let N denote the unipotent radical of the Heisenberg parabolic P. Then the center Z of N is the exponential of the one-dimensional space Fe_0. The group Z is also the commutator subgroup $[N, N]$ of N. Consequently, the abelianization of N is naturally identified with W_J, via the map $w \mapsto \exp(w) \mapsto \overline{\exp(w)}$, where $\exp(w) \in N$ and for $n \in N$, $\overline{n}$ denotes the image of n in N/Z.

In case $G = G_2$, the Heisenberg parabolic is the (standard) maximal parabolic with two-step unipotent radical.

6.3. The Cartan involution

In this section, we discuss the Cartan involutions on various relevant groups.

6.3.1. The Cartan involution on Sp_{2n}. Suppose $W = F^{2n}$, $J_n = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$, and $\mathrm{Sp}(W) = \{g \in Aut(W) : {}^t g J_n g = J_n\}$. In other words, assume that the sympletic form on W is defined by $\langle w_1, w_2 \rangle = {}^t w_1 J_n w_2$ for w_1, w_2 column vectors in $W = F^{2n}$. Then $J_n \in \mathrm{Sp}(W)$. This induces an involution Θ on $Sym^2(W)$ via $\Theta(ww') = (J_n w)(J_n w')$. If the ground field $F = \mathbf{R}$, this is a Cartan involution and $(w_1, w_2) := \langle J_n w_1, w_2 \rangle$ defines a symmetric positive definite form on W.

6.3.2. The Cartan involution on $\mathrm{SO}(V)$. The Killing form is

$$B_{\mathfrak{so}}(w \wedge x, y \wedge z) = (x, y)(w, z) - (w, y)(x, z).$$

This is a symmetric $\mathfrak{so}(V)$ invariant form on $\mathfrak{so}(V)$; if the ground field $F = \mathbf{R}$, it is a positive multiple of the Killing form.

Suppose $F = \mathbf{R}$. Suppose $\iota : V \to V$ is an involution, for which the quadratic form (v, v) is positive definite on the subspace of V for which ι is $+1$, and is negative definite where ι is -1. Further assume that ι defines an element of the orthogonal group $\mathrm{O}(V)$. Then $(v, \iota(w))$ is a positive definite symmetric bilinear form on V. Associated to ι, one can define a Cartan involution Θ_ι on the Lie algebra $\mathfrak{so}(V) \simeq \wedge^2 V$. Namely, one sets $\Theta_\iota : \wedge^2 V \to \wedge^2 V$ via $\Theta_\iota(v \wedge w) = \iota(v) \wedge \iota(w)$.

6.3.3. The Cartan involution on M_J. In the next several subsections, we assume the trace pairing on J is positive definite. This occurs if $J = H_3(C)$, with C a composition algebra with positive-definite norm form n_C.

Recall that the pairing on J gives rise to an $\iota : J \to J^\vee$ and thus an involution $\Theta_{\mathfrak{m}}$ on $\mathfrak{m}(J)$ via $\Theta_{\mathfrak{m}}(\phi) = \iota^{-1} \circ \widetilde{\phi} \circ \iota$, where $\widetilde{\phi}$ denotes the action of ϕ on $J^\vee$. One computes immediately that $\Theta_{\mathfrak{m}}(\Phi_{\gamma,x}) = -\Phi_{\iota(x),\iota(\gamma)}$. If the ground field $F = \mathbf{R}$, $\Theta_{\mathfrak{m}}$ is a Cartan involution on $\mathfrak{m}(J)$.

6.3.4. The Cartan involution on H_J. Suppose the ground field $F = \mathbf{R}$. Consider the map J_2 on W_J, given by $J_2(a, b, c, d) = (d, -\iota(c), \iota(b), -a)$. Define a symmetric pairing on W_J via $(v_1, v_2) := \langle J_2 v_1, v_2 \rangle$. Since J_2 is in H_J^1, there is an associated involution on $\mathfrak{h}_J$ given by $\Theta_{\mathfrak{h}}(\phi) = J_2 \phi J_2^{-1}$. One has $\Theta_{\mathfrak{h}}(\Phi_{w,w'}) = \Phi_{J_2 w, J_2 w'}$. Then $\Theta_{\mathfrak{h}}$ is a Cartan involution on $\mathfrak{h}(J)^0$. .

6.3.5. The Cartan involution on G_J I. We abuse notation and also write $J_2 = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) \in \mathrm{SL}_2$. (There is a natural map $\mathrm{SL}_2 \to H_J^1$, and the image of $J_2 \in \mathrm{SL}_2$ is the $J_2 \in H_J^1$.)

Using J_2, we define an involution $\Theta_{\mathfrak{g}}$ on $\mathfrak{g}(J)$ as

$$\Theta_{\mathfrak{g}}(\phi_2 + \phi_J, v \otimes w) = (J_2 \phi_2 J_2^{-1} + J_2 \phi_J J_2^{-1}, J_2 v \otimes J_2 w).$$

Here $\phi_2 \in \mathfrak{sl}_2$, $\phi_J \in \mathfrak{h}(J)^0$, $v \in V_2$ and $w \in W_J$. It is clear that $\Theta_{\mathfrak{g}}$ is an involution on $\mathfrak{g}(J)$. If the ground field $F = \mathbf{R}$ then $\Theta_{\mathfrak{g}}$ defines a Cartan involution on $\mathfrak{g}(J)$.

6.3.6. The Cartan involution on G_J II. We now express the Cartan involution $\Theta_{\mathfrak{g}}$ on $\mathfrak{g}(J)$ via the definition of $\mathfrak{g}(J)$ in terms of its $\mathbf{Z}/3$-grading. To do this, we endow V_3 with the positive definite symmetric form given by $(v, v') = {}^t v v'$. In other words, we make the standard basis v_1, v_2, v_3 of V_3 orthonormal. This induces an identification ι between V_3 and $V_3^\vee$.

Define an involution $\Theta_{\mathfrak{g}}$ on $\mathfrak{g}(J)$ as follows: On $\mathfrak{sl}_3$ it is $X \mapsto -X^t$. On $\mathfrak{m}(J)^0$ it is Θ_m. On $V_3 \otimes J$ it is $v \otimes X \mapsto \iota(v) \otimes \iota(X) \in V_3^\vee \otimes J^\vee$ and on $V_3^\vee \otimes J^\vee$ it is $\delta \otimes \gamma \mapsto \iota(\delta) \otimes \iota(\gamma) \in V_3 \otimes J$. The map $\Theta_{\mathfrak{g}}$ is a Cartan involution.

CHAPTER 7

Quaternionic modular forms

In this chapter, we explain a bit about what is known regarding quaternionic modular forms beyond the case of G_2.

7.1. The differential equation

We begin with the definition of quaternionic modular forms. Thus suppose J is a cubic norm structure with positive definite trace pairing, and G_J is the associated quaternionic group. Recall that we have identified a Cartan involution $\Theta_{\mathfrak{g}}$ on $\mathfrak{g}(J) \otimes \mathbf{R}$. Let $K \subseteq G_J(\mathbf{R})$ be the associated maximal compact subgroup. We assume G_J is of an exceptional Dynkin type; this insures that $G_J(\mathbf{R})$ is connected. Our assumptions let us uniformly describe the maximal compact subgroup K as $K = (\mathrm{SU}(2) \times L)/\mu_2$ for a certain group L (that depends on J). We write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition. As a representation of K, one has $\mathfrak{p} = V_2 \boxtimes W$, where V_2 is the standard representation of $\mathrm{SU}(2)$ and W is a certain symplectic representation of L. See [**GW96**] and [**Pol20a**].

For an integer $\ell \geq 1$, let $\mathbf{V}_\ell = Sym^{2\ell}(V_2) \boxtimes \mathbf{1}$, as a representation of K. Now suppose $\varphi : G_J(\mathbf{R}) \to \mathbf{V}_\ell$ is a smooth function, satisfying $\varphi(gk) = k^{-1}\varphi(g)$ for all $g \in G_J(\mathbf{R})$ and $k \in K$. We define a differential operator $D_\ell\varphi : G(\mathbf{R}) \to Sym^{2\ell-1}(V_2) \boxtimes W$ as follows.

First, we define $\widetilde{D}_\ell\varphi : G(\mathbf{R}) \to \mathbf{V}_\ell \otimes \mathfrak{p}^\vee$ as:

$$\widetilde{D}_\ell\varphi = \sum_j X_j\varphi \otimes X_j^\vee$$

where $\{X_j\}$ is a basis of $\mathfrak{p}$ and $\{X_j^\vee\}$ is the dual basis of $\mathfrak{p}^\vee$. Here $X_j\varphi$ is the right regular action of $\mathfrak{p} \subseteq \mathfrak{g}$ on φ. One checks easily that $\widetilde{D}_\ell\varphi$ is still K-equivariant, i.e., if $\varphi' = \widetilde{D}_\ell\varphi$, then $\varphi'(gk) = k^{-1}\varphi(g)$ for all $k \in K$ and $g \in G_J(\mathbf{R})$.

Now, because $\mathfrak{p} \simeq \mathfrak{p}^\vee$, we have

$$\mathbf{V}_\ell \otimes \mathfrak{p}^\vee \simeq Sym^{2\ell-1}(V_2) \boxtimes W \oplus Sym^{2\ell+1}(V_2) \boxtimes W.$$

Let $pr : \mathbf{V}_\ell \otimes \mathfrak{p}^\vee \to Sym^{2\ell-1}(V_2) \boxtimes W$ be a K equivariant projection (unique up to scalar multiple). We define $D_\ell = pr \circ \widetilde{D}_\ell$.

Definition 7.1.1. Suppose $\ell \geq 1$ is a non-negative integer. A smooth function $\varphi : G_J(\mathbf{A}) \to \mathbf{V}_\ell$ is a **quaternionic modular form** of weight ℓ if

(1) φ is of moderate growth
(2) φ is right invariant under an open compact subgroup of $G_J(\mathbf{A}_f)$
(3) φ is left $G(\mathbf{Q})$-invariant, i.e., $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbf{Q})$
(4) φ is K-equivariant, i.e., $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in K$ and
(5) $D_\ell\varphi \equiv 0$.

7.2. Representation theory

One can also define quaternionic modular forms from the lens of representation theory. In this section, we briefly explain this representation-theoretic approach and its relationship to the definition above.

To begin, we start with the *quaternionic representations* of the group $G_J(\mathbf{R})$. These were defined and studied by Gross-Wallach [**GW94, GW96**]. We say[1] an irreducible representation π of $G_J(\mathbf{R})$ is quaternionic of weight ℓ if

(1) π contains the K-type $\mathbf{V}_\ell^\vee \simeq \mathbf{V}_\ell$ with multiplicity one
(2) π does not contain the K-type $Sym^{2\ell-1}(V_2) \boxtimes W$.

Gross-Wallach [**GW94, GW96**] constructed quaternionic representations π_ℓ of $G_J(\mathbf{R})$, which at least for ℓ sufficiently large are moreover discrete series representations of $G_J(\mathbf{R})$. Let $S \simeq \mathrm{SU}(2) \subseteq K$ be the normal subgroup that is the kernel of the map $K \to L/\mu_2$. Gross-Wallach also showed that the π_ℓ are admissible when restricted to S. The above properties make the π_ℓ analogous to the so-called holomorphic (discrete series) representations of the groups that have an associated Hermitian symmetric space.

Suppose π is a quaternionic representation of $G_J(\mathbf{R})$ of weight ℓ. Suppose $\Phi_\pi : \pi \to \mathcal{A}(G_J)$ is homomorphism of $(\mathfrak{g}, K)$-modules, from π to the space of automorphic forms on G_J. In the parlance of [**GGS02**], such a map is a modular form of weight ℓ. To relate the maps Φ_π to the φ's of the previous section, one proceeds as follows. Restricting Φ_π to the K-type $\mathbf{V}_\ell^\vee$ of π, one obtains a map $\Phi_\pi : \mathbf{V}_\ell^\vee \to \mathcal{A}(G_J)$, or equivalently, a function $\varphi : G_J(\mathbf{Q})\backslash G_J(\mathbf{A}) \to \mathbf{V}_\ell$. Because Φ_π is K-equivariant, so is the function φ. Moreover, because π does not contain the K-type $Sym^{2\ell-1}(V_2) \boxtimes W$, one can check that $D_\ell \varphi = 0$. Consequently, out of a map $\Phi_\pi : \pi \to \mathcal{A}(G_J)$, one obtains a modular form φ of weight ℓ.

7.3. The Fourier expansion

In this section, we describe the Fourier expansion of quaternionic modular forms. Suppose φ is a quaternionic modular form. One can take the constant term of φ along Z, and then Fourier expand the result along N/Z:

$$\varphi_Z(g) = \varphi_N(g) + \sum_{\omega \in W_J, \omega \neq 0} \varphi_\omega(g) \tag{1}$$

where $\varphi_\omega(g) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \psi(\langle \omega, \overline{n} \rangle)^{-1} \varphi(ng)\, dn$. The Fourier expansion we explain in this section is a refinement of (1) for quaternionic modular forms.

The existence of the Fourier expansion that we describe is based on the following theorem, which is the main result of [**Pol20a**]. To state the theorem, we need a couple definitions. Say a nonzero element $\omega \in W_J(\mathbf{R})$ is *positive semi-definite*, written $\omega \geq 0$, if $\langle \omega, (1, -Z, Z^\#, -N(Z)) \rangle$ is never 0 for Z in the upper half space $\mathfrak{h}_J = \{Z = X + iY : X, Y \in J, Y > 0\}$. (It is mildly remarkable that such ω exist!)

We now define generalized Whittaker functions. If χ is a character of $N(\mathbf{R})$, we say a function $F : G_J(\mathbf{R}) \to \mathbf{V}_\ell$ is a generalized Whittaker function of type χ if it satisfies

- $F(ng) = \chi(n)F(g)$ for all $n \in N(\mathbf{R})$
- $F(gk) = k^{-1}F(g)$ for all $k \in K$

[1] This is an ad-hoc definition that is suitable for our purposes

- $D_\ell F \equiv 0$

If $\omega \in W_J$ and $\chi(n) = \chi_\omega(n) = e^{i\langle \omega, \overline{n}\rangle}$, we alternatively say that F is a generalized Whittaker function of type ω.

THEOREM 7.3.1. *Suppose $\omega \in W_J(\mathbf{R})$ is nonzero. Then the space of moderate growth generalized Whittaker functions of type ω is at most one-dimensional. Moreover:*

(1) *Suppose ω is not positive semi-definite. Then every moderate growth generalized Whittaker function of type ω is 0.*
(2) *Suppose ω is positive semi-definite. Then there is a completely explicitly function $W_\omega : G_J \to \mathbf{V}_\ell$ satisfying:*
    (a) *W_ω is a moderate growth generalized Whittaker function of type ω*
    (b) *If F is a moderate growth generalized Whittaker function of type ω then $F = \lambda_F W_\omega$ for some $\lambda_F \in \mathbf{C}$.*

When $\omega \in W_J(\mathbf{R})$ is rank four, the first two statements of the Theorem (i.e., the theorem without the explicit function W_ω) are due to Wallach [**Wal03**].

An immediate corollary is the Fourier expansion of quaternionic modular forms:

COROLLARY 7.3.2. *Let $\omega \in W_J$ be nonzero, let φ be a quaternionic modular form of weight ℓ, and let φ_ω be as in* (1). *Then*

(1) *If ω is not positive semi-definite, $\varphi_\omega(g) \equiv 0$.*
(2) *If ω is positive semi-definite, then there is locally constant function $c_\omega : G_J(\mathbf{A}_f) \to \mathbf{C}$ so that $\varphi_\omega(g_f g_\infty) = c_\omega(g_f) W_{2\pi\omega}(g_\infty)$.*

The locally constant functions $c_\omega(g_f)$, or sometimes their values at $g_f = 1$, are called the Fourier coefficients of φ. When ω is rank four, Gan-Gross-Savin [**GGS02**] had defined these Fourier coefficients without the use of the explicit functions W_ω, but instead using Wallach's results in [**Wal03**].

We now describe the functions W_ω. Because of the equivariance properties of W_ω, to describe it completely it suffices to give a formula for W_ω on the Levi H_J of the Heisenberg parabolic. Let e_ℓ, h_ℓ, f_ℓ be the basis of the long root $\mathfrak{sl}_2$ of $\mathfrak{k}$ from [**Pol20a**]. Let x, y be a basis of its standard representation $\mathbf{C}^2$, so that $h_\ell x = x$, $h_\ell y = -y$ and $f_\ell x = y$. We write $\mathbf{V}_\ell$ for the 2ℓth symmetric power representation of this $\mathfrak{sl}_2$, which we also consider as a representation of K. The space $\mathbf{V}_\ell$ has as basis the elements $x^{\ell+v}y^{\ell-v}$ for $-\ell \leq v \leq \ell$.

Let $r_0(i) \in W_J \otimes \mathbf{C}$ be the element $r_0(i) = (1, -i, -1, i)$. Then if $m \in H_J$,

$$W_\omega(m) = \nu(m)^\ell |\nu(m)| \sum_v \left(\frac{|\langle \omega, m \cdot r_0(i)\rangle|}{\langle \omega, m \cdot r_0(i)\rangle}\right)^v K_v(|\langle \omega, m \cdot r_0(i)\rangle|) \frac{x^{\ell+v}y^{\ell-v}}{(\ell+v)!(\ell-v)!}.$$

7.4. Examples of quaternionic modular forms

In this section, we give results and examples about quaternionic modular forms and their Fourier coefficients that go beyond G_2.

7.4.1. Eisenstein series. The easiest family of examples of quaternionic modular forms is the degenerate Heisenberg Eisenstein series. Thus let $P \subseteq G_J$ be the Heisenberg parabolic, with $\nu : P \to \mathrm{GL}_1$ the character given by $p \cdot e_0 = \nu(p) e_0$. There is a weight ℓ modular form associated to inducing sections in $Ind_P^G(\nu^\ell|\nu|)$. In more detail, suppose $\ell > 0$ is even, and let $f_\ell(g,s) \in Ind_{P(\mathbf{R})}^{G(\mathbf{R})}(|\nu|^s)$ be the unique K-equivariant, $\mathbf{V}_\ell$-valued flat section whose value at $g = 1$ is $x^\ell y^\ell$. Let now $f_{fte} \in Ind_{P(\mathbf{A}_f)}^{G(\mathbf{A}_f)}(|\nu|^s)$ be an arbitrary flat section. Set $f(g,s) = f_{fte}(g_f,s)f_\ell(g_\infty,s)$ and $E(g,f,s) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f(\gamma g, s)$. The sum converges absolutely if $Re(s) > 1 + dim(W_J)/2$. Suppose $s = \ell + 1$ is in the range of absolute convergence, so that $\ell > dim(W_J)/2$. Then it can be shown that the value $E(g,f,s=\ell+1)$ is a quaternionic modular form of weight ℓ.

Another way to create modular forms is to use Heisenberg Eisenstein series with nontrivial inducing data. One can take a classical holomorphic modular form Φ of weight n on H_J (weight $3n$ on G_2), and from it produce a weight n modular form $E(g,\Phi)$ on G_J, if n is sufficiently large. This is spelled out in [**Pol20a**].

7.4.2. Small representations. Because absolutely convergent degenerate Eisenstein series always define modular forms, it makes sense to ask about the automorphic forms defined by these Eisenstein series outside the range of absolute convergence. In other words, suppose $\ell \leq \dim(W_J)/2$. Then one can ask:

- Is the Eisenstein series $E(g,f,s)$ regular at $s = \ell + 1$?
- If so, is the resulting automorphic function a modular form of weight ℓ?

In some cases, these questions have been answered.

Theorem 7.4.1. *Let $E_\ell(g,s)$ be the Eisenstein series on $G = E_8$ with f_{fte} spherical at every finite place.*

(1) *(Gan* [**Gan00a**]*, Gan-Savin* [**GS05**]*) Suppose $\ell = 4$. Then $E_4(g,s)$ is regular at $s = 5$, and $\theta_{min}(g) := E_4(g, s=5)$ defines a square integrable, non-cuspidal, modular form of weight* 4*.*
(2) *(Pollack* [**Pol20c**]*) Suppose $\ell = 8$. Then $E_8(g,s)$ is regular at $s = 9$, and $\theta_{ntm}(g) := E_8(g, s=9)$ defines a square-integrable, non-cuspidal, modular form of weight* 8*.*

The choice of $\ell = 4, 8$ are inspired by the work [**GW94**].

About the modular forms θ_{min}, θ_{ntm}, one can prove that their Fourier coefficients are rational:

Theorem 7.4.2 ([**Pol20b**],[**Pol20c**]). *The modular forms θ_{min} and θ_{ntm} have rational Fourier coefficients.*

7.4.3. Theta lifts. Using θ_{min}, one can construct new, interesting modular forms. The first such examples were considered by Gan-Gross-Savin [**GGS02**]. Recall that, via the correspondence between binary cubic forms and cubic rings, the Fourier coefficients of modular forms on G_2 correspond to totally real cubic rings.

Let $I \in J$ be the three-by-three identity matrix. For a cubic ring A, and an element $X \in J$ with $N_J(X) = 1$ (such as $X = I$), let $N(A,X)$ be the number of maps $f : A \to J$ satisfying

(1) $f(1) = X$
(2) $N_J(f(x)) = N_A(x)$ for all $x \in A$. Here N_A is the cubic norm on A.

This is the number of embeddings of pointed cubic spaces of A into J. Besides $X = I$, there another second element $E \in J$ with $N_J(E) = 1$, with the property that the pointed cubic spaces (J, I) and (J, E) are not globally equivalent; see [**GGS02**] for a discussion. See also [**GG99**], [**EG97**].

THEOREM 7.4.3. [**GGS02**] *There are level one modular forms θ_I and θ_E on G_2 of weight 4 whose Ath Fourier coefficient is $N(A, I)$, respectively, $N(A, E)$, if A is a non-degenerate totally real cubic ring.*

Let H be a certain group of type F_4, which is compact at the archimedean place and split at every finite place. Then there is a dual pair $G_2 \times H \subseteq E_8$. It is a theorem that the automorphic double quotient $H(\mathbf{Q})\backslash H(\mathbf{A})/H(\mathbf{R})H(\widehat{\mathbf{Z}})$ has size two: see [**EG97**] and [**GG99**]. There is thus a two-dimensional space of special automorphic forms on $H(\mathbf{A})$. The modular forms θ_I, θ_E are lifts from this two-dimensional space, using θ_{min} as the kernel function.

The linear combination $91\theta_I + 600\theta_E$ is the theta lift of the trivial function on H. A Siegel-Weil theorem of Gan identifies this lift with the Eisenstein series of weight 4 on G_2.

THEOREM 7.4.4 (Gan [**Gan00b**]). *The linear combination $91\theta_I + 600\theta_E$ is the spherical weight 4 Eisenstein series on G_2.*

We mention that the archimedean theta correspondence between $G_2(\mathbf{R})$ and $H(\mathbf{R})$ has been determined in Huang-Pandzic-Savin [**HPS96**].

7.4.4. Distinguished modular forms. Suppose $f(Z) = \sum_T a_f(T)e^{2\pi i\,\mathrm{tr}(TZ)}$ is a Siegel modular form. Then f is said to be distinguished if it satisfies the following condition:

(1) There exists T_0 with $\det(T_0) \neq 0$ so that $a_f(T_0) \neq 0$
(2) If T is such that $\det(T) \neq 0$ and $a_f(T) \neq 0$, then $\det(T) \equiv \det(T_0) \bmod (\mathbf{Q}^\times)^2$.

One can make an analogous definition for quaternionic modular forms:

DEFINITION 7.4.5. A quaternionic modular form φ with Fourier coefficients $a_\varphi(\omega)$ is said to be distinguished if

(1) There exists a rank four ω_0 with $a_\varphi(\omega_0) \neq 0$
(2) If ω is rank four and $a_\varphi(\omega) \neq 0$, then $q(\omega) \equiv q(\omega_0)$ modulo $(\mathbf{Q}^\times)^2$.

In [**Pol20b**], by restricting θ_{min} to quaternionic groups of type E_6, we constructed distinguished modular forms:

THEOREM 7.4.6. [**Pol20b**] *Let $E = \mathbf{Q}(\sqrt{-d})$ be an imaginary quadratic field, and G_E the quaternionic group of type E_6 defined from E. Then there is a weight 4 distinguished modular form φ_E on G_E. The form φ_E satisfies: if ω is rank four and $a_\varphi(\omega) \neq 0$, then $q(\omega) \equiv -d$ modulo $(\mathbf{Q}^\times)^2$.*

There is an embedding $G_E \to E_8$, and φ_E is defined as the pullback of θ_{min} via this embedding.

CHAPTER 8

The Fourier expansion on orthogonal groups

In this chapter, we give an explicit Fourier expansion on $\mathrm{SO}(4, n+2)$, to show how it may be done for a classical group. More precisely, we state and prove the formula for the generalized Whittaker function on $\mathrm{SO}(4, n+2)(\mathbf{R})^0$. To state the result, we use the notation of the following section. Throughout this chapter, we work over the ground field $\mathbf{R}$ of real numbers.

8.1. Setups

We begin by setting up notation. Set $V_{2,n} = V_2 \oplus V_n$. Here V_2 is a positive definite quadratic space with orthonormal basis v_1, v_2 and V_n is a negative definite quadratic space with basis v_{-j} for $1 \leq j \leq n$. Now set $U = \mathrm{Span}(b_1, b_2)$ and $U^\vee = \mathrm{Span}(b_{-1}, b_{-2})$, i.e., U, respectively $U^\vee$, is a two-dimensional vector space with basis b_1, b_2, respectively b_{-1}, b_{-2}. Let $V = U \oplus V_{2,n} \oplus U^\vee$. We put a symmetric bilinear form $(\ ,\ )$ on V in such a way that $U, U^\vee$ are two-dimensional and isotropic with pairing $(b_i, b_{-j}) = \delta_{ij}$ and $U \oplus U^\vee$ is orthogonal to $V_{2,n}$. We identify the Lie algebra of $G := \mathrm{SO}(V)$ with $\wedge^2 V$, so that $v_1 \wedge v_2(v) = (v_2, v)v_1 - (v_1, v)v_2$.

We define the Heisenberg parabolic $P = MN$ to be the stabilizer of U inside $\mathrm{SO}(V)$. The Levi subgroup M is defined to be the subgroup of P that also stabilizes $U^\vee$. Then $M \simeq \mathrm{GL}_2 \times \mathrm{SO}(V_{2,n})$, and we write $r = \mathrm{diag}(m, h, {}^t m^{-1})$ for a typical element of M. Observe that the Lie algebra $\mathfrak{n}$ of N is $U \wedge (U + V_{2,n})$ and the Lie algebra $\mathfrak{m}$ of M is $U \wedge U^\vee + \wedge^2 V_{2,n}$.

Now, for $j = 1, 2$, one sets $u_j = (b_j + b_{-j})/\sqrt{2}$ and $u_{-j} = (b_j - b_{-j})/\sqrt{2}$. Thus u_1, u_2, u_{-1}, u_{-2} are orthonormal (up to sign). Define a Cartan involution on $\mathrm{SO}(V)$ as conjugation by ι, where $\iota(b_j) = b_{-j}$, $\iota(b_{-j}) = b_j$, ι is $+1$ on V_2 and ι is -1 on V_n. With this Cartan involution, $\mathfrak{p} = \mathrm{Span}\{u_1, u_2, v_1, v_2\} \wedge \mathrm{Span}\{u_{-1}, u_{-2}, v_{-1}, \ldots, v_{-n}\} =: V_4 \wedge V_{n+2}$ and $\mathfrak{k} = \wedge^2 V_4 \oplus \wedge^2 V_{n+2}$. We write K for the associated maximal compact subgroup of $G = \mathrm{SO}(V)$ and K^0 for its identity component. Then $K^0 = \mathrm{SO}(V_4) \times \mathrm{SO}(V_{n+2})$.

Recall that $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)/\mu_2$. Consequently, the complexified Lie algebra $\mathfrak{k}$ of K has two $\mathfrak{sl}_2$ pieces. The first $\mathfrak{sl}_2$ is given as

- $e^+ = \frac{1}{2}(u_1 - iu_2) \wedge (v_1 - iv_2)$
- $h^+ = i(u_1 \wedge u_2 + v_1 \wedge v_2) = \frac{1}{2}(u_1 - iu_2) \wedge (u_1 + iu_2) + \frac{1}{2}(v_1 - iv_2) \wedge (v_1 + iv_2)$
- $f^+ = -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 + iv_2)$.

The other $\mathfrak{sl}_2$ is obtained by replacing v_2 with $-v_2$ in the above formulas: That is, it has basis

- $e'^+ = \frac{1}{2}(u_1 - iu_2) \wedge (v_1 + iv_2)$
- $h'^+ = i(u_1 \wedge u_2 - v_1 \wedge v_2)$
- $f'^+ = -\frac{1}{2}(u_1 + iu_2) \wedge (v_1 - iv_2)$.

Let V_2 denote the standard representation of one of these $\mathfrak{sl}_2$'s. We write x, y for a weight basis of it. We may identify $V_4 \otimes \mathbf{C}$ with $V_2 \otimes V_2$ for these two $\mathfrak{sl}_2$'s as follows:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} u_1 - iu_2 & v_1 + iv_2 \\ -(v_1 - iv_2) & u_1 + iu_2 \end{pmatrix} = \begin{pmatrix} x \otimes x & y \otimes x \\ x \otimes y & y \otimes y \end{pmatrix}.$$

Identifying $\mathfrak{sl}_2 = Sym^2(V_2)$, this leads to the identifications

- $e^+ = (x \otimes x) \wedge (-x \otimes y) \mapsto -x^2$
- $h^+ = (x \otimes x) \wedge (y \otimes y) - (x \otimes y) \wedge (y \otimes x) \mapsto 2xy$
- $f^+ = -(y \otimes y) \wedge (y \otimes x) \mapsto y^2$

Finally, we set $\mathbf{V}_\ell = Sym^{2\ell}(V_2) \boxtimes \mathbf{1} \boxtimes \mathbf{1}$ as a representation of K^0. It has basis $x^{2\ell}, \cdots, y^{2\ell}$.

8.2. Statement of theorem

Our main result in this chapter is a formula for a generalized Whittaker function on $\mathrm{SO}(V)^0$ for the Heisenberg parabolic. In this section, we state that theorem.

In more detail, suppose T_1, T_2 are in $V_{2,n}$. A generalized Whittaker function φ of type (T_1, T_2) is a function on $\mathrm{SO}(V)(\mathbf{R})^0$ that satisfies:

- φ is valued in $\mathbf{V}_\ell$.
- $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in \mathrm{SO}(V)(\mathbf{R})^0$ and $k \in K^0$.
- $\varphi(\exp(b_1 \wedge y_1 + b_2 \wedge y_2) \exp(z b_1 \wedge b_2) g) = e^{i(T_1,y_1)+i(T_2,y_2)} \varphi(g)$.
- $D_\ell \varphi \equiv 0$, where D_ℓ is the so-called Schmid operator, defined exactly as it was in previous sections.

For $r = (m, h, {}^t m^{-1})$, and $T_1, T_2 \in V_{2,n}$ define $\beta(r) = \sqrt{2} i (T_1, T_2) m \begin{pmatrix} h(v_1 + iv_2) \\ ih(v_1 + iv_2) \end{pmatrix}$. The notation $(T_1, T_2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ means $(T_1, w_1) + (T_2, w_2)$. Say that the pair (T_1, T_2) is positive semi-definite, written $(T_1, T_2) \geq 0$, if $\beta(r) \neq 0$ for all $r \in \mathrm{GL}_2(\mathbf{R})^+ \times \mathrm{SO}(V_{2,n})^0$.

Suppose $\epsilon' \in \mathrm{SO}(V_{2,n})$ takes $v_1 \mapsto -v_1$ and $v_2 \mapsto v_2$ and define $\epsilon = \mathrm{diag}(\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right), \epsilon', \left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right))$. Then $\epsilon \in \mathrm{SO}(4, n+2)(\mathbf{R})^0$. Observe that $\beta(r\epsilon) = -\beta^*(r)$.

With notation as above, and (T_1, T_2) positive semi-definite, define

$$W_{(T_1,T_2)}(r) = \det(m)^\ell |\det(m)| \sum_{-\ell \leq v \leq \ell} \left(\frac{|\beta(r)|}{\beta(r)^*} \right)^v K_v(|\beta(r)|) \frac{x^{\ell+v} y^{\ell-v}}{(\ell+v)!(\ell-v)!}.$$

We will need the following result.

Exercise 8.2.1. *The function $W_{(T_1,T_2)}$ satisfies $W_{(T_1,T_2)}(rk) = k^{-1} W_{(T_1,T_2)}(r)$ for all $k \in K^0 \cap M(\mathbf{R})$.*

Here is the theorem.

Theorem 8.2.2. *Suppose F is a moderate growth generalized Whittaker function for the pair (T_1, T_2), and T_1, T_2 are not both 0. Then, if (T_1, T_2) is not positive semi definite, $F = 0$. If (T_1, T_2) is positive semi-definite, then F is proportional to $W_{(T_1,T_2)}$.*

8.3. The Iwasawa decomposition

We need detailed forms of the Iwasawa decomposition of elements of $\mathfrak{p}$. Recall $\mathfrak{n} = U \wedge (U + V_{2,n}) \subseteq \wedge^2 V \simeq \mathfrak{g}$ and $\mathfrak{m} = U \wedge U^\vee + \wedge^2 V_{2,n}$. Then we have $\mathfrak{g} = \mathfrak{n} + \mathfrak{m} + \mathfrak{k}$, and we will write elements of $\mathfrak{p}$ in terms of this decomposition:

- $(u_1 + iu_2) \wedge (u_{-1} + iu_{-2}) = (b_{-1} + ib_{-2}) \wedge (b_1 + ib_2)$ is in $\mathfrak{m}$.
- $(u_1 - iu_2) \wedge (u_{-1} - iu_{-2}) = (b_{-1} - ib_{-2}) \wedge (b_1 - ib_2)$ is in $\mathfrak{m}$.
- $(u_1 + iu_2) \wedge (u_{-1} - iu_{-2}) = (b_{-1} + ib_2) \wedge (b_1 + ib_{-2}) = (b_{-1} \wedge b_1 + b_{-2} \wedge b_2) + 2ib_2 \wedge b_1 + i(b_{-1} \wedge b_{-2} + b_1 \wedge b_2)$. Moreover, $b_{-1} \wedge b_{-2} + b_1 \wedge b_2 = u_1 \wedge u_2 + u_{-1} \wedge u_{-2}$ so that $i(b_{-1} \wedge b_{-2} + b_1 \wedge b_2)$ has projection into $\wedge^2 V_4$ equal to $\frac{1}{2}h^+ + \frac{1}{2}h'^+$.
- $(u_1 - iu_2) \wedge (u_{-1} + iu_{-2}) = (b_{-1} - ib_2) \wedge (b_1 - ib_{-2}) = (b_{-1} \wedge b_1 + b_{-2} \wedge b_2) - 2ib_2 \wedge b_1 - i(b_{-1} \wedge b_{-2} + b_1 \wedge b_2)$. Moreover, $b_{-1} \wedge b_{-2} + b_1 \wedge b_2 = u_1 \wedge u_2 + u_{-1} \wedge u_{-2}$ so that $-i(b_{-1} \wedge b_{-2} + b_1 \wedge b_2)$ has projection into $\wedge^2 V_4$ equal to $-\frac{1}{2}h^+ - \frac{1}{2}h'^+$.

Some more Iwasawa decompositions:

- $u_i \wedge v_{-j} = \sqrt{2} b_i \wedge v_{-j} - u_{-i} \wedge v_{-j}$ is in $\mathfrak{n} + \mathfrak{k}$. Consequently $\frac{1}{\sqrt{2}}(u_1 + iu_2) \wedge v_{-j} = (b_1 + ib_2) \wedge v_{-j} - \frac{1}{\sqrt{2}}(u_{-1} + iu_{-2}) \wedge v_{-j}$ and $\frac{1}{\sqrt{2}}(u_1 - iu_2) \wedge v_{-j} = (b_1 - ib_2) \wedge v_{-j} - \frac{1}{\sqrt{2}}(u_{-1} - iu_{-2}) \wedge v_{-j}$. These decompositions are in $\mathfrak{n} + \mathfrak{k}$.
- $v_i \wedge v_{-j}$ is in $\mathfrak{m}$

One has $v_i \wedge u_{-j} = \sqrt{2} v_i \wedge b_j - v_i \wedge u_j$ is in $\mathfrak{n} + \mathfrak{k}$. This leads to the following decompositions:

- $\frac{1}{2}(v_1 + iv_2) \wedge (u_{-1} + iu_{-2}) = \frac{1}{\sqrt{2}}(v_1 + iv_2) \wedge (b_1 + ib_2) + \frac{1}{2}(u_1 + iu_2) \wedge (v_1 + iv_2) = \frac{1}{\sqrt{2}}(v_1 + iv_2) \wedge (b_1 + ib_2) - f^+$.
- $\frac{1}{2}(v_1 + iv_2) \wedge (u_{-1} - iu_{-2}) = \frac{1}{\sqrt{2}}(v_1 + iv_2) \wedge (b_1 - ib_2) + \frac{1}{2}(u_1 - iu_2) \wedge (v_1 + iv_2) = \frac{1}{\sqrt{2}}(v_1 + iv_2) \wedge (b_1 - ib_2) + e'^+$
- $\frac{1}{2}(v_1 - iv_2) \wedge (u_{-1} + iu_{-2}) = \frac{1}{\sqrt{2}}(v_1 - iv_2) \wedge (b_1 + ib_2) + \frac{1}{2}(u_1 + iu_2) \wedge (v_1 - iv_2) = \frac{1}{\sqrt{2}}(v_1 - iv_2) \wedge (b_1 + ib_2) - f'^+$
- $\frac{1}{2}(v_1 - iv_2) \wedge (u_{-1} - iu_{-2}) = \frac{1}{\sqrt{2}}(v_1 - iv_2) \wedge (b_1 - ib_2) + \frac{1}{2}(u_1 - iu_2) \wedge (v_1 - iv_2) = \frac{1}{\sqrt{2}}(v_1 - iv_2) \wedge (b_1 - ib_2) + e^+$.

8.4. The differential equations

We assume φ is a generalized Whittaker function of type (T_1, T_2).

Suppose $r = (m, h, {}^t m^{-1})$ is in the Heisenberg Levi, so that $m = \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{R})$ and $h \in \mathrm{SO}(V_{2,n})$. Then if $z_1, z_2 \in V_{2,n}$,

$$\begin{aligned}(b_1 \wedge z_1 + b_2 \wedge z_2)\varphi(M) &= i((T_1, m_{11}h(z_1) + m_{12}h(z_2)) + (T_2, m_{21}h(z_1) + m_{22}h(z_2)))\varphi(M) \\ &= i(T_1, T_2)m(h(z_1), h(z_2))^t \varphi(M).\end{aligned}$$

Recall the operator $\widetilde{D}$ with $D_\ell = \mathrm{pr} \circ \widetilde{D}$. We now have:

$$\begin{aligned}\widetilde{D}\varphi = &\frac{1}{2}(u_1 + iu_2) \wedge (u_{-1} + iu_{-2})\varphi \otimes x \boxtimes x \otimes \frac{1}{\sqrt{2}}(u_{-1} - iu_{-2}) \\ &+ \frac{1}{2}(u_1 + iu_2) \wedge (u_{-1} - iu_{-2})\varphi \otimes x \boxtimes x \otimes \frac{1}{\sqrt{2}}(u_{-1} + iu_{-2})\end{aligned}$$

$$+\frac{1}{2}(u_1 - iu_2) \wedge (u_{-1} + iu_{-2})\varphi \otimes y \boxtimes y \otimes \frac{1}{\sqrt{2}}(u_{-1} - iu_{-2})$$
$$+\frac{1}{2}(u_1 - iu_2) \wedge (u_{-1} - iu_{-2})\varphi \otimes y \boxtimes y \otimes \frac{1}{\sqrt{2}}(u_{-1} + iu_{-2})$$
$$+\sum_{1\le j\le n} ((b_1 + ib_2) \wedge v_{-j}\varphi) \otimes x \boxtimes x \otimes v_{-j}$$
$$+\sum_{1\le j\le n} ((b_1 - ib_2) \wedge v_{-j}\varphi) \otimes y \boxtimes y \otimes v_{-j}$$
$$+\sum_{1\le j\le n} \frac{1}{\sqrt{2}}((v_1 + iv_2) \wedge v_{-j}\varphi) \otimes -x \boxtimes y \otimes v_{-j}$$
$$+\sum_{1\le j\le n} \frac{1}{\sqrt{2}}((v_1 - iv_2) \wedge v_{-j}\varphi) \otimes y \boxtimes x \otimes v_{-j}$$
$$+(\frac{1}{\sqrt{2}}(v_1 + iv_2) \wedge (b_1 + ib_2) - f^+)\varphi \otimes -x \boxtimes y \otimes \frac{1}{\sqrt{2}}(u_{-1} - iu_{-2})$$
$$+(\frac{1}{\sqrt{2}}(v_1 + iv_2) \wedge (b_1 - ib_2) + e'^+)\varphi \otimes -x \boxtimes y \otimes \frac{1}{\sqrt{2}}(u_{-1} + iu_{-2})$$
$$+(\frac{1}{\sqrt{2}}(v_1 - iv_2) \wedge (b_1 + ib_2) - f'^+)\varphi \otimes y \boxtimes x \otimes \frac{1}{\sqrt{2}}(u_{-1} - iu_{-2})$$
$$+(\frac{1}{\sqrt{2}}(v_1 - iv_2) \wedge (b_1 - ib_2) + e^+)\varphi \otimes y \boxtimes x \otimes \frac{1}{\sqrt{2}}(u_{-1} + iu_{-2})$$

Simplifying and contracting, one finds that in $D_\ell\varphi$ the coefficient of the various terms are as follows:

Define $[x^a] = \frac{x^a}{a!}$ and similarly $[y^b] = \frac{y^b}{b!}$. We write $\varphi = \sum_{-\ell\le v\le \ell} \varphi_v [x^{\ell+v}][y^{\ell-v}]$.

THEOREM 8.4.1. *Let the notation be as above. Then*

(1) *The coefficient of* $[x^{\ell+v}][y^{\ell-v-1}] \otimes x \otimes \frac{1}{\sqrt{2}}(u_{-1} - iu_{-2})$*:*

$$-\frac{1}{2}(u_1 + iu_2) \wedge (u_{-1} + iu_{-2})\varphi_v - \frac{i}{\sqrt{2}}(T_1, T_2)m(h(v_1) - ih(v_2), ih(v_1) + h(v_2))^t\varphi_{v+1}$$

(2) *The coefficient of* $[x^{\ell+v-1}][y^{\ell-v}] \otimes y \otimes \frac{1}{\sqrt{2}}(u_{-1} + iu_{-2})$*:*

$$\frac{1}{2}(u_1 - iu_2) \wedge (u_{-1} - iu_{-2})\varphi_v - \frac{i}{\sqrt{2}}(T_1, T_2)m(h(v_1) + ih(v_2), -ih(v_1) + h(v_2))^t\varphi_{v-1}$$

(3) *The coefficient of* $[x^{\ell+v}][y^{\ell-v-1}] \otimes x \otimes v_{-j}$*:*

$$-i(T_1, T_2)m(h(v_{-j}), ih(v_{-j}))^t\varphi_v + \frac{1}{\sqrt{2}}((v_1 - iv_2) \wedge v_{-j})\varphi_{v+1}$$

(4) *The coefficient of* $[x^{\ell+v-1}][y^{\ell-v}] \otimes y \otimes v_{-j}$*:*

$$i(T_1, T_2)m(h(v_{-j}), -ih(v_{-j}))^t\varphi_v + \frac{1}{\sqrt{2}}((v_1 + iv_2) \wedge v_{-j})\varphi_{v-1}$$

(5) *The coefficient of* $[x^{\ell+v}][y^{\ell-v-1}]\otimes x\otimes \frac{1}{\sqrt{2}}(u_{-1}+iu_{-2})$*:*

$$\frac{1}{2}(b_1\wedge b_{-1}+b_2\wedge b_{-2}-2(\ell+1)-v)\varphi_v-\frac{i}{\sqrt{2}}(T_1,T_2)m(h(v_1)-ih(v_2),-ih(v_1)-h(v_2))^t\varphi_{v+1}$$

(6) *The coefficient of* $[x^{\ell+v-1}][y^{\ell-v}]\otimes y\otimes \frac{1}{\sqrt{2}}(u_{-1}-iu_{-2})$*:*

$$-\frac{1}{2}(b_1\wedge b_{-1}+b_2\wedge b_{-2}-2(\ell+1)+v)\varphi_v-\frac{i}{\sqrt{2}}(T_1,T_2)m(h(v_1)+ih(v_2),ih(v_1)-h(v_2))^t\varphi_{v-1}$$

8.5. Solving the differential equations

Abusing notation, write $\varphi_v(w,x,y,h)=\varphi_v(r)$ where $r=\mathrm{diag}(m,h,{}^tm^{-1})$ and $m=\left(\begin{smallmatrix} w & \\ & w\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x\\ & 1\end{smallmatrix}\right)\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}$. Define $f_v(w,x,y,h)$ as $\varphi_v=w^{2\ell+2}f_v$. Here $w,y\in\mathbf{R}^\times_{>0}$, $x\in\mathbf{R}$ and $h\in\mathrm{SO}(V_{2,n})^0$. We will find the formula φ by solving the differential equations above in terms of the explicit coordinates w,x,y,h.

Recall

$$\beta=\sqrt{2}i(T_1,T_2)m\begin{pmatrix} h(v_1)+ih(v_2)\\ ih(v_1)-h(v_2)\end{pmatrix}.$$

Note that $b_1\wedge b_{-1}+b_2\wedge b_{-2}$ becomes the differential operator $w\partial_w$. Also note that $w\partial_w(w^Af)=w^A(w\partial_w+A)f$.

With these definitions, the final two differential equations of Theorem 8.4.1 become

- $(w\partial_w-v)f_v+\beta^*f_{v+1}=0$ and
- $(w\partial_w+v)f_v+\beta f_{v-1}=0$.

Note that β depends linearly on w. Solving these two equations on a domain where $\beta\neq 0$ gives that, as a function of w, ϕ_v is proportional to $w^{2\ell+2}K_v(|\beta|)$. (This uses that ϕ_v is of moderate growth.)

Define $Y_v(x,y,h)$ so that $f_v=Y_vK_v(|\beta|)$; i.e., Y_v is independent of w. Then the differential equations imply

$$\begin{aligned}\beta Y_{v-1}K_{v-1}(|\beta|)&=\beta f_{v-1}=-(w\partial_w+v)f_v\\ &=Y_v\left(-(w\partial_w+v)K_v(|\beta|)\right)\\ &=|\beta|Y_vK_{v-1}(|\beta)|.\end{aligned}$$

One obtains that $Y_v=\frac{\beta}{|\beta|}Y_{v-1}=\frac{|\beta|}{\beta^*}Y_{v-1}$. It follows that $\phi_v(w,x,y,h)=Y_0(x,y,h)w^{2\ell+2}\left(\frac{|\beta|}{\beta^*}\right)^vK_v(|\beta|)$. We will now use the other differential equations of Theorem 8.4.1 to prove that Y_0 is constant.

Suppose $m=\left(\begin{smallmatrix} 1 & x\\ 0 & 1\end{smallmatrix}\right)\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}w$. Set $z=x+iy$. Then one computes

$$\beta=-\sqrt{2}y^{-1/2}w(z^*T_1+T_2,h(v_1+iv_2)).$$

We will now prove the following proposition, which will be used in deducing that Y_0 is constant.

Proposition 8.5.1. *One has*

$$y(\partial_x-i\partial_y)(|\beta|)=\frac{\beta}{|\beta|}\left(-\frac{i}{\sqrt{2}}wy^{-1/2}(z^*T_1+T_2,h(v_1-iv_2))\right).$$

PROOF. Set $\beta' = (z^*T_1 + T_2, h(v_1 + iv_2))$, so that $\beta = -\sqrt{2}wy^{-1/2}\beta'$ and $-\frac{\beta'}{|\beta'|} = \frac{\beta}{|\beta|}$. We first compute that

$$y(\partial_x - i\partial_y)(|\beta|) = \frac{i\sqrt{2}}{2}wy^{-1/2}|\beta'| + \sqrt{2}wy^{1/2}(\partial_x - i\partial_y)(|\beta'|).$$

Now

$$(\partial_x - i\partial_y)(|\beta'|) = \frac{1}{2|\beta'|}(\beta'^*(\partial_x - i\partial_y)(\beta') + \beta'(\partial_x - i\partial_y)(\beta'^*)).$$

One computes $(\partial_x - i\partial_y)(\beta') = 0$ and $(\partial_x - i\partial_y)(\beta'^*) = 2(T_1, h(v_1 - iv_2))$. Thus

$$\begin{aligned} y(\partial_x - i\partial_y)(|\beta|) &= (\sqrt{2}wy^{-1/2})\left(\frac{i}{2}|\beta'| + \frac{y}{|\beta'|}\beta'(T_1, h(v_1 - iv_2))\right) \\ &= \frac{\sqrt{2}wy^{-1/2}\beta'}{|\beta|}\left(\frac{i}{2}(zT_1 + T_2, h(v_1 - iv_2)) + y(T_1, h(v_1 - iv_2))\right) \\ &= \frac{i}{2}\frac{\sqrt{2}wy^{-1/2}\beta'}{|\beta'|}(zT_1 + T_2 - 2iyT_1, h(v_1 - iv_2)) \\ &= \left(-\frac{i}{\sqrt{2}}wy^{-1/2}(z^*T_1 + T_2, h(v_1 - iv_2))\right)\left(-\frac{\beta'}{|\beta'|}\right). \end{aligned}$$

The proposition follows. $\square$

Note that if $X \in \mathfrak{g}$ and $v \in V$ then $Xh(v) = \frac{d}{dt}(he^{tX})(v)|_{t=0} = h(X(v))$.

LEMMA 8.5.2. *One has*

(1) $(v_1 - iv_2) \wedge v_{-j}(|\beta|) = \frac{\beta^*}{|\beta|}(\sqrt{2}y^{-1/2}w(z^*T_1 + T_2, h(v_{-j})))$

(2) $(v_1 + iv_2) \wedge v_{-j}(|\beta|) = \frac{\beta}{|\beta|}(\sqrt{2}y^{-1/2}w(zT_1 + T_2, h(v_{-j})))$.

PROOF. We have

$$(v_1 - iv_2) \wedge v_{-j}(|\beta|) = \frac{1}{2|\beta|}(\beta^*((v_1 - iv_2) \wedge v_{-j})(\beta) + \beta((v_1 - iv_2) \wedge v_{-j})(\beta^*))$$

and similarly for $(v_1 + iv_2) \wedge v_{-j}$. Now, one has

- $(v_1 - iv_2) \wedge v_{-j}(\beta) = 2\sqrt{2}y^{-1/2}w(z^*T_1 + T_2, h(v_{-j}))$
- $(v_1 + iv_2) \wedge v_{-j}(\beta) = 0$
- $(v_1 - iv_2) \wedge v_{-j}(\beta^*) = 0$
- $(v_1 + iv_2) \wedge v_{-j}(\beta^*) = 2\sqrt{2}y^{-1/2}w(zT_1 + T_2, h(v_{-j}))$.

The lemma follows. $\square$

PROPOSITION 8.5.3. *The function* $Y_0(x, y, h)$ *above is constant.*

PROOF. We first consider Y_0 as a function of x, y the coordinates on the complex upper half-plane. Note that

$$-\frac{1}{2}(u_1 + iu_2) \wedge (u_{-1} + iu_{-2}) = \frac{1}{2}(b_1 + ib_2) \wedge (b_{-1} + ib_{-2}).$$

By acting on b_1 and b_2, one sees that this is the element of $\mathfrak{gl}_2$ (via our usual isomorphism) with matrix $\frac{1}{2}\left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right)$. As a differential operator, this Lie algebra element gives $iy(\partial_x - i\partial_y)$ on functions that are right invariant under SO(2). Using the proposition above and the differential equations of the theorem, one obtains that $y(\partial_x - i\partial_y)Y_0 = 0$. Similarly, one obtains $y(\partial_x + i\partial_y)Y_0 = 0$. Thus Y_0 is constant as a function of x, y.

Using the lemma above, and again the differential equations of the theorem, one obtains $XY_0 = 0$ for all $X \in \mathfrak{p} \cap \wedge^2 V_{2,n} = V_2 \wedge V_n$. Because Y_0 is right invariant under $K^0 \cap \mathrm{SO}(V_{2,n})^0$, we see that Y_0 is constant. $\square$

We now put everything together. From the work above, one obtains that on a domain where $\beta \neq 0$, there is a unique line of generalized Whittaker functions of type (T_1, T_2), spanned by $W_{(T_1,T_2)}$. Indeed, we solved the differential equations on $\mathrm{GL}_2(\mathbf{R})^+ \times \mathrm{SO}(V_{2,n})^0$, and then one must observe that $W_{(T_1,T_2)}$ is appropriately equivariant by ϵ.

One can now argue as in [**Pol20a**] to prove that if (T_1, T_2) is not positive semi-definite, then the only generalized Whittaker function is 0. The idea is that one first solves the differential equations on a domain where $\beta \neq 0$, and then observes that this unique solution blows up as M approaches a point where $\beta = 0$. Finally, one can check that the $W_{(T_1,T_2)}$ satisfy the Schmid equations of the theorem above; we omit this aspect. This completes our work.

CHAPTER 9

Proposed projects

There are numerous interesting unanswered questions about quaternionic modular forms. In the previous chapters, I have tried to mention a few such questions. Basically, whatever is your favorite result about holomorphic modular forms probably has an interesting, open analogue for quaternionic modular forms. For the projects, I am proposing develop some of the first results about quaternionic modular forms on the groups $U(2,n)$. However, perhaps you have a different idea for a project; that would be great to work on too!

9.1. Project 1: The generalized Whittaker function on $U(2,n)$

The first project is to find the generalized Whittaker function for the Heisenberg parabolic of quaternionic modular forms on $U(2,n)$. The Heisenberg parabolic $P = MN$ of $U(2,n)$ is the maximal parabolic subgroup stabilizing an isotropic line in the defining representation of $U(2,n)$; its Levi subgroup is $\mathrm{GL}_{1,E} \times U(1,n-1)$ where E is the quadratic imaginary extension used to define the unitary group. In one sentence, the goal is to do for the group $U(2,n)$ what is done for the group $\mathrm{SO}(4,n)$ in Chapter 8. Thus references for this project are Chapter 8 of these notes, and, to a lesser extent, [**Pol20a**]. The case of $U(2,n)$ was excluded from [**Pol20a**] because it could not be handled by the same uniform argument that handled the quaternionic exceptional groups. However, a direct argument, similar to the work of Chapter 8, should yield the desired generalized Whittaker function.

Here are some problems to organize the work on this project.

Problem 9.1.1. *Find an explicit realization of the Lie algebra of $G = U(2,n)$ in terms of the standard representation V of $U(2,n)$.*

Problem 9.1.2. *Define a Cartan involution on $\mathfrak{g}$ and identify the maximal compact subgroup K of G. Identify explicitly $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

Problem 9.1.3. *Define quaternionic modular forms on $G = U(2,n)$.*

Problem 9.1.4. *Find the Iwasawa decomposition ($g = \mathfrak{n} + \mathfrak{m} + \mathfrak{k}$) of a weight basis of $\mathfrak{p}$.*

Problem 9.1.5. *Write down explicitly the Schmid differential equations for satisfied by quaternionic modular forms on G.*

Problem 9.1.6. *Solve these differential equations, both when the character of N is trivial, and when it is nontrivial.*

Problem 9.1.7. *Use the case of trivial character to show that Heisenberg Eisenstein series with appropriate cuspidal inducing data become quaternionic modular forms.*

9.2. Project 2: An Oda-Rallis-Schiffman lift for $U(2,n)$

The second project is to give an analogue of the Oda-Rallis-Schiffmann [**Oda78**], [**RS81**] lift to $U(2,n)$. The Oda-Rallis-Schiffmann lift is a theta lift from holomorphic modular forms f on SL_2 or its double cover to holomorphic modular forms $\theta(f)$ on the group $O(2,n)$. It is a very special instance of the general theta lift from symplectic to orthogonal groups, but in this special case one can write simple formulas for the Fourier coefficients of $\theta(f)$ in terms of those of f. See the introduction to [**Pol21**] for more historical references.

The goal of the second project is produce an analogue of this lift from holomorphic modular forms g on $U(1,1)$ to quaternionic modular forms $\theta(g)$ on $U(2,n)$, and to give simple formulas for the Fourier coefficients of $\theta(g)$ in terms of those of g. In [**Pol21**] I considered a special theta lift from holomorphic Siegel modular forms f on $\mathrm{Sp}(4)$ to quaternionic modular forms $\theta(f)$ on $\mathrm{SO}(4,n)$, for some n, and found the quaternionic Fourier coefficients of the lift $\theta(f)$ in terms of the classical Fourier coefficients of f. Thus this project is an analogue to unitary groups of the work done for orthogonal groups in [**Pol21**]. So, [**Pol21**] is a good reference for working on this project.

This project assumes a successful completion of Project 1, but can mostly be worked on independently of that project. Also note that it would be a good idea to pay close attention to Wee Teck Gan's lectures on the theta correspondence to work on this project.

Here are some problems to organize our work on this project.

PROBLEM 9.2.1. *Understand general formulas for the Schrodinger model of the Weil representation for the pair* $(U(1,1),U(2,n))$.

PROBLEM 9.2.2. *Work out a general (soft) formula for Heisenberg Fourier coefficients of* $\theta(g)$ *in terms of the standard Whittaker Fourier coefficients of* g. *Here by a soft formula I mean one where the data for the Weil representation is arbitrary at the archimedean place.*

PROBLEM 9.2.3. *Find special data for the Weil representation for the pair* $(U(1,1),U(2,n))$ *so that holomorphic representations on* $U(1,1)$ *lift to quaternionic representations on* $U(2,n)$.

PROBLEM 9.2.4. *Prove that the quaternionic Fourier coefficients of* $\theta(g)$ *can be given in terms of the Fourier coefficients of* g *via simple formulas.*

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