Mathematics 103B Exam 1

Instructions: This is a closed book, closed note exam. You have 60 minutes to read the questions, solve the questions, and write your solutions. Please time yourself, and mark your starting and ending time on your solutions. Asking others for help, using the internet to help you, consulting your notes or lectures, or spending longer than 60 minutes is academic misconduct.

Once your 60 minutes is up, please scan and upload your solutions to Gradescope. The time it takes to upload your solutions to Gradescope is not part of the 60 minutes you have for the exam. You must upload your solutions to Gradescope by 11:59pm on Friday April 23, US Pacific time. Do not wait until the last minute, in case you have technical difficulties!

For the questions below, please write your solutions clearly and be sure to prove your answers.

1. (16 points) Give an example of a commutative ring $R$, and units $u, v \in R$ so that
   
   • $u + v \neq 0$
   • $u + v$ is a zero-divisor of $R$.

2. (16 points) Give an example of a ring of characteristic 5 that has finite size strictly bigger than 5. That is, write down a specific ring $R$ satisfying
   
   • $5 < |R| < \infty$
   • $\text{char} R = 5$.

3. (16 points) Is $\mathbb{Z}[x]/\langle x^3 \rangle$ an integral domain? (Recall that $\langle x^3 \rangle$ denotes the principal ideal generated by $x^3$.) Be sure to prove your answer.

4. (20 points) Let $R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$. $R$ is a ring (you do not need to prove this fact.) Set $I = \{a + b\sqrt{5} : a \in 5\mathbb{Z}, b \in \mathbb{Z}\}$.
   
   (a) (8 points) Prove that $I$ is an ideal of $R$.
   (b) (12 points) What is the characteristic of $R/I$?

5. (16 points) Suppose $R$ is a commutative ring with unity of characteristic 3. Suppose moreover that $a$ is an element of $R$ satisfying $a^3 = 0$. Prove that $1 + a$ is a unit of $R$.

6. (16 points) Let $R = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$. $R$ is a ring (you do not need to prove this fact.) Prove that $R$ has infinitely many units. **Hint:** Find a unit $u = a + b\sqrt{3}$ in $R$ with $u \neq 1$ by trying small values of $a$ and $b$. 