EXCEPTIONAL THETA FUNCTIONS AND ARITHMETICITY OF MODULAR FORMS ON G_2

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ABSTRACT. Quaternionic modular forms on the split exceptional group $G_2 = G_2^s$ were defined by Gan-Gross-Savin. A remarkable property of these automorphic functions is that they have a robust notion of Fourier expansion and Fourier coefficients, similar to the classical holomorphic modular forms on Shimura varieties. In this paper we prove that in even weight ℓ at least 6, there is a basis of the space of cuspidal modular forms of weight ℓ such that all the Fourier coefficients of elements of this basis are in the cyclotomic extension of $\mathbf{Q}$.

CONTENTS

1. INTRODUCTION

Holomorphic modular forms on Shimura varieties have a good notion of Fourier coefficients. It is theorem, going back to Shimura, and finalized in work of Harris [Har86], that the Fourier coefficients of holomorphic modular forms on Hermitian tube domains give these modular forms an algebraic structure: there is a basis of the space of holomorphic modular forms so that all the Fourier coefficients of elements of this basis are algebraic.

Outside the realm of holomorphic modular forms on Shimura varieties, there is little one can say[1] about the arithmeticity of Fourier coefficients of automorphic functions. In fact, at present, it is not clear in general how one might define a good notion of Fourier coefficients for spaces of non-holomorphic automorphic forms.

Nevertheless, the quaternionic exceptional groups possess a special class of automorphic functions called the quaternionic modular forms, which do have a good notion of Fourier expansion and

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[1]One exception might be the case of globally generic cohomological automorphic forms, for which the Whittaker coefficients can be directly related to Satake paremeters.

Fourier coefficients. These automorphic forms go back to Gross-Wallach [GW94, GW96] and Gan-Gross-Savin [GGS02]. The precise shape of their Fourier expansion was determined in [Pol20a], extending and refining earlier work of Wallach [Wal03]. In particular, it is possible to define what it means for a quaternionic modular form to have Fourier coefficients in some ring $R \subseteq \mathbf{C}$. In [Pol22], we conjectured that the space of quaternionic modular forms of some fixed weight ℓ on a quaternionic exceptional group G has a basis consisting of elements all of whose Fourier coefficients are algebraic numbers.[2]

In this paper, we provide substantial evidence toward this conjecture in the case of G the split exceptional group G_2. We setup the result now.

Suppose φ is a cuspidal quaternionic modular form on G_2 of weight $\ell \geq 1$. Let N denote the unipotent radical of the Heisenberg parabolic of G_2 and Z its center. Then

$$\varphi_Z(g_f g_\infty) = \sum_{\chi} a_\chi(g_f) W_\chi(g_\infty)$$

is the Fourier expansion of φ. Here the χ range over non-degenerate characters of $N(\mathbf{Q})\backslash N(\mathbf{A})$, φ_Z is the constant term of φ along Z, and the W_χ are the generalized Whittaker functions of [Pol20a]. The functions $a_\chi : G_2(\mathbf{A}_f) \to \mathbf{C}$ are locally constant and called the Fourier coefficients of φ. We say that φ has Fourier coefficients in a ring R if $a_\chi(g_f) \in R$ for all characters χ of $N(\mathbf{Q})\backslash N(\mathbf{A})$ and all $g_f \in G_2(\mathbf{A}_f)$. We write $S_\ell(G_2; R)$ for the space cuspidal quaternionic modular forms on G_2 of weight ℓ with Fourier coefficients in R.

Let $\mathbf{Q}_{cyc} = \mathbf{Q}(\mu_\infty)$ be the cyclotomic extension of $\mathbf{Q}$.

Theorem 1.0.1. *Suppose $\ell \geq 6$ is even. Then there is a basis of the cuspidal quaternionic modular forms on G_2 of weight ℓ with all Fourier coefficients in $\mathbf{Q}_{cyc}$. In other words, $S_\ell(G_2, \mathbf{C}) = S_\ell(G_2, \mathbf{Q}_{cyc}) \otimes_{\mathbf{Q}_{cyc}} \mathbf{C}$.*

Our main tool for proving Theorem 1.0.1 is a notion of "exceptional" theta functions, that mirrors the classical theory of Siegel modular theta functions associated to pluriharmonic polynomials. Recall that these classical pluriharmonic theta functions are (often cuspidal) Siegel modular forms, with completely explicit Fourier expansions, that can be considered as arising from the Weil representation restricted to $\mathrm{Sp}_{2g} \times O(V)$ where V is a rational quadratic space whose quadratic form is positive definite. In other words, they are the theta lifts from algebraic modular forms on $O(V)$ with *nontrivial archimedean weight* to holomorphic Siegel modular forms. From the perspective of theta lifts and algebraic modular forms, the remarkable fact is that one can give the Fourier expansions of these lifts completely explicitly.

The proof of Theorem 1.0.1 can be broken into three steps, two of which are in this paper and the other of which is in the companion paper [Pol23]. These steps are as follows:

(1) We develop a notion of exceptional theta functions on the split group G_2. These are quaternionic modular forms on G_2 whose Fourier coefficients we can tightly control. They arise as theta lifts from an anisotropic group of type F_4. This step is contained in this paper.
(2) We prove a Siegel-Weil theorem for a family of dual pairs of type $D_4 \times D_4$ in quaterionic E_8. This step is the content of the paper [Pol23].
(3) Using the Siegel-Weil theorem [Pol23] of the previous step, the Rankin-Selberg integral of [GS15] and [Seg17], the computation of unitary dual of p-adic G_2 in [Mui97], and the construction (see Corollary 1.2.3) of certain special cuspidal modular forms on G_2, we prove that every cuspidal quaternionic modular form of even weight at least 6 is one of our exceptional theta functions. Combined with the tight control of their Fourier expansions

[2] It seems useful to point out that there has been other recent work, specifically [PV21], that conjectures the existence of surprising algebraic structures on spaces of non-holomorphic automorphic forms.

that we prove in step one, this establishes the theorem. This step is contained in this paper, see section 8.

Our theory of exceptional theta functions on G_2 has a parallel–but easier–development on Sp_6. As further consequences of this theory of exceptional theta functions on G_2 and Sp_6 we also obtain the following corollaries, which are detailed below.

(1) Corollary 1.1.3: there is an algorithm to determine if any cuspidal, level one Siegel modular form on Sp_6 of most weights is a lift from G_2^a;
(2) Corollary 1.2.2: the level one theta lifts from F_4^I possess an integral structure, in the sense of Fourier coefficients;
(3) Corollary 1.2.3: in every even weight $k \geq 6$, there is a nonzero, level one Hecke eigen quaternionic cusp form on split G_2 with nonzero $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ Fourier coefficient.

We also obtain evidence (see Corollary 1.2.4) for a conjecture of Gross relating Fourier coefficients of certain G_2 quaternionic modular forms to L-values.

1.1. **Holomorphic theta functions.** We now describe in more detail our results on exceptional theta functions on Sp_6 and G_2, beginning with the case of Sp_6.

Let G_2^a denote the the algebraic $\mathbf{Q}$ group of type G_2 that is split at every finite place and compact at the archimedean place. Let H_J^1 denote the simply connected group of type E_7 that is split at every finite place and the group $E_{7,3}$ at the archimedean place. There is a dual pair $\mathrm{Sp}_6 \times G_2^a \subseteq H_J^1$, and a corresponding theta lift from G_2^a to Sp_6 using the automorphic minimal representation on H_J^1 [Kim93] that was studied by Gross-Savin [GS98]. This lift produces (in general) vector-valued holomorphic Siegel modular forms on Sp_6. It makes sense to ask if, given an algebraic modular form on G_2^a, one can give an explicit Fourier expansion of its theta lift to Sp_6. This is easy if the weight of the algebraic modular form is trivial, but is not immediate (at least to us) if the weight is nontrivial. Theorem 1.1.1 computes this Fourier expansion in the level one case.

We now setup this theorem. Let Θ denote the octonions over $\mathbf{Q}$ with positive definite norm. Denote by $J = H_3(\Theta)$ the 27-dimensional exceptional cubic norm structure consisting of the 3×3 Hermitian matrices with coefficients in Θ. Let V_7 denote the 7-dimensional space of trace 0 octonions, let W_3 denote the standard representation of GL_3 and set $V_3 = \wedge^2 W_3$. There is a projection $\mathcal{P} : J^\vee \simeq J \to V_3 \otimes V_7$ given by taking the trace 0 projections of the off-diagonal entries of an element of J. Consider the natural map

$$\begin{aligned}(V_3 \otimes V_7)^{\otimes k_1+2k_2} &\to S^{k_1}(V_3) \otimes V_7^{\otimes k_1} \otimes S^{k_2}(\wedge^2 V_3) \otimes (\wedge^2 V_7)^{\otimes k_2} \\ &\simeq S^{k_1}(\wedge^2 W_3) \otimes V_7^{\otimes k_1} \otimes S^{k_2}(W_3) \otimes (\wedge^2 V_7)^{\otimes k_2} \otimes \det(W_3)^{k_2},\end{aligned}$$

and denote by $\mathcal{P}_{k_1,k_2}(T)$ the image of $(T)^{\otimes(k_1+2k_2)}$ under this map.

Let ω_1 denote the highest weight of the representation V_7 of G_2, and let ω_2 denote the highest weight of the 14-dimensional adjoint representation $\mathfrak{g}_2 \subseteq \wedge^2 V_7$. For non-negative integers k_1, k_2, let $W(k_1, k_2)$ denote the representation of $G_2(\mathbf{C})$ with highest weight $k_1\omega_1 + k_2\omega_2$, embedded in $V_7^{\otimes k_1} \otimes (\wedge^2 V_7)^{\otimes k_2} \otimes \mathbf{C}$. This representation is the one generated by $v_{\omega_1}^{\otimes k_1} \otimes v_{\omega_2}^{\otimes k_2}$ where v_{ω_1} and v_{ω_2} are highest weight vectors for V_7 and $\mathfrak{g}_2$ for the same Borel subgroup of $G_2(\mathbf{C})$. Given $\beta \in W(k_1, k_2)$, and $T \in J$, we can form the pairing

$$\{P_{k_1,k_2}(T), \beta\} \in S^{k_1}(\wedge^2 W_3) \otimes S^{k_2}(W_3) \otimes \det(W_3)^{k_2+4},$$

where we have shifted the $k_2 \mapsto k_2 + 4$ in the exponent of $\det(W_3)$.

Denote by $R_\Theta \subseteq \Theta$ Coxeter's order of integral octonions, and set $J_R \subseteq J$ the elements whose diagonal entries are in $\mathbf{Z}$ and off-diagonal entries are in R_Θ. Recall that Kim's modular form on H_J^1 has Fourier expansion

$$\Theta_{Kim}(Z) = \sum_{T \geq 0, T \in J_R, rk(T) \leq 1} a(T) e^{2\pi i (T,Z)}$$

where $a(0) = 1$ and if T is rank one then $a(T) = 240\sigma_3(d_T)$ where d_T is the largest integer with $d_T^{-1}T \in J_R$.

Finally, set $\Gamma_{G_2} = G_2^a(\mathbf{Z})$.

Theorem 1.1.1. *Suppose $\beta \in W(k_1,k_2)$ and α is the level one algebraic modular form on G_2^a with $\alpha(1) = \frac{1}{|\Gamma_{G_2}|}\sum_{\gamma\in\Gamma_{G_2}} \gamma\cdot\beta$. Then the theta lift $\Theta(\alpha)$ of α is a vector-valued Siegel modular form of weight $(k_1+2k_2+4, k_1+k_2+4, k_2+4)$ with Fourier expansion*

$$\Theta(\alpha)(Z) = \sum_{T\geq 0, T\in J_R, rk(T)\leq 1} a(T)\{P_{k_1,k_2}(T),\beta\}e^{2\pi i(T,Z)}.$$

When $k_2 > 0$ [GS98] proves that $\Theta(\alpha)$ is a cusp form.

A simple restatement of Theorem 1.1.1 is as follows. Consider the projection map $J \to H_3(\mathbf{Q})$ given by sending

$$\begin{pmatrix} c_1 & a_3 & a_2^* \\ a_3^* & c_2 & a_1 \\ a_2 & a_1^* & c_3 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} 2c_1 & \mathrm{tr}(a_3) & \mathrm{tr}(a_2) \\ \mathrm{tr}(a_3) & 2c_2 & \mathrm{tr}(a_1) \\ \mathrm{tr}(a_2) & \mathrm{tr}(a_1) & 2c_3 \end{pmatrix}.$$

Then if $T_0 \in H_3(\mathbf{Q})$ (Hermitian 3×3 matrices with $\mathbf{Q}$ coefficients) is half-integral, the T_0 Fourier coefficient of $\Theta(\alpha)(Z)$ is

$$a_{\Theta(\alpha)}(T_0) = \sum_{T\geq 0, T\in J_R, rk(T)\leq 1, T\mapsto T_0} a(T)\{P_{k_1,k_2}(T),\beta\}. \quad (1)$$

(The sum is finite and explicitly determinable.) We verify that the $a_{\Theta(\alpha)}(T_0)$ live in the highest weight submodule

$$V_{[k_1,k_2]} := V_{(k_1+2k_2+4,k_1+k_2+4,k_2+4)} \subseteq S^{k_1}(\wedge^2 W_3)\otimes S^{k_2}(W_3)\otimes \det(W_3)^{k_2+4}$$

so that $\Theta(\alpha)$ really is a vector-valued Siegel modular form for the representation $V_{[k_1,k_2]}$.

An important computational aspect of Theorem 1.1.1 is that one can use β in the pairing $\{P_{k_1,k_2}(T),\beta\}$ instead of $\alpha(1) = \frac{1}{|\Gamma_{G_2}|}\sum_{\gamma\in\Gamma_{G_2}}\gamma\cdot\beta$. This enables one to compute theta lifts much more quickly than if one had to use the algebraic modular form $\alpha(1) \in W(k_1,k_2)^{\Gamma_{G_2}}$. Moreover, *even if one does not know a priori that $W(k_1,k_2)^{\Gamma_{G_2}} \neq 0$*, the theorem still holds as stated. In particular, verifying that the right hand side of equation (1) is nonzero for a single T_0 shows that $\frac{1}{|\Gamma_{G_2}|}\sum_{\gamma\in\Gamma_{G_2}}\gamma\cdot\beta \neq 0$. The reader can, of course, easily check this claim directly.

The Fourier expansion in Theorem 1.1.1 can be seen as completely analogous to the Fourier expansion of classical pluriharmonic theta functions. Indeed, the $\beta \in W(k_1,k_2)$ becomes the pluriharmonic polynomial, and the rank one T's become the lattice vectors over which one sums.

When the Siegel modular form $\Theta(\alpha)$ is a Hecke eigenform, it has Satake parameters c_p, one for each prime number p, which are semisimple elements in $\mathrm{Spin}_7(\mathbf{C})$. (Because we work with level one forms, we blur the distinction between Sp_6 and PGSp_6.) It is proved by Gross-Savin [GS98], Maagard-Savin [MS97], and Gan-Savin [GS21] that the theta lift is functorial for spherical representations; in fact, it is functorial for all representations, see Gan-Savin [GS22]. In particular, these conjugacy classes c_p are in $G_2(\mathbf{C}) \subseteq \mathrm{Spin}_7(\mathbf{C})$. Thus Theorem 1.1.1 can produce numerous explicit examples of level one vector-valued Siegel modular forms all of whose Satake parameters are in $G_2(\mathbf{C})$. We have taken the liberty of providing some small illustration of this, as follows. Let $\lambda_1 = (12,8,8)$ and $\lambda_2 = (14,10,8)$. Then it is known from Chenevier-Taibi [CT20] that the space of vector-valued, level one cuspidal Siegel modular forms of these weights are one dimensional.

Corollary 1.1.2. *For both $\lambda_1 = (12,8,8)$ and $\lambda_2 = (14,10,8)$, the cuspidal Siegel modular forms of these weights are lifts from G_2^a. In particular, their Satake parameters all lie in $G_2(\mathbf{C})$.*

Indeed, the proof of Corollary 1.1.2 is to produce a single $\beta_1 \in W(0,4)$ and $\beta_2 \in W(2,4)$, and a single T_0 so that the right hand side of (1) is nonzero for these β_i. It then follows that the $\Theta(\alpha)$

are nonzero level one Siegel modular forms, and thus by the dimension computation of [CT20] are the unique cuspidal level one eigenforms of weights λ_1 and λ_2. Producing many more such explicit examples would be possible. We remark that one can find conjectures here [BCFvdG17] of which small weight level one Siegel modular forms have their Satake parameters in $G_2(\mathbf{C})$.

More than just a couple of examples, however, Theorem 1.1.1 produces an algorithm to determine if any fixed level one Siegel modular cusp form of most weights is a lift from G_2^a. To setup the result, note that for a weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, it is known [Ibu02] that there exists explicitly determinable finite sets $\mathcal{C}_\lambda$ of half-integral symmetric matrices T so that if F is a cuspidal level one Siegel modular form of weight λ, and if $a_F(T) = 0$ for all $T \in \mathcal{C}_\lambda$, then $F = 0$.

Corollary 1.1.3. *Suppose $\lambda = (k_1 + 2k_2 + 4, k_1 + k_2 + 4, k_2 + 4)$ with $k_2 > 0$, and F is a level one Siegel modular cusp form of weight λ on Sp_6, whose Fourier coefficients $a_F(T)$ are given for all $T \in \mathcal{C}_\lambda$. Then there is an algorithm to determine if $F = \Theta(\alpha)$ for some algebraic modular form α on G_2^a.*

Let us indicate now some of the ingredients that go into the proof of Corollary 1.1.3. First, let us clarify that the theta lifts $\Theta(\alpha)$ of level one algebraic modular forms $\alpha(1) \in W(k_1,k_2)^{\Gamma_{G_2}}$ that appear in Theorem 1.1.1 are defined as integrals

$$\Theta(\alpha)(g) = \int_{G_2^a(\mathbf{Q})\backslash G_2^a(\mathbf{A})} \{\Theta_{k_1,k_2}(g,h), \alpha(h)\}\, dh$$

for a certain specific vector-valued element of Θ_{k_1,k_2} in the minimal representation Π_{min} on H_J^1. Here, in the integral, $\{\,,\,\}$ is a $G_2^a(\mathbf{R})$-equivariant pairing valued in $S^{k_1}(V_3) \otimes S^{k_2}(\wedge^2 V_3)$. Let $\{\beta_1, \ldots, \beta_N\}$ be a spanning set of $W(k_1,k_2)$, and $\alpha_j = \frac{1}{|\Gamma_{G_2}|}\sum_{\gamma\in\Gamma_{G_2}} \gamma\beta_j$. One can use Theorem 1.1.1 to explicitly compute the Fourier coefficients of $\Theta(\alpha_j)$ associated to the various $T's$ in $\mathcal{C}_\lambda$, where $\lambda = (k_1 + 2k_2 + 4, k_1 + k_2 + 4, k_2 + 4)$. Using linear algebra, one can check algorithmically if F is some linear combination of the $\Theta(\alpha)$'s. Thus Theorem 1.1.1 gives an algorithm to determine if the Siegel modular form F is a $\Theta(\alpha)$ for some algebraic modular form α on G_2^a. Thus Corollary 1.1.3 will be proved if one knew the following claim:

Claim 1.1.4. *Suppose F is a level one cuspidal Siegel modular form of weight $\lambda = (k_1 + 2k_2 + 4, k_1 + k_2 + 4, k_2 + 4)$ with $k_2 > 0$. Suppose F is in the image of the theta correspondence for $G_2^a \times \mathrm{Sp}_6 \subseteq H_J^1$. Then $F = \Theta(\alpha)$ for some algebraic modular form $\alpha(1) \in W(k_1,k_2)^{\Gamma_{G_2}}$.*

We prove this claim. The proof uses some powerful ingredients: The Howe Duality theorem of Gan-Savin [GS21]; an analysis of the minimal representation, as provided by Gross-Savin [GS98], Magaard-Savin [MS97] and Gan-Savin [GS21]; the existence of enough nonvanishing Fourier coefficients of F of a certain form, as proved by Böcherer-Das [BD21]; and another argument from Gross-Savin [GS98].

Our proof of Theorem 1.1.1 is very simple. Let $\mathfrak{p}_J$ be the complexification of the -1 eigenspace for the Cartan involution on the Lie algebra of $E_{7,3} = H_J^1(\mathbf{R})$. Write $\mathfrak{p}_J = \mathfrak{p}_J^+ \oplus \mathfrak{p}_J^-$, the natural decomposition, so that $\mathfrak{p}_J^-$ annihilates the automorphic form on $E_{7,3}$ associated to any holomorphic modular form for this group. Let $\{X_\alpha\}_\alpha$ be a basis of $\mathfrak{p}_J^+$ and $\{X_\alpha^\vee\}_\alpha$ be the dual basis of $\mathfrak{p}_J^{+,\vee} \simeq \mathfrak{p}_J^-$. For an automorphic form φ on H_J^1, set $D\varphi(g) = \sum_\alpha X_\alpha\varphi \otimes X_\alpha^\vee$. For an integer $m \geq 0$, let $D^m\varphi = D \circ D \circ \cdots \circ D\varphi$, so that

$$D^m\varphi = \sum_{\alpha_1,\ldots,\alpha_m} X_{\alpha_m}\cdots X_{\alpha_1}\varphi \otimes X_{\alpha_1}^\vee \otimes \cdots \otimes X_{\alpha_m}^\vee.$$

If (abusing notation) Θ_{Kim} denotes the automorphic form on H_J^1 associated to Kim's holomorphic modular form, we set $\Theta_{k_1,k_2}(g) = D^{k_1+2k_2}\Theta_{Kim}(g)$.

With this definition, using Kim's expansion of Θ_{Kim}, we compute the Fourier expansion of $\Theta_{k_1,k_2}(g)$. This is the main step in the proof of Theorem 1.1.1. We then use this to compute the Fourier expansion of $\Theta(\alpha)$. One obtains (the automorphic form associated to) a holomorphic function with the Fourier expansion as given in Theorem 1.1.1. We remark that the use of differential operators as we do has some overlap with the works [Ibu99, Cle22] of Ibukiyama and Clerc.

1.2. **Quaternionic theta functions.** Denote by G_J the group of type E_8 which is split at every finite place and $E_{8,4}$ at the archimedean place and by G_2 the split exceptional group of this Dynkin type. Analogous to the dual pair $\mathrm{Sp}_6 \times G_2^a \subseteq H_J^1$ is the dual pair $G_2 \times F_4^I \subseteq G_J$, where F_4^I is a specific form of F_4 that is split at all finite places and compact at the archimedean place, defined as the stabilizer of the identity matrix $I \in J$. We compute the theta lifts of certain algebraic modular forms on F_4^I to G_2, and obtain cuspidal quaternionic modular forms on G_2 together with their exact Fourier expansions.

More precisely, let J^0 denote the trace 0 elements of J. In other words, J^0 consists of the $X \in J$ with $(I,X)_I = 0$, where $(\,,\,)_I$ is the symmetric non-degenerate pairing on J determined by I. There is a surjective F_4^I-equivariant map $\wedge^2 J^0 \to \mathfrak{f}_4$ from $\wedge^2 J^0$ to the Lie algebra $\mathfrak{k}_4$ of F_4^I. Denote by V_{λ_3} the kernel of this map. It is an irreducible representation of F_4 of dimension 273. For an integer $m > 0$, let $V_{m\lambda_3} \subseteq (\wedge^2 J^0)^{\otimes m}$ denote the irreducible representation of F_4 with highest weight $m\lambda_3$, generated by the tensor product of a highest weight vector of V_{λ_3}. It follows from the archimedean theta correspondence calculated in [HPS96] that algebraic modular forms on F_4^I for the representation $V_{m\lambda_3}$ should lift to quaternionic modular forms on G_2 of integer weight $4+m$. We explicitly compute the Fourier expansion of this lift, and as a result, obtain exceptional "pluriharmonic" cuspidal quaternionic theta functions on G_2.

We setup the statement of the result. To do so, recall that the group F_4^I has not one but two integral structures [Gro96], [EG96]. More precisely, if $F_4^I(\widehat{\mathbf{Z}})$ denotes one of these integral structures, then the double coset space $F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A}_f)/F_4^I(\widehat{\mathbf{Z}})$ has size two. Because of this, algebraic modular forms for F_4^I can be described as follows. Denote by M_J^1 the subgroup of $\mathrm{GL}(J)$ fixing the cubic norm. Set Γ_I to be the subgroup of M_J^1 preserving the lattice J_R and fixing the element I. Recall the element $E \in J_R$ of norm one from [EG96] or [Gro96]. Let Γ_E denote the subgroup of M_J^1 preserving the lattice J_R and fixing the element E. Fix an element $\delta_E^{\mathbf{Q}} \in M_J^1(\mathbf{Q})$ with $\delta_E^{\mathbf{Q}} E = I$. If V is a representation of $F_4^I(\mathbf{R})$, let Γ_E act on V via $\gamma \cdot_E v = (\delta_E^{\mathbf{Q}} v \delta_E^{\mathbf{Q},-1})v$, where the conjugation takes place in M_J^1. Then a level one algebraic modular form for F_4^I can be considered as a pair

$$(\alpha_I, \alpha_E') \in V^{\Gamma_I} \oplus V^{\Gamma_E}.$$

In the case of $V = V_{m\lambda_3}$, we rephrase this as follows. Let $(\,,\,)_E$ be the symmetric non-degenerate pairing on J determined by E; one has

$$(u,v)_E = \frac{1}{4}(E,E,u)(E,E,v) - (E,u,v)$$

where $(\cdot,\cdot,\cdot)$ is the symmetric trilinear form on J satisfying $(x,x,x) = 6n(x)$, 6 times the cubic norm on J. Set J_E^0 to be the perpendicular space to E under the pairing $(\,,\,)_E$. One easily verifies that $\delta_E^{\mathbf{Q},-1} J^0 = J_E^0$. Thus

$$\alpha_E := \delta_E^{\mathbf{Q},-1} \alpha_E' \subseteq (\wedge^2 J_E^0)^{\otimes m}$$

and is Γ_E invariant for the natural action of Γ_E. It will be convenient for us to consider the algebraic modular form to be the pair (α_I, α_E).

Now, for $x, y, b \in J$ and $c \in J^\vee$, write

$$\langle x \wedge y, b \wedge c \rangle_I = (x,b)_I (y,c) - (x,c)(y,b)_I.$$

We use the same notation for the pairing $(\wedge^2 J)^{\otimes m} \otimes (\wedge^2 J)^{\otimes m} \to \mathbf{C}$ that extends this one via $\langle z_1 \otimes \cdots \otimes z_m, z_1' \otimes \cdots \otimes z_m' \rangle_I = \prod_{j=1}^m \langle z_j, z_j' \rangle_I$. Thus if $\beta \in V_{m\lambda_3}$, $b \in J$ and $c \in J^\vee$, we can compute the quantity $\langle \beta, (b \wedge c)^{\otimes m} \rangle \in \mathbf{C}$. We similarly define $\langle\,,\,\rangle_E$, by replacing the pairing $(\,,\,)_I$ with $(\,,\,)_E$.

Set $W_{J_R} = \mathbf{Z} \oplus J_R \oplus J_R^\vee \oplus \mathbf{Z}$. For $w \in W_{J_R}$ of rank one, set $a(w) = \sigma_4(d_w)$ where d_w is the largest integer with $d_w^{-1}w \in W_{J_R}$. In [Pol20b], we proved that the minimal modular form Θ_{Gan} on G_J has Fourier expansion

$$\Theta_{Gan,Z}(g) = \Theta_{Gan,N}(g) + \sum_{w \in W_{J_R}, rk(w)=1} a(w)W_{2\pi w}(g).$$

Here $N \supseteq Z \supseteq 1$ is the unipotent radical of the Heisenberg parabolic of $E_{8,4}$, $\Theta_{Gan,Z}$, $\Theta_{Gan,N}$ denote constant terms, and $W_{2\pi w}$ is the generalized Whittaker function of [Pol20a].

Now, if $w \in W_J$ and $m > 0$, set $P_m(w) = (b \wedge c)^{\otimes m} \in (\wedge^2 J)^{\otimes m}$ if $w = (a, b, c, d)$. Moreover, with this notation, let $p_I(w)$ and $p_E(w)$ be the binary cubic forms given as

$$p_I(w)(u,v) = au^3 + (b, I^\#)u^2v + (c, I)uv^2 + dv^3; \;\; p_E(w)(u,v) = au^3 + (b, E^\#)u^2v + (c, E)uv^2 + dv^3.$$

We prove the following.

Theorem 1.2.1. *Suppose α is a level one algebraic modular form on F_4^I for the representation $V_{m\lambda_3}$ with $m > 0$. Represent α as a pair (α_I, α_E) with $\alpha_I \in (\wedge^2 J)^{\otimes m}$ being Γ_I invariant and $\alpha_E \in (\wedge^2 J)^{\otimes m}$ being Γ_E invariant. Let β_I, β_E be in $V_{m\lambda_3}$, respectively $\delta_E^{\mathbf{Q},-1} V_{m\lambda_3}$ so that $\alpha_I = \frac{1}{|\Gamma_I|}\sum_{\gamma \in \Gamma_I} \gamma\beta_I$ and $\alpha_E = \frac{1}{|\Gamma_E|}\sum_{\gamma\in\Gamma_E} \gamma\beta_E$. Then the theta lift $\Theta(\alpha)$ is a cuspidal, level one, quaternionic modular form on G_2 of weight $4 + m$ with Fourier expansion*

$$\Theta(\alpha)_Z(g) = \frac{1}{|\Gamma_I|} \sum_{w \in W_{J_R}, rk(w)=1} a(w)\langle P_m(w), \beta_I\rangle_I W_{2\pi pr_I(w)}(g)$$
$$+ \frac{1}{|\Gamma_E|} \sum_{w \in W_{J_R}, rk(w)=1} a(w)\langle P_m(w), \beta_E\rangle_E W_{2\pi pr_E(w)}(g).$$

A simple restatement of Theorem 1.2.1 is as follows. If w_0 is an integral binary cubic form, then the w_0 Fourier coefficient of $\Theta(\alpha)$ is

$$a_{\Theta(\alpha)}(w_0) = \frac{1}{|\Gamma_I|} \sum_{w \in W_{J_R}, pr_{w,I}(w)=w_0} a(w)\langle P_m(w), \beta_I\rangle_I + \frac{1}{|\Gamma_E|} \sum_{w \in W_{J_R}, pr_{w,E}(w)=w_0} a(w)\langle P_m(w), \beta_E\rangle_E.$$

These sums are finite.

The reason we do not state Theorem 1.2.1 in the case $m = 0$ is because then the theta lifts will be non-cuspidal. In fact, the theta lifts obtained for $m = 0$ are exactly the automorphic forms obtained in [GGS02, Section 10]. Thus Theorem 1.2.1 may be considered a generalization of [GGS02, Section 10].

Again, just like Theorem 1.1.1, the Fourier expansion given by Theorem 1.2.1 is completely parallel to the classical pluriharmonic theta functions: The $\beta' s$ in $V_{m\lambda_3}$ are the pluriharmonic polynomial, and the sum over $w \in W_{J_R}$ of rank one is the sum over lattice vectors.

We now state a few corollaries of Theorem 1.2.1. For the first corollary, we can partially refine Theorem 1.0.1 in the case of level one.

Corollary 1.2.2. *There is a lattice $L_m^I \subseteq V_{m\lambda_3}$ and a lattice $L_m^E \subseteq \delta_E^{\mathbf{Q},-1} V_{m\lambda_3}$ so that the level one theta lifts of elements of these lattices to G_2 have Fourier coefficients that are integers when evaluated at $g_f = 1$.*

For the second corollary, recall that it was proved in [cDD+22] that if π is a cuspidal automorphic representation on $G_2(\mathbf{A})$ that corresponds to a level one quaternionic modular form φ_π of even weight ℓ, then the completed standard L-function $\Lambda(\pi, Std, s)$ satisfies the exact functional equation $\Lambda(\pi, Std, s) = \Lambda(\pi, Std, 1 - s)$, so long as the $w_0 = u^2v - uv^2$ Fourier coefficient of φ_π is nonzero. At the time of the writing of [cDD+22], it was not known whether such π exist. Using Theorem 1.2.1, one easily obtains the following.

Corollary 1.2.3. *Suppose $\ell \geq 6$ is even. Then there is a cuspidal automorphic representation π on $G_2(\mathbf{A})$ that corresponds to a level one quaternionic modular form φ_π of weight ℓ with nonzero $w_0 = u^2v - uv^2$ Fourier coefficient.*

This corollary is, in fact, an ingredient in the proof of Theorem 1.0.1.

The third corollary we state has to do with the Fourier expansion of a particular cuspidal quaternionic modular form. To setup this corollary, recall that Dalal [Dal21] has recently given an explicit formula for the dimension of the level one quaternionic cuspidal modular forms of weights at least 3. From his dimension formula, one has that the first such nonzero cusp form appears in weight 6, and the space of weight 6 cuspidal quaternionic modular forms is one-dimensional, spanned by a cusp form Δ_{G_2}. Combining the (proof of) Corollary 1.2.3 with Corollary 1.2.2, one obtains that Δ_{G_2} can be normalized to have integer Fourier coefficients, and that the Fourier coefficient associated to the cubic ring $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ is nonzero.

Now, Benedict Gross has suggested that the Fourier coefficients of certain non-tempered cuspidal quaternionic modular forms on G_2 of weight ℓ should be related to square-roots of twists of L-values of classical modular forms of weight 2ℓ by Artin motives associated to totally real cubic fields. As pointed out to the author by Mundy, the quaternionic modular form Δ_{G_2} should be one of these non-tempered lifts, to which Gross's conjecture applies. Now, for D congruent to 0 or 1 modulo 4, denote by $\mathbf{Z}_D$ the quadratic ring of discriminant D. Then on the one hand, in the case of Δ_{G_2}, Gross's conjecture implies that the square Fourier coefficients $a_{\Delta_{G_2}}(\mathbf{Z} \times \mathbf{Z}_D)^2$ associated to the cubic ring $\mathbf{Z} \times \mathbf{Z}_D$ should be related to the central[3] L-value $L(\Delta, D, 6)$ of the twist of Ramanujan's Δ function by the quadratic character associated to D. Denote by $\delta(z) \in S_{13/2}(\Gamma_0(4))^+$, $\delta(z) = \sum_{D \equiv 0,1 \pmod 4} \alpha(D) q^D$ the Shimura lift of $\Delta(z)$. On the other hand, following Waldspurger [Wal81], Kohnen-Zagier [KZ81] relate the squares of Fourier coefficients $\alpha(D)$ to the same L-value, $L(\Delta, D, 6)$. It thus makes sense to ask, in light of Gross's conjecture, if there is some relationship between $a_{\Delta_{G_2}}(\mathbf{Z} \times \mathbf{Z}_D)$ and $\alpha(D)$. It turns out, the numbers are equal:

Corollary 1.2.4. *Normalize Δ_{G_2} so that $a_{\Delta_{G_2}}(\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}) = 1$, and normalize $\delta(z)$ so that $\alpha(1) = 1$. Then $a_{\Delta_{G_2}}(\mathbf{Z} \times \mathbf{Z}_D) = \alpha(D)$ for all D.*

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2. Groups and embeddings

In this section, we explain various group theoretic facts regarding the groups with which we work.

2.1. **Some exceptional groups.** We begin by defining the group H_J^1. Thus let J be our exceptional cubic norm structure, $W_J = \mathbf{Q} \oplus J \oplus J^\vee \oplus \mathbf{Q}$. A typical element of W_J we write as an ordered four-tuple (a,b,c,d), so that $a, d \in \mathbf{Q}$, $b \in J$ and $c \in J^\vee$. We put on W_J Freudenthal's symplectic form, and quartic form. The group H_J^1 is defined as the algebraic $\mathbf{Q}$-group preserving these two forms. We let H_J be the group that preserves the forms on W_J up to similitude, and let $\nu : H_J \to \mathrm{GL}_1$ be this similitude.

The Siegel parabolic P_J of H_J^1 is defined as the stabilizer of the line $\mathbf{Q}(0,0,0,1)$. Write M_J for the group of linear automorphisms of J that preserve the norm on J, up to scaling. Let $\lambda : M_J \to \mathrm{GL}_1$ be this scaling factor, and M_J^1 the kernel of λ. A Levi subgroup of P_J can be defined as the subgroup

[3] We here use the classical normalization of L-functions, instead of the automorphic normalization.

that also stabilizes the line $\mathbf{Q}(1,0,0,0)$. This is isomorphic to the group of pairs $(\delta, m) \in \mathrm{GL}_1 \times M_J$ with $\delta^2 = \lambda(m)$. Such a pair acts on W_J as $(a,b,c,d) \mapsto (\delta^{-1}a, \delta^{-1}m(b), \delta\widetilde{m}(c), \delta d)$. We write $M(\delta, m)$ for this group element of H_J^1.

2.2. **The first dual pair.** We now explain how $\mathrm{GSp}_6 \times G_2^a$ embeds in H_J. To accomplish this, we describe a linear isomorphism between $\wedge_0^3 W_6 \otimes \nu^{-1} \oplus W_6 \otimes \Theta^0$ and W_J. Here $\Theta^0 = V_7$ is the trace 0 octonions, W_6 is the 6-dimensional defining representation of GSp_6, and $\wedge_0^3 W_6$ is the kernel of the contraction map $\wedge^3 W_6 \to W_6 \otimes \nu$. We let $e_1, e_2, e_3, f_1, f_2, f_3$ be the standard symplectic basis of W_6.

- If $v_1, v_2, v_3, v_1', v_2', v_3' \in \Theta^0$, we map $v_1'e_1 + v_2'e_2 + v_3'e_3 + v_1f_1 + v_2f_2 + v_3f_3$ to $(0, Y, Y', 0)$ where $Y = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix} \in J$ and $Y' = -\begin{pmatrix} 0 & v_3' & -v_2' \\ -v_3' & 0 & v_1' \\ v_2' & -v_1' & 0 \end{pmatrix}$.
- We map $f_1 \wedge f_2 \wedge f_3$ to $(1,0,0,0)$ and $e_1 \wedge e_2 \wedge e_3$ to $(0,0,0,1)$.
- Set $f_i^* = f_{i+1} \wedge f_{i-1}$ and $e_i^* = e_{i+1} \wedge e_{i-1}$ (with indices taken modulo 3). We map $\sum_{i,j} b_{ij} f_i^* \wedge e_j$ to $(0, (b_{ij}), 0, 0)$ and $\sum_{i,j} c_{ij} e_i^* \wedge f_j$ to $(0, 0, (c_{ij}), 0)$.

Via the natural action of $\mathrm{GSp}_6 \times G_2^a$ on $\wedge_0^3 W_6 \otimes \nu^{-1} \oplus W_6 \otimes \Theta^0$, we obtain an action of this group on W_J. It is clear that the action is faithful. To obtain the embedding into H_J, we must check that $\mathrm{GSp}_6 \times G_2^a$ preserves the symplectic and quartic form on W_J, up to scaling:

Proposition 2.2.1. *The defined action of* $\mathrm{GSp}_6 \times G_2^a$ *on* W_J *gives an embedding* $\mathrm{GSp}_6 \times G_2^a \subseteq H_J$.

To prove the proposition, we first make a few lemmas. We work over a general field F of characteristic 0.

Lemma 2.2.2. *Suppose* $X = \begin{pmatrix} s_1 & u_1 & u_2 \\ u_3 & s_2 & u_1 \\ u_2 & u_1 & s_3 \end{pmatrix}$ *is in* $H_3(F)$ *and* $v_j \in \Theta^0$ *so that the element* $Y = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix}$ *is in* J. *Then* $X \times Y = -\begin{pmatrix} 0 & v_3' & -v_2' \\ -v_3' & 0 & v_1' \\ v_2' & -v_1' & 0 \end{pmatrix}$ *with*

(1) $v_1' = s_1v_1 + u_3v_2 + u_2v_3$
(2) $v_2' = u_3v_1 + s_2v_2 + u_1v_3$
(3) $v_3' = u_2v_1 + u_1v_2 + s_3v_3$.

Proof. This is direct computation. □

Let now $X = (X_{ij}) \in H_3(F)$ and consider the element $n_{L,\mathrm{Sp}_6}(X) = \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right)$ in the Lie algebra of Sp_6. Recall the element $n_L(X)$, in the Lie algebra of H_J^1, that acts on $(a,b,c,d) \in W_J$ as $n_L(X)(a,b,c,d) = (0, aX, b \times X, (c, X))$.

Lemma 2.2.3. *Under the above identification* $\wedge_0^3 W_6 \otimes \nu^{-1} \oplus W_6 \otimes \Theta^0 \simeq W_J$, *the operator* $n_{L,\mathrm{Sp}_6}(X)$ *acts as* $n_L(X)$.

Proof. We first compute how $n_{L,\mathrm{Sp}_6}(X)$ acts on $(0, (b_{ij}), 0, 0) = \sum_{ij} b_{ij} f_i^* \wedge e_j$. Writing out the action of $n_{L,\mathrm{Sp}_6}(X)$ on this element, we obtain

$$\sum_{ij} b_{ij} \Big(\sum_m X_{m,i-1} f_{i+1} \wedge e_m \wedge e_j + \sum_\ell X_{\ell,i+1} e_\ell \wedge f_{i-1} \wedge e_j\Big).$$

The coefficient of $e_1^* \wedge f_1 = e_2 \wedge e_3 \wedge f_1$ comes from 4 terms:

- $i = 3, m = 2, j = 3$: $b_{33}X_{22}$
- $i = 3, m = 3, j = 2$: $-b_{32}X_{32}$
- $i = 2, \ell = 3, j = 2$: $b_{22}X_{33}$

- $i=2,\ell=2,j=3$: $-b_{23}X_{23}$

The coefficient of $e_2^*\wedge f_3 = e_3\wedge e_1\wedge f_3$ again comes from 4 terms:

- $i=2,m=3,j=1$: $b_{21}X_{31}$
- $i=2,m=1,j=3$: $-b_{23}X_{11}$
- $i=1,\ell=3,j=1$: $-b_{11}X_{32}$
- $i=1,\ell=1,j=3$: $b_{13}X_{12}$.

Putting the above computations together, one obtains $n_{L,\mathrm{Sp}_6}(X)(0,b,0,0)=(0,0,c,0)$ where $c=b\times X$.

Finally, one immediately obtains that $n_{L,\mathrm{Sp}_6}(X)(0,0,c,0)=(0,0,0,(c,X))$ and $n_{L,\mathrm{Sp}_6}(X)(1,0,0,0)=(0,X,0,0)$. The lemma follows. □

The following lemma is immediate.

Lemma 2.2.4. *The element* $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ *of* Sp_6 *acts on* W_J *as* $(a,b,c,d)\mapsto(d,-c,b,-a)$. *The element* $\left(\begin{smallmatrix}\nu&\\&1\end{smallmatrix}\right)$ *acts on* (a,b,c,d) *as* $(\nu^{-1}a,b,\nu c,\nu^2 d)$.

Lemma 2.2.5. *Suppose* $\left(\begin{smallmatrix}m&\\&n\end{smallmatrix}\right)\in\mathrm{Sp}_6$, *so that* $n={}^tm^{-1}$. *The action of this element on* W_J *is in* M_J^1; *in particular, it preserves the symplectic and quartic form on* W_J.

Proof. We compute the action of $\left(\begin{smallmatrix}m&\\&n\end{smallmatrix}\right)$ on (a,b,c,d) when $n={}^tm^{-1}$. First, one computes that n acts on f_i^* taking f_i^* to $\sum_k(n_{k+1,i+1}n_{k-1,i-1}-n_{k-1,i+1}n_{k+1,i-1})f_k^*=\sum_k c(n)_{ki}f_k^*$, where $c(n)=\det(n)\,{}^tn^{-1}$ is the cofactor matrix of n. This gives that the $\wedge_0^3W_6\otimes\nu^{-1}$ part of b maps to $c(n)b\,{}^tm=\det(n)\,{}^tn^{-1}bn^{-1}$. The vector (i.e., Θ^0) part of b moves as $\sum_j v_jf_j\mapsto\sum_k(\sum_j n_{kj}v_j)f_k$.

Completely similarly (it is the same calculation), the $\wedge_0^3W_6\otimes\nu^{-1}$ part of c maps to $c(m)c\,{}^tn=\det(m)\,{}^tm^{-1}cm^{-1}$. The vector part of c moves as $\sum_j v_j'e_j\mapsto\sum_k(\sum_j m_{kj}v_j')e_k$.

To finish the proof of the lemma, one must verify that the map $J\simeq H_3(F)\oplus\Theta^0\otimes V_3\to J$ given by

$$(X,v)\mapsto({}^tn^{-1}Xn^{-1},\det(n)^{-1}nv)$$

scales the norm on J by $\det(n)^{-2}$. This can be done, for example, by repeatedly applying the Cayley-Dickson construction and using the formulas of [Pol18, Section 8.1]. □

We can now prove Proposition 2.2.1.

Proof of Proposition 2.2.1. Exponentiating the action of $n_{L,\mathrm{Sp}_6}(X)$, we see that the action of $\left(\begin{smallmatrix}1&X\\&1\end{smallmatrix}\right)\in\mathrm{Sp}_6$ lands in H_J^1. The group GSp_6 is generated by these elements, together with the $\left(\begin{smallmatrix}\nu&\\&1\end{smallmatrix}\right)$ and with $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. The proposition thus follows from the above lemmas. □

2.3. **Action of the maximal compact.** Let K_{Sp_6} denote the standard maximal compact subgroup of $\mathrm{Sp}_6(\mathbf{R})$, so that

$$K_{\mathrm{Sp}_6}=\left\{\left(\begin{array}{cc}A&-B\\B&A\end{array}\right):A+iB\in U(3)\right\}.$$

Let K_{E_7} denote the subgroup of $H_J^1(\mathbf{R})$ that fixes the line spanned by $(1,i,-1,-i)\in W_J(\mathbf{C})$. This is a maximal compact subgoup of $H_J^1(\mathbf{R})$. Let $\mathfrak{p}_J$ be the complexification of the -1 part for the Cartan involution on the Lie algebra of $H_J^1(\mathbf{R})$ for this choice of maximal compact. Write $\mathfrak{p}_J^{\pm}$ for its two K_{E_7} factors. We now work out how K_{Sp_6} acts on $\mathfrak{p}_J^{\pm}$.

We begin by discussing the Cayley transform $C_h\in H_J^1(\mathbf{C})$ [Pol20a, section 5]. To do, we review some notation from [Pol20a]:

- $n_G(X):=\exp(n_L(X))$;
- $n_G^\vee(\gamma):=\exp(n_L^\vee(\gamma))$, with, for $\gamma\in J^\vee$, $n_L^\vee(\gamma)(a,b,c,d)=((b,\gamma),c\times\gamma,d\gamma,0)$;
- for $\lambda\in\mathrm{GL}_1$, $\eta(\lambda)(a,b,c,d)=(\lambda^3a,\lambda b,\lambda^{-1}c,\lambda^{-3}d)$;
- $\mathfrak{m}_J$ is the Lie algebra of M_J, which acts on W_J as given in [Pol20a, section 3.4].

The Cayley transform is defined as $C_h = n_G(-i)n_G^\vee(-i/2)\eta(2^{-1/2})$, so that $C_h^{-1} = \eta(2^{1/2})n_G^\vee(i/2)n_G(i)$. It satisfies:

(1) $C_h^{-1} n_L(J \otimes \mathbf{C})C_h = \mathfrak{p}_J^+$
(2) $C_h^{-1} n_L^\vee(J \otimes \mathbf{C})C_h = \mathfrak{p}_J^-$
(3) $C_h^{-1}(\mathfrak{m}_J \otimes \mathbf{C})C_h = \mathfrak{k}_{E_7}$.

For $Z \in J \otimes \mathbf{C}$, we set $r_1(Z) \in W_J \otimes \mathbf{C}$ as

$$r_1(Z) = (1, Z, Z^\#, n(Z)) = r_0(-Z).$$

For $g \in H_J(\mathbf{R})$ and $Z \in \mathcal{H}_J$, the factor of automorphy $j(g, Z) \in \mathbf{C}^\times$ and the action of g on $\mathcal{H}_J$ is defined as $gr_1(Z) = j(g, Z)r_1(g \cdot Z)$.

Recall that we let $M(\delta, m)$, for $m \in M_J$ and $\delta \in \mathrm{GL}_1$ such that $\delta^2 = \lambda(m)$, act on W_J as

$$M(\delta, m)(a, b, c, d) = (\delta^{-1}a, \delta^{-1}m(b), \delta\widetilde{m}(c), \delta(d)).$$

For $Y \in J_{>0}$ set $M_Y = M(n(Y)^{1/2}, U_{Y^{1/2}})$. Here $Y^{1/2}$ is the positive definite square root of Y, and for $x \in J$, $U_x : J \to J$ is the map defined as $U_x(z) = -x^\# \times z + (x, z)x$. For $Z = X + iY \in \mathcal{H}_J$ set $g_Z = n_G(X)M_Y$. Then $g_Z r_1(i) = n(Y)^{-1/2}r_1(Z)$.

Lemma 2.3.1. *One has*

(1) $C_h^{-1}(1, 0, 0, 0) = \frac{1}{2\sqrt{2}} r_1(i)$
(2) $C_h^{-1}(0, 0, 0, 1) = \frac{1}{2\sqrt{2}} r_1(-i)$.

Consequently, $\mathfrak{k} = C_h^{-1}\mathfrak{m}_J C_h$. *Moreover,*

$$C_h^{-1}(0, z, 0, 0) = \frac{1}{2\sqrt{2}}(i(1, z), 2z - (1, z)1, i(-2z + (1, z)1), (-1, z))$$

and

$$C_h^{-1}(0, 0, E, 0) = \frac{1}{2\sqrt{2}}(-(1, E), i((1, E) - 2E), 2E - (1, E), i(1, E)).$$

Proof. The first parts are direct verifications. For the second, recall that K is the subgroup of $H_J^1(\mathbf{R})$ that stabilizes the lines $\mathbf{C}r_1(i)$ and $\mathbf{C}r_1(-i)$, while M_J is the subgroup of $H_J^1(\mathbf{R})$ that stabilizes the lines spanned by $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$. The last part is again a direct verification. □

Here is the statement of the result.

Proposition 2.3.2. *Suppose* $k = \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \in K_{\mathrm{Sp}_6}$, *so that* $A + iB \in U(3)$. *Let* $E \in J$, *with* E *perpendicular to* $H_3(\mathbf{R})$, *so that* $E = v_1 \otimes x_1 + v_2 \otimes x_2 + v_3 \otimes x_3$. *Then* $Ad(k)C_h^{-1}n_L^\vee(E)C_h = C_h^{-1}n_L^\vee(E')C_h$ *with* $E' = \det(A + iB)\,{}^t(A + iB)^{-1}E$.

Proof. We know from lemmas above that $Ad(k)C_h^{-1}n_L^\vee(E)C_h = C_h^{-1}n_L^\vee(E')C_h$ for some $E' \in J$. To compute it, we apply both sides to $r_0(i) = r_1(-i)$. The right-hand side gives $2\sqrt{2}C_h^{-1}(0, 0, E', 0)$, which is $(0, -2iE', 2E', 0)$. The left-hand side gives $j(k^{-1}, i)^* k \cdot (0, -2iE, 2E, 0)$.

To further compute this left-hand side, note that we have $(0, iE, -E, 0)$ is identified with $(e_1 + if_1) \otimes x_1 + (e_2 + if_2) \otimes x_2 + (e_3 + if_3) \otimes x_3$. Applying k, one gets the action of $A - iB = {}^t(A + iB)^{-1}$ on the $e_j + if_j$. □

2.4. **More exceptional groups.** We define the group F_4^I to be the stabilizer of I inside of M_J^1. For the group G_J, recall the Lie algebra $\mathfrak{g}(J)$ of [Pol20a], so that G_J is the identity component of the group of automorphisms of $\mathfrak{g}(J)$. Let us remark that an integral model of the group G_J is described in [Gan00]. We use this integral model.

2.5. **The second and third dual pairs.** We will now define two more dual pairs, $\mathrm{GL}_2 \times F_4^I \subseteq H_J$ and $G_2 \times F_4^I \subseteq G_J$.

For the second dual pair, $\mathrm{GL}_2 \times F_4^I \subseteq H_J$, we explicate an identification $W_{\mathbf{Q}} \oplus V_2 \otimes J^0 \simeq W_J$, as follows. Here V_2 is the standard representation of GL_2, with standard basis e, f. Identify $(a,b,c,d) + f \otimes B + e \otimes C$ with $(a, bI + B, cI - C, d)$. We claim that under this identification, and the natural action of $\mathrm{GL}_2 \times F_4^I$ on the left, this group action preserves the symplectic and quartic form on W_J.

It is clear that F_4^I preserves the symplectic and quartic form. For the action of GL_2, the proof is similar to (but much easier than) that given above for $\mathrm{GSp}_6 \times G_2$. One simply needs to observe that $n_L(b)(0,B,0,0) = (0,0,bI \times B, 0) = (0,0,-bB,0)$ and $n_L(b)(f \otimes B) = be \otimes B$.

For the third dual pair, $G_2 \times F_4^I \subseteq G_J$, we proceed as follows. Observe that $F_4^I \subseteq G_J$ from the action of F_4^I on J and $J^\vee$. Now notice that $\mathfrak{g}(J)^{F_4^I} = \mathfrak{g}_2$. Because G_2 is simply connected, we thus obtain a map from G_2 to the centralizer of the group F_4^I in G_J. Thus we have a unique map $G_2 \times F_4^I \to G_J$ so that F_4^I acts on $J, J^\vee$ naturally, and the differential of the map $G_2 \to G_J$ is the Lie algebra embedding $\mathfrak{g}_2 \to \mathfrak{g}(J)$.

We now relate these two dual pairs. Thus let GL_2^s be the Levi of the Heisenberg parabolic in G_2. Observe that GL_2^s fixes the line spanned by E_{13} in $\mathfrak{g}(J)$, because it does so in $\mathfrak{g}_2$. Thus because H_J is the Levi of the Heisenberg parabolic in G_J, we obtain a map $\mathrm{GL}_2^s \times F_4^I \to H_J \to \mathrm{GL}(W_J)$.

Proposition 2.5.1. *The above-described two maps* $\mathrm{GL}_2^s \times F_4^I \to H_J$ *are identical.*

Proof. Indeed, it is clear that the F_4^I's act exactly the same way. As for GL_2^s, we have two algebraic representations of GL_2^s on W_J. By the formulas for the Lie bracket on $\mathfrak{g}(J)$, it is easy to see that the differential of these representations of GL_2^s are identical. Thus, they agree on the level of algebraic groups, as desired. □

3. Facts about F_4^I

We set down some notations and results we will need concerning the representations $V_{m\lambda_3}$ and algebraic modular forms on F_4^I.

3.1. **Special elements of representations of F_4.** Let $J^0 \subseteq J$ be the trace 0 subspace. Let $\iota : J \to J^\vee$ be the identification of the J with its dual given by the trace pairing. Recall that if $\gamma \in J^\vee$ and $x \in J$ then $\Phi_{\gamma,x} \in End(J)$ is defined as

$$\Phi_{\gamma,x}(z) = -\gamma \times (x \times z) + (\gamma, z)x + (\gamma, x)z.$$

If $X, Y \in J$ then one defines $\Phi_{X \wedge Y} = \Phi_{\iota(X),Y} - \Phi_{\iota(Y),X}$. This defines a map $\wedge^2 J \to \mathfrak{a}(J) = \mathfrak{f}_4$. (Here $\mathfrak{a}(J) = f_4$ is the Lie algebra of the subgroup of M_J^1 that preserves the trace pairing, or equivalently, fixes the element $I \in J$.) As $\Phi_{\iota(1),x} = \Phi_{\iota(x),1}$, this map factors through the projection $\wedge^2 J \to \wedge^2 J^0$.

Set

$$V_{\lambda_3} = \ker\{\wedge^2 J^0 \to \mathfrak{f}_4\}.$$

One can construct special elements of V_{λ_3} using the following two lemmas.

Lemma 3.1.1. *Suppose $x \in J$ is rank one. Suppose $z \in J$ and $\gamma = z \times x$. Then $\Phi_{\gamma,x} = 0$.*

Proof. Observe that $(\gamma, x) = (z \times x, x) = 2(z, x^\# = 0)$ so that

$$\Phi_{\gamma,x}(z') = -(x \times z) \times (x \times z') + (x, z, z')x.$$

Keeping this expression in mind, we now recall the identity

$$(u \times v)^\# + u^\# \times v^\times = (u, v^\#)u + (v, u^\#)v$$

valid for all u, v in J. If $u^\# = 0$, then symmetrizing this identity in v gives

$$(u \times v_1) \times (u \times v_2) = (u, v_1, v_2)u.$$

Consequently, taking $u = x$, $v_1 = z$ and $v_2 = z'$ we obtain

$$(x \times z) \times (x \times z') = (x, z, z')x.$$

This gives the lemma. □

Lemma 3.1.2. *Suppose $x \in J$ and $\gamma \in J^\vee$ are such that $\Phi_{\gamma,x} = 0$. Then $\Phi_{\iota(x),\iota(\gamma)} = 0$.*

Proof. The point is that one has $(x, \gamma) = 0$ and $\Phi_{\gamma,x} = 0$ (in fact, $\Phi_{\gamma,x} = 0$ implies $(\gamma, x) = 0$) if and only if $(0, x, \gamma, 0)$ is rank at most one in W_J. But $J_2(0, x, \gamma, 0) = (0, -\iota(\gamma), \iota(x), 0)$ so the lemma follows.

Of course, one can also give a more direct proof: One has

$$\begin{aligned}(\iota(y), \Phi_{\iota(x),\iota(\gamma)}(\iota(\mu))) &= (\iota(y), -\iota(x) \times (\iota(\gamma) \times \iota(\mu)) + (x, \mu)\iota(\gamma) + (x, \gamma)\iota(\mu)) \\ &= (-\gamma \times (x \times y), \mu) + ((y, \gamma)x, \mu) + ((x, \gamma)y, \mu) \\ &= (\Phi_{\gamma,x}(y), \mu).\end{aligned}$$

Thus if $\Phi_{\gamma,x} = 0$, then $\Phi_{\iota(x),\iota(\gamma)} = 0$. □

We now write down some specific elements $x, y \in J^0$ with the following property:

- The four dimensional space

  $$E_{x,y} := \{(0, \alpha_1 x + \alpha_2 y, \alpha_3 \iota(x) + \alpha_4 \iota(y), 0) : \alpha_i \in F\}$$

  is isotropic and singular in the sense that symplectic form on W_J restricted to $E_{x,y}$ is 0 and every element of $E_{x,y}$ has rank at most one.

Note that $E_{x,y}$ is isotropic and singular is equivalent to:

- $(x, x) = (x, y) = (y, y) = 0$
- x, y are rank at most one, and $x \times y = 0$
- $\Phi_{\iota(x),x} = 0$, $\Phi_{\iota(x),y} = 0$, $\Phi_{\iota(y),x} = 0$ and $\Phi_{\iota(y),y} = 0$.

If x, y satisfy the above properties, then $x \wedge y \in V_{\lambda_3}$ is a highest weight vector for some Borel. Indeed, one can show that the span of x, y is "amber", in the sense of Aschbacher [Asc87, 9.3-9.5], see also [MS97, Definition 7.2, Proposition 7.3, Proposition 7.4(1)]. Consequently, such x, y will allow us to construct explicit elements of $V_{m\lambda_3}$.

Lemma 3.1.3. *Suppose $x \in J^0$ is rank one, and that $z \in J$ is such that $(z, x) = 0$ and $(x, z^\#) = 0$. Set $y = \iota(z \times x)$. Then $x, y \in J^0$ and $E_{x,y}$ is isotropic and singular.*

Proof. First suppose $x \in J^0$ is rank one and $z \in J$ is arbitrary. Then if $y = \iota(z \times x)$, then $\Phi_{\iota(y),x} = 0$ and $\Phi_{\iota(x),y} = 0$.

Moreover, observe that if $v \in J^0$ is rank one then $1 \times v = (1, v)1 - v = -v$, so that $\Phi_{\iota(v),v} = 0$. Thus $\Phi_{\iota(x),x} = 0$, and if we arrange that $y \in J^0$ is rank one, then $\Phi_{\iota(y),y} = 0$.

To ensure that $y \in J^0$ we use $(z, x) = 0$. Indeed,

$$(y, 1) = (z \times x, 1) = (z, x \times 1) = (z, (1, x)1 - x) = -(z, x).$$

To ensure y has rank at most one, we use $(x, z^\#) = 0$. Indeed,

$$y^\# = (x \times z)^\# = (x, z^\#)x.$$

Because $\Phi_{\iota(x),x} = 0$ and $\Phi_{\iota(y),x} = 0$ and $\Phi_{\iota(y),y} = 0$, we get for free that $(x, x) = (x, y) = (y, y) = 0$. (Of course, one can also check this directly.)

To complete the proof, we must verify that $x \times y = 0$. For this, we compute

$$\begin{aligned}(z', x \times y) &= (z' \times x, y)\\ &= (z' \times x, -1 \times y)\\ &= ((z \times x) \times (z' \times x), -1)\\ &= ((z, z', x)x, -1) = 0.\end{aligned}$$

□

Example 3.1.4. As an example of such $x, y, z \in J \otimes \mathbf{C}$ we can take:

- $x = \begin{pmatrix} 1 & a_3 & a_2^* \\ a_3^* & -1 & a_3^* a_2^* \\ a_2 & a_2 a_3 & 0 \end{pmatrix}$ with $n(a_2) = 0$, $a_2 \neq 0$, and $n(a_3) = -1$.
- $z = \begin{pmatrix} 0 & 0 & (a_2')^* \\ 0 & 1 & 0 \\ a_2' & 0 & 0 \end{pmatrix}$ with $n(a_2') = 1$ and $(a_2', a_2) = 1$. Note that $z^\# = -z$ and $(z, x) = 0$, so that $(z^\#, x) = 0$ as well.
- Then $y = z \times x = \begin{pmatrix} 0 & (a_2')^*(a_2 a_3) & * \\ * & -1 & a_3^*(a_2')^* \\ a_2' - a_2 & * & 1 \end{pmatrix}$.

We will use this example to prove Corollaries 1.2.3 and 1.2.4.

3.2. **Algebraic modular forms.** Suppose V is a representation of $F_4^I(\mathbf{R})$. By an algebraic modular form for F_4^I, we mean an automorphic form $\alpha : F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A}) \to V$ satisfying $\alpha(gk) = k^{-1} \cdot \alpha(g)$ for all $g \in F_4^I(\mathbf{A})$ and $k \in F_4^I(\mathbf{R})$. If α has level one, then because the double coset $F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A}_f)/F_4^I(\widehat{\mathbf{Z}})$ has size two, such α can be described by two elements of V. In this subsection, we make this identification explicit.

Recall the elements $I, E \in J_R$ of norm 1, see [EG96]. Define F_4^I to be the stabilizer of I in $M_J^1 \simeq E_6$ and F_4^E to be the stabilizer of E in M_J^1. From the point of view of double cosets, the element $E \in J_R$ arises as follows. Let $\{1, \gamma_E\}$ be representatives for $F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A}_f)/F_4^I(\widehat{\mathbf{Z}})$. Using strong approximation on M_J^1, we can write $\gamma_E = \delta_E^{\mathbf{Q}}(\delta_E^{\mathbf{R}})^{-1}\delta_E^{\widehat{\mathbf{Z}}}$ with $\delta_E^{\mathbf{Q}} \in M_J^1(\mathbf{Q})$ etc. We can choose γ_E and $\delta_E^?$ so that $E = (\delta_E^{\mathbf{Q}})^{-1} \cdot I$. Indeed, observe that

(1) $E = (\delta_E^{\mathbf{Q}})^{-1} I \in J_R \otimes \mathbf{Q}$
(2) $(\delta_E^{\mathbf{Q}})^{-1} = (\delta_E^{\mathbf{R}})^{-1}\delta_E^{\widehat{\mathbf{Z}}}\gamma_E^{-1}$ so that $(\delta_E^{\mathbf{Q}})^{-1} I$ has finite part in $J_R \otimes \widehat{\mathbf{Z}}$.

We have $F_4^E = (\delta_E^{\mathbf{Q}})^{-1} F_4^I \delta_E^{\mathbf{Q}}$. If V is a representation of $F_4^I(\mathbf{R})$, we let $F_4^E(\mathbf{R})$ act on V via $g \cdot_E v = (\delta_E^{\mathbf{R}} g (\delta_E^{\mathbf{R}})^{-1})v$.

We make the following notations:

- Let Γ_I be the image of $F_4^I(\mathbf{Q}) \cap (F_4^I(\widehat{\mathbf{Z}})F_4^I(\mathbf{R}))$ in $F_4^I(\mathbf{R})$. Thus
$$\Gamma_I = (F_4^I(\mathbf{Q})F_4^I(\widehat{\mathbf{Z}})) \cap F_4^I(\mathbf{R}).$$
- Let Γ_E be the image of $F_4^E(\mathbf{Q}) \cap (F_4^E(\widehat{\mathbf{Z}})F_4^E(\mathbf{R}))$ in $F_4^E(\mathbf{R})$. Thus
$$\Gamma_E = (F_4^E(\mathbf{Q})F_4^E(\widehat{\mathbf{Z}})) \cap F_4^E(\mathbf{R}).$$
- Let Γ_{γ_E} be the image in $F_4^I(\mathbf{R})$ of $\gamma_E^{-1}F_4^I(\mathbf{Q})\gamma_E \cap (F_4^I(\widehat{\mathbf{Z}})F_4^I(\mathbf{R}))$. Thus
$$\Gamma_{\gamma_E} = ((\gamma_E^{-1}F_4^I(\mathbf{Q})\gamma_E)F_4^I(\widehat{\mathbf{Z}})) \cap F_4^I(\mathbf{R}).$$

Lemma 3.2.1. *One has* $\delta_E^{\mathbf{R}}\Gamma_E\delta_E^{\mathbf{R},-1} = \Gamma_{\gamma_E}$.

Proof. Observe

$$\Gamma_E = (F_4^E(\mathbf{Q})M_J^1(\widehat{\mathbf{Z}})) \cap F_4^E(\mathbf{R})$$

and

$$\Gamma_{\gamma_E} = ((\gamma_E^{-1}F_4^I(\mathbf{Q})\gamma_E)M_J^1(\widehat{\mathbf{Z}})) \cap F_4^I(\mathbf{R}).$$

Using that $\delta_E^{\mathbf{R}} = \delta_E^{\widehat{\mathbf{Z}}}\gamma_E^{-1}\delta_E^{\mathbf{Q}}$, the lemma follows. ☐

Suppose V is a representation of $F_4^I(\mathbf{R})$ and

$$\alpha : F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A}) \to V$$

satisfies $\alpha(gk_fk_{\mathbf{R}}) = k_{\mathbf{R}}^{-1}\alpha(g)$ for all $g \in F_4^I(\mathbf{A})$, $k_f \in F_4^I(\widehat{\mathbf{Z}})$ and $k_{\mathbf{R}} \in F_4^I(\mathbf{R})$. Then α is determined by its values at $g = 1$ and $g = \gamma_E$. Moreover, because $F_4(\widehat{\mathbf{Z}})$ acts freely on $F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A})$, we have

$$F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A}) = \left(\Gamma_I\backslash F_4^I(\mathbf{R})\right) \cdot F_4^I(\widehat{\mathbf{Z}}) \bigsqcup \left(\Gamma_{\gamma_E}\backslash F_4^I(\mathbf{R})\right) \cdot F_4^I(\widehat{\mathbf{Z}}).$$

Note that the measures of the two open sets are in the proportion $\frac{1}{|\Gamma_I|} : \frac{1}{|\Gamma_E|}$. Consequently, if $V = V_{m\lambda_3}$, one has

$$\int_{[F_4^I]} \{D^{2m}\Theta(g,h), \alpha(h)\}\, dh = \frac{1}{|\Gamma_I|}\{D^{2m}\Theta(g,1), \alpha(1)\} + \frac{1}{|\Gamma_E|}\{D^{2m}\Theta(g,\gamma_E), \alpha(\gamma_E)\}.$$

Set $\alpha_I = \alpha(1)$. Consider $V_{m\lambda_3} \subseteq (\wedge^2 J^0)^{\otimes m} \subseteq (\wedge^2 J)^{\otimes m}$, and let $\alpha_E = \delta_E^{\mathbf{Q},-1}\alpha(\gamma_E)$. Observe the following lemma:

Lemma 3.2.2. *If $b, x \in J$ then $(\delta_E^{\mathbf{Q}}b, x)_I = (b, \delta_E^{\mathbf{Q},-1}x)_E$.*

Proof. One has

$$\begin{aligned}(\delta_E^{\mathbf{Q}}b, x)_I &= \frac{1}{4}(I,I,\delta_E^{\mathbf{Q}}b)(I,I,x) - (I,\delta_E^{\mathbf{Q}}b,x)\\ &= \frac{1}{4}(E,E,b)(E,E,\delta_E^{\mathbf{Q},-1}x) - (E,b,\delta_E^{\mathbf{Q},-1}x)\\ &= (b,\delta_E^{\mathbf{Q},-1}x)_E.\end{aligned}$$

☐

If one writes J_E^0 to be the perpendicular space to E under the pairing $(\ ,\ )_E$, then $\alpha_E \in (\wedge^2 J_E^0)^{\otimes m}$. This follows from the lemma with $b = E$. Moreover, observe that, for the action of M_J^1 on $(\wedge^2 J)^{\otimes m}$, α_E is stabilized by the action of Γ_E. Thus, we can think of our algebraic modular form as being the pair

$$(\alpha_I, \alpha_E) \in [(\wedge^2 J)^{\otimes m}]^{\Gamma_I} \oplus [(\wedge^2 J)^{\otimes m}]^{\Gamma_E}.$$

4. Theorems on Siegel modular forms

In this section, we give the proof of Theorem 1.1.1 and its corollaries, Corollary 1.1.2 and Corollary 1.1.3. We do this assuming Theorem 4.0.1 stated below, which is the main technical ingredient in the proof of Theorem 1.1.1. Theorem 4.0.1 will be proved in section 6.

To setup the statement of Theorem 4.0.1, suppose V is a K_{E_7} representation, and $\varphi : H_J^1(\mathbf{R}) \to V$ is a function satisfying $\varphi(gk) = k^{-1}\cdot\varphi(g)$. In this scenario, let $D\varphi$ be the $V \otimes \mathfrak{p}_J^{+,\vee}$-valued function defined as

$$D\varphi = \sum_\alpha X_\alpha\varphi \otimes X_\alpha^\vee$$

where $\{X_\alpha\}_\alpha$ is a $\mathbf{C}$-basis of $\mathfrak{p}_J^+$. It is easily checked that $D\varphi$ is again K_{E_7}-equivariant. Recall also that $j(g,Z) : H_J^1(\mathbf{R}) \times \mathcal{H}_J \to \mathbf{C}^\times$ is the factor of automorphy defined in section 2. Let $\rho_{[k_1,k_2]}$ be the representation of $\mathrm{GL}_3(\mathbf{C})$ on $S^{k_1}(V_3) \otimes S^{k_2}(\wedge^2 V_3)$.

Theorem 4.0.1. *Suppose $g \in \mathrm{Sp}_6(\mathbf{R})$ and $\beta \in W(k_1,k_2)$. Let $\ell \geq 0$ be an integer. There is a nonzero constant B_{k_1,k_2}, independent of g and T, so that*

$$j(g,i)^{\ell}\rho_{[k_1,k_2]}(J(g,i))\{D^{k_1+2k_2}(j(g,i)^{-\ell}e^{2\pi i(T,g\cdot i)}),\beta\} = B_{k_1,k_2}\{P_{k_1,k_2}(T),\beta\}e^{2\pi i(pr(T),g\cdot i)}.$$

Moreover, $\{P_{k_1,k_2}(T),\beta\}$ lies in the highest weight submodule $S^{[k_1,k_2]}$ of $S^{k_1}(V_3)\otimes S^{k_2}(\wedge^2 V_3)$.

Recall that if $\alpha \in \mathcal{A}(G_2^a)\otimes W(k_1,k_2)$ is a level one algebraic modular form for the representation $W(k_1,k_2)$, then we defined the theta lift of α as

$$\Theta(\alpha)(g) := \int_{G_2^a(\mathbf{Q})\backslash G_2^a(\mathbf{A})}\{D^{k_1+2k_2}\Theta_{Kim}((g,h)),\alpha(h)\}\,dh = \frac{1}{|\Gamma_{G_2}|}\{D^{k_1+2k_2}\Theta_{Kim}((g,1)),\alpha(1)\}$$

where we have normalized the measure so that $G_2^a(\widehat{\mathbf{Z}})G_2^a(\mathbf{R})$ has measure 1. By rescaling α or this measure, we can (and will) ignore the term $\frac{1}{|\Gamma_{G_2}|}$.

We first state the fact that $\Theta(\alpha)$ is the automorphic form corresponding to a Siegel modular form of weight $[k_1,k_2]$.

Proposition 4.0.2. *For $g \in \mathrm{Sp}_6(\mathbf{R})$ and $Z \in \mathcal{H}_3$ the Siegel upper half-space of degree three, define*

$$f_{\Theta(\alpha)}(Z) = \rho_{[k_1,k_2]}(J(g,i))\Theta(\alpha)(g)$$

for any g with $g\cdot(i1_3) = Z$ in $\mathcal{H}_3$. Then $f_{\Theta(\alpha)}(Z)$ is well-defined, and is a level one Siegel modular form of weight $[k_1,k_2]$. If $k_2 > 0$, it is a cusp form.

Proof. The fact that $f_{\Theta(\alpha)}(Z)$ is well-defined comes from the $K_{\mathrm{Sp}_6} \simeq U(3)$ action on $\mathfrak{p}_J^{+,\vee}$, which was determined in section 2. To see that it's a holomorphic modular form of the correct weight, we use [GS98, Theorem 3.5]. The cuspidality when $k_2 > 0$ is proved in [GS98, Corollary 4.9].

In more detail, if $\ell \in S^{[k_1,k_2],\vee}$ is a linear form on $S^{[k_1,k_2]}$, then one can write $\ell(\Theta(\alpha)(g))$ as a sum of terms of the form $\Theta(v_j,\alpha_j)$, where

$$\Theta(v_j,\alpha_j) = \int_{[G_2^a]}\Theta_{v_j}((g,h))\alpha_j(h)\,dh$$

is a usual scalar-valued theta lift. By [GS98, Theorem 3.5], as functions of g, these lifts all lie in the holomorphic discrete series representation $\pi(k_1,k_2)$ with minimal K_{Sp_6}-type $S^{[k_1,k_2]}\det^4$. Moreover, by the K_{Sp_6}-equivariance that proves that $f_{\Theta(\alpha)}(Z)$ is well-defined, the vector-valued function $\Theta(\alpha)$ exactly encompasses the minimal K_{Sp_6}-type in $\pi(k_1,k_2)$. Here we use that this minimal K_{Sp_6}-type appears in $\pi(k_1,k_2)$ with multiplicity one. Consequently, $f_{\Theta(\alpha)}(Z)$ is a holomorphic Siegel modular form. It is clearly level one. Finally, [GS98, Corollary 4.9] shows that all the $\Theta(v_j,\alpha_j)$ are cusp forms if $k_2 > 0$, thus so is $\Theta(\alpha)$.

This completes the proof. □

Proof of Theorem 1.1.1. Theorem 1.1.1 follows directly from Proposition 4.0.2 and Theorem 4.0.1 by plugging in the Fourier expansion of Kim's modular form $\Theta_{Kim}(Z)$. □

We now explain the proofs of Corollary 1.1.2 and Corollary 1.1.3, especially that of Claim 1.1.4.

Proof of Corollary 1.1.2. Let u,v span a null, two-dimensional subspace of the trace zero elements $\Theta^0\otimes\mathbf{C}$. That they are null means that $u^2 = uv = vu = v^2 = 0$. We set $\beta = u^{\otimes k_1}\otimes(u\wedge v)^{\otimes k_2}$. It is easy to see that $\beta \in W(k_1,k_2)$. Indeed, we can choose a Borel subgroup of $G_2(\mathbf{C})$ to be the one that stabilizes the flag $\mathbf{C}u \subseteq \mathbf{C}u\oplus\mathbf{C}v$, and then it is clear that β is a highest weight vector in $W(k_1,k_2)$. A computer calculation shows that, if β_1 is defined as above with $k_1 = 0$ and $k_2 = 4$, then the

$$T_0 := \frac{1}{2}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Fourier coefficient of $\Theta(\beta_1)$ is nonzero. Similarly, if β_2 is defined as above with $k_1 = 2$ and $k_2 = 4$, then the T_0 Fourier coefficient of $\Theta(\beta_2)$ is nonzero.

To actually do the computation on a computer, we proceed as follows. First, we set H to be the quaternion algebra over $\mathbf{Q}$ ramified at 2 and the archimedean place. Let $1, i, j, k$ be its usual basis. We obtain the octonion algebra $\Theta = H \oplus H$ via the Cayley-Dickson construction using $\gamma = -1$. This means that the addition in Θ is component-wise and the multiplication is

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 + \gamma y_2^* y_1, y_2 x_1 + y_1 x_2^*).$$

Set $e = (0, 1)$ and $h = \frac{1}{2}(i + j + k + e)$. Then, the following are a $\mathbf{Z}$ basis of R_Θ, Coxeter's ring [Cox46]: $jh, e, -h, j, ih, 1, eh, ke$. These are the simple roots of the E_8 root lattice, with jh the extended node, 1 the branch vertex, and $e, -h, j, ih, 1, eh, ke$ going along longways.

For u, v we take elements inside of $\Theta \otimes \mathbf{Q}(\sqrt{-1})$ as

$$u = \frac{1}{2}((0,1) - \sqrt{-1}(0, i)); \;\; v = \frac{1}{2}((0, j) - \sqrt{-1}(0, k)).$$

Finally, to compute the T_0 Fourier coefficient, where

$$T_0 = \frac{1}{2}\begin{pmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{pmatrix},$$

we must explain how to enumerate the rank one $T \in J_R$ with $pr(T) = T_0$. The point is that

$$T = \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix}$$

being rank one with $pr(T) = T_0$ implies $n(x_1) = c_2 c_3 = bc$, $n(x_2) = c_3 c_1 = ca$, $n(x_3) = c_1 c_2 = ab$. Thus one only must search through a finite list of x_i (namely, those with these norms) to find all such T. □

In order to prove Corollary 1.1.3, we require Claim 1.1.4, which we prove now.

Proof of Claim 1.1.4. Recall that we assume $k_2 > 0$, that F is a level one Siegel modular form of weight $[k_1, k_2]$ and we wish to prove that if F is in the image of the theta correspondence from G_2^a, then $F = \Theta(\alpha)$ for a level one algebraic modular form $\alpha(1) \in W(k_1, k_2)^{\Gamma_{G_2}}$.

We first use the Howe Duality theorem of Gan-Savin [GS21] to reduce to the case of eigenforms. Thus write $F = \sum_j F_j$ as am orthogonal sum of eigenforms with each F_j nonzero. Then the Petersson inner product $(F_j, F) \neq 0$ for each j. Let π_{F_j} be the automorphic cuspidal representation of PGSp_6 generated by F_j. Then, by changing the order of the theta integral, one sees that the big theta lift $\Theta(\pi_{F_j})$ of π_{F_j} to G_2^a is nonzero. By Howe Duality [GS21] and the argument in [Gan22, Proposition 3.1], $\tau_j := \Theta(\pi_{F_j})$ is an irreducible representation of G_2^a.

We now argue that $\Theta(\tau_j) = \pi_{F_j}$. Let $W(k_1', k_2')$ the archimedean component of τ_j, which a priori could depend upon j. For any fixed vector y_j in the finite part of $\tau_j \simeq \tau_{j,f} \otimes W(k_1', k_2')$, and any fixed vector v_j in the finite part of $\Pi_{min} \simeq \Pi_{min,f} \otimes \Pi_{min,\infty}$ one can consider the map

$$\Pi_{min,\infty} \otimes W(k_1', k_2') \to \mathcal{A}(\mathrm{Sp}_6)$$

given by the theta lift. By the archimedean correspondence proved in Gross-Savin [GS98], the image is a holomorphic discrete series representation $\pi(k_1', k_2')$ with lowest K_{Sp_6}-type $S^{[k_1', k_2']}$. Because for some choice of y_j and v_j we must obtain a theta lift that is not orthogonal to F_j, we must have $k_1' = k_1$ and $k_2' = k_2$.

It now follows from [GS98], because $k_2' = k_2 > 0$, that $\Theta(\tau_j)$ consists of cusp forms. Thus again by Howe Duality [GS21] and the argument in [Gan22, Proposition 3.1], $\Theta(\tau_j)$ is irreducible, so $\Theta(\tau_j) = \pi_{F_j}$. By the results of [GS98, MS97, GS21] that apply to spherical representations, it must be that τ_j is unramified at every finite place. Thus we have a level one algebraic modular form $\alpha_j(1) \in W(k_1,k_2)^{\Gamma_{G_2}}$, and we must show that $\Theta(\alpha_j)$ is a nonzero multiple of F_j. We have now reduced ourselves to the case of eigenforms, so will drop the j from α_j, F_j and τ_j.

Let $b_1,\ldots,b_N$ be a basis of $W(k_1,k_2)$, φ_j the level one automorphic form on $G_2^a(\mathbf{A})$ corresponding to b_j, and let $w_1,\ldots,w_M$ be a basis of $S^{[k_1,k_2]}$. By changing variables in the integral defining the theta lift from G_2^a to Sp_6, one sees that there exists $v_{i,j}$ in Π_{min} so that

$$F = \sum_{i,j} \Theta(v_{i,j},\varphi_j)\otimes w_i.$$

Set $v' = \sum_{i,j} v'_{i,j}\otimes w_i\otimes b_j \in \Pi_{min}\otimes S^{[k_1,k_2]}\otimes W(k_1,k_2)$ and $\alpha = \sum_j \varphi_j \otimes b_j^\vee$. Then we have

$$F = \int_{[G_2^a]} \{\Theta_{v'}((g,h)),\alpha(h)\}\,dh.$$

Let

$$v = \int_{K_{\mathrm{Sp}_6}\times G_2^a(\mathbf{R})} (k_1,k_2)\cdot v'\,dk_1\,dk_2$$

where the action is diagonal on the minimal representation and the vector space $S^{[k_1,k_2]}\otimes W(k_1,k_2)$. Because α is $G_2^a(\mathbf{R})$-equivariant and F is K_{Sp_6}-equivariant, one has

$$F = \int_{[G_2^a]} \{\Theta_{v}((g,h)),\alpha(h)\}\,dh.$$

Because the $K_{\mathrm{Sp}_6}\times G_2^a(\mathbf{R})$-type $(S^{[k_1,k_2]}\otimes W(k_1,k_2))^\vee$ appears in $\Pi_{min,\infty}$ with multiplicity one [GS98], we can write

$$v = \sum_{i,j} v_{i,j,f}\otimes v_{i,j,\infty}\otimes w_i\otimes b_j$$

where $v_{i,j,\infty}\in\Pi_{min,\infty}$ is the basis of $(S^{[k_1,k_2]}\otimes W(k_1,k_2))^\vee \subseteq \Pi_{min,\infty}$ dual to the basis $w_i\otimes b_j$. We wish to show that we can set $v_{i,j,f}$ to be the spherical vector in $\Pi_{min,f}$ for all i,j in this equality.

Fixing the vectors $v_{i,j,\infty}$ and letting the $v_{i,j,f}$ vary, the theta integral gives an equivariant map

$$\Pi_{min,f}\otimes\tau_f \to \pi_{F,f}.$$

This map is nonzero by assumption, so by p-adic Howe Duality again [GS21] the map is uniquely determined up to scalar multiple. Our goal is to show that the image of the spherical vector on the left-hand side is nonzero on the right-hand side. We will do this by a global argument.

It suffices to show that, for each prime p, in the unique up to scalar map $\Pi_{min,p}\otimes\tau_p\to\pi_{F,p}$, the image of the spherical vector is nonzero. Because this map is unique up to scalar multiple, we must only find some $v\in\Pi_{min}$ which is spherical at p, some $\varphi\in\tau$ which is spherical at p, and so that $\Theta(v,\varphi)\neq 0$. By [GS98, Proposition 4.5], it suffices to check that α has an appropriate period. More precisely, let q be an odd prime with $q\neq p$, and let B_q be the quaternion algebra over $\mathbf{Q}$ ramified at q and infinity. Then it suffices to show that α has a B_q period. But finally, by Bocherer-Das [BD21], for some odd prime $q\neq p$, F has a nonzero Fourier coefficient associated to the maximal order in B_q. Consequently, α does have a B_q period and we have shown that the unique map $\Pi_{min,p}\otimes\tau_p\to\pi_{F,p}$ is nonzero on the spherical vector.

Let

$$v_0 = \sum_{i,j} v_{0,f}\otimes v_{i,j,\infty}\otimes w_i\otimes b_j$$

where $v_{0,f}$ is the spherical vector in $\Pi_{min,f}$. We have shown

$$\int_{[G_2^a]} \{\Theta_{v_0}((g,h)),\alpha(h)\}\,dh.$$

is nonzero, and thus a nonzero multiple of F. Because

$$\Theta(\alpha)(g) := \int_{[G_2^a]} \{D^{k_1+2k_2}\Theta_{Kim}((g,h)), \alpha(h)\}\, dh = \int_{[G_2^a]} \{\Theta_{v_0}((g,h)), \alpha(h)\}\, dh,$$

we obtain $\Theta(\alpha)$ is nonzero, and thus is nonzero multiple of F. The claim is proved. □

Proof of Corollary 1.1.3. We have explained, given $\beta \in W(k_1,k_2) \subseteq V_7^{\otimes k_1} \otimes (\wedge^2 V_7)^{\otimes k_2}$, how to compute individual Fourier coefficients of $\Theta(\beta)$. It remains to explain how to enumerate a spanning set of $W(k_1,k_2)$. To do this, we define elements $e_1, e_2, e_3, e_1^*, e_2^*, e_3^*, u_0 = \epsilon_1 - \epsilon_2$ which are a basis of $V_7 \otimes \mathbf{Q}(\sqrt{-1})$, as follows.

- $e_2 = \frac{1}{2}((0,1) - \sqrt{-1}(0,i))$
- $e_3^* = \frac{1}{2}((0,j) - \sqrt{-1}(0,k))$
- $e_3 = \frac{1}{2}((0,-j) - \sqrt{-1}(0,k))$.
- $e_2^* = \frac{1}{2}((0,-1) - \sqrt{-1}(0,i))$
- $\epsilon_1 = \frac{1}{2}((1,0) - \sqrt{-1}(i,0))$
- $\epsilon_2 = \frac{1}{2}((1,0) + \sqrt{-1}(i,0))$
- $e_1 = \frac{1}{2}((j,0) - \sqrt{-1}(k,0))$
- $e_1^* = \frac{1}{2}((-j,0) - \sqrt{-1}(k,0))$

Now, in terms of these elements, a basis for root spaces of $\mathfrak{g}_2$ can be found in [Pol19]. In particular, for the standard Borel chosen in [Pol19], $v(k_1,k_2) := e_1^{\otimes k_1} \otimes (e_1 \wedge e_3^*)^{\otimes k_2}$ will be a highest weight vector for $W(k_1,k_2)$. By the Poincare-Birkoff-Witt theorem, the representation $W(k_1,k_2)$ is spanned by $\{y_1^{m_1} y_2^{m_2} v(k_1,k_2) | m_1, m_2 \leq N(k_1,k_2)\}$ where here

- y_1, y_2 span the negative root spaces of the simple roots of $\mathfrak{g}_2$;
- the element $y_1^{m_1} y_2^{m_2} v(k_1,k_2)$ can be explicitly computed using the formulas of [Pol19];
- one can come up with easy bounds for the integer $N(k_1,k_2)$.

The corollary follows. □

5. Theorems on quaternionic modular forms

In this section we prove Theorem 1.2.1, assuming a crucial technical result, Theorem 5.0.1, which is proved in section 7. We also prove Corollaries 1.2.2, 1.2.3, and 1.2.4.

For $w \in W_J(\mathbf{R})$ that is positive semi-definite, and $\ell \geq 1$ an integer, let $W_{w,\ell}$ be the generalized Whittaker function [Pol20a] on $G_J(\mathbf{R}) = E_{8,4}$ associated to w and ℓ. Similarly, for w_0 a real binary cubic form that is positive semi-definite, let $W_{w_0,\ell}$ be the associated generalized Whittaker function on $G_2(\mathbf{R})$.

Let now $\mathfrak{p}$ denote the complexification of the -1 part for the Cartan involution on $\mathfrak{g}(J) \otimes \mathbf{R}$ for the Cartan involution defined in [Pol20a]. Recall that if $\varphi : G_J(\mathbf{R}) \to V$ is a smooth function, then $D\varphi \in C^\infty(G_J(\mathbf{R}); V \otimes \mathfrak{p}^\vee)$ is defined as $D\varphi = \sum_\alpha X_\alpha \varphi \otimes X_\alpha^\vee$, where $\{X_\alpha\}_\alpha$ is a basis of $\mathfrak{p}$.

Theorem 5.0.1. *Suppose $\ell = 4$ and $\beta \in V_{m\lambda_3}$ with $m \geq 0$. Then there is a nonzero constant B_m so that for all $w \in W_J$ rank one and $g \in G_2(\mathbf{R})$, one has*

$$\{D^{2m} W_{w,\ell}(g), \beta\} = B_m \langle P_m(w), \beta\rangle_I W_{pr_I(w),\ell+m}(g).$$

Here, the F_4-equivariant pairing $\{\,,\,\}$ is defined as follows. By virtue of the exceptional Cayley transform of [Pol20a] and the explanations of subsection 2.5, one has $\mathfrak{p} \simeq V_2^\ell \otimes W_J$, and $W_J = W_{\mathbf{Q}} \oplus V_2^s \otimes J^0$. Thus

$$\mathfrak{p}^{\otimes 2m} \to (V_2^\ell)^{\otimes 2m} \otimes (V_2^s \otimes J^0)^{\otimes 2m} \to S^{2m}(V_2^\ell) \otimes \det(V_2^s)^{\otimes m} \otimes (\wedge^2 J^0)^{\otimes m}.$$

Thus if $\beta \in V_{m\lambda_3} \subseteq (\wedge^2 J^0)^{\otimes m}$, and $r \in S^{2\ell}(V_2^\ell) \otimes \mathfrak{p}^{\otimes 2m}$ we obtain an element $\{r, \beta\}$ in $S^{2m+2\ell}(V_2^\ell)$.

We can use the theorem to compute the Fourier expansion of the theta lift $\Theta(\alpha)$ of a level one algebraic modular form α on F_4^I. We begin with the statement that these lifts are quaternionic modular forms of weight $4+m$ on G_2.

Proposition 5.0.2. *Suppose $m \geq 0$, and $\beta \in V_{m\lambda_3}$. Then*

$$\Theta(\beta) := \{D^{2m}\Theta_{Gan}(g,1), \beta\}$$

is a quaternionic modular form on G_2 of weight $4+m$. If $m>0$, it is a cusp form.

Proof. First note that, if $\gamma \in \Gamma_I$, then

$$\{D^{2m}\Theta_{Gan}(g,1), \gamma\cdot\beta\} = \{\gamma^{-1}\cdot D^{2m}\Theta_{Gan}(g,1),\beta\} = \{D^{2m}\Theta_{Gan}(g,\gamma),\beta\} = \{D^{2m}\Theta_{Gan}(g,1),\beta\}.$$

Thus if α is the level one algebraic modular form on F_4^I with $\alpha(1) = \sum_{\gamma\in\Gamma_I}\gamma\cdot\beta$ and $\alpha(\gamma_E) = 0$, then

$$\Theta(\alpha) = \int_{[F_4^I]}\{D^{2m}\Theta_{Gan}(g,h),\alpha(h)\}\,dh = \frac{1}{|\Gamma_I|}\{D^{2m}\Theta_{Gan}(g,1),\alpha(1)\} = \Theta(\beta).$$

Now, for $v \in \Pi_{min,\infty}$ and $w \in V_{m\lambda_3}$, consider the theta lift

$$\Theta(v,\varphi_w)(g) = \int_{[F_4^I]}\Theta_v(g,h)\varphi_w(h)\,dh$$

where $\varphi_w(h) = \{\alpha(h), w\}$ is the level one automorphic function on $F_4^I(\mathbf{A})$ associated to $w \in V_{m\lambda_3}$. This lift gives an equivariant pairing $\Pi_{min,\infty}\otimes V_{m\lambda_3} \to \mathcal{A}(G_2)$. By [HPS96], it thus gives a map $\pi_{4+m} \to \mathcal{A}(G_2)$. Finally, by the K_{G_2}-equivariance of $\Theta(\alpha)$, we see that $\Theta(\alpha)$ is the minimal K-type of this copy of π_{4+m}, so it is a quaternionic modular form of weight $4+m$.

We now show the cuspidality of the theta lifts if $m>0$. Let P and Q, respectively, be the two standard maximal parabolic subgroups of G_2, so that P is the Heisenberg parabolic. Let P_J be the Heisenberg parabolic of G_J, and Q_J the standard maximal parabolic with $Q_J \cap G_2 = Q$. In terms of the F_4 root system underlying the group G_J, with long roots α_1,α_2 and short roots α_3,α_4, Q_J is the parabolic for which α_2 is in its unipotent radical. The Levi subgroup of Q_J is of absolute Dynkin type $A_1\times E_6$. Write $P = MN$, $Q = LV$, $P_J = M_JN_J$, and $Q_J = L_JV_J$ for the Levi decompositions.

We must check that the constant terms $\Theta(v,\varphi)_N$ and $\Theta(v,\varphi)_V$ are 0, if φ is an automorphic form in a representation τ with $\tau_\infty = V_{m\lambda_3}$ with $m>0$. We first observe the following claim:

Claim 5.0.3. *One has an equality of constant terms $\Theta_{v,N} = \Theta_{v,N_J}$ and $\Theta_{v,V} = \Theta_{v,V_J}$.*

Granting the claim for the moment, we obtain that $\Theta(v,\varphi)_N(g) = \int_{[F_4^I]}\Theta_{v,N_J}(g,h)\varphi(h)\,dh$ and $\Theta(v,\varphi)_V(g) = \int_{[F_4^I]}\Theta_{v,V_J}(g,h)\varphi(h)\,dh$. The constant terms Θ_{v,N_J} and Θ_{v,V_J}, restricted to their Levi subgroups, were determined in [Gan00]. For the first one, see page 174 of [Gan00], it is a sum of terms from a one-dimensional representation of M_J and the minimal representation of M_J. Both of these have integral 0 against φ, because by [HPS96], the representation $V_{m\lambda_3}$ with $m>0$ does not participate in the theta correspondence for the dual pair $\mathrm{SL}_2\times F_4^I \subseteq H_J^1$. For the second one, see page 176 of [Gan00], the constant term restricted to F_4^I is the trivial representation. Thus this too has integral 0 against φ.

It remains to explain the proof of Claim 5.0.3. For the equality $\Theta_{v,N} = \Theta_{v,N_J}$, note that $\Theta_{v,N}$ is a sum of terms of the form $\sum_{w\in W_J: rk(w)\leq 1, pr_I(w)=0}\Theta_w(g)$. But, using that if $T \in J$ is rank one with $\mathrm{tr}(T) = 0$ then $T = 0$, we find that the only $w \in W_J$ with $rk(w)\leq 1$ and $pr_I(w) = 0$ is $w = 0$. Thus $\Theta_{v,N}(g) = \Theta_{v,0}(g) = \Theta_{v,N_J}(g)$. One makes a completely similar argument for the constant term $\Theta_{v,V}(g)$. The proposition is proved. □

Proof of Theorem 1.2.1. Suppose $\alpha_I \in V_{m\lambda_3}^{\Gamma_I}$, and α is the level one algebraic modular form on F_4^I with $\alpha(1) = \alpha_I$ and $\alpha_E = 0$. Then it follows from Proposition 5.0.2 that the theta lift $\Theta(\alpha)$ of α

to G_2 is a quaternionic modular form of weight $4+m$, and cuspidal if $m > 0$. Up to the constant B_m, its Fourier expansion is given exactly as in the statement of Theorem 1.2.1. The general case, where $\alpha_E \neq 0$, is explained in subsection 5.1 below. ☐

We explain the proof of Corollary 1.2.2.

Proof of Corollary 1.2.2. Over $\mathbf{C}$, we have a decomposition $(\wedge^2 J^0)^{\otimes m} \otimes \mathbf{C} = V_{m\lambda_3} \oplus V'$, where V' is F_4^I-stable. Because F_4^I is a pure inner form of split F_4, one can use the results of [BGW15, sections 2.1, 2.2] to give such a decomposition over $\mathbf{Q}$: $(\wedge^2 J^0)^{\otimes m} = V_{m\lambda_3,\mathbf{Q}} \oplus V'_{\mathbf{Q}}$, where $V_{m\lambda_3,\mathbf{Q}}$ is a rational representation of the algebraic group F_4^I whose complexification is $V_{m\lambda_3}$. Now let L_m be the intersection of $V_{m\lambda_3,\mathbf{Q}}$ with $(\wedge^2 J_R)^{\otimes m}$; it is immediately seen to be an integral lattice in $V_{m\lambda_3}$, so that $L_m \otimes \mathbf{C} = V_{m\lambda_3}$. But now, from the explicit formula from Theorem 1.2.1, it is clear that if $\beta_I \in |\Gamma_I| L_m$, then $\Theta(\beta_I)$ has integral Fourier coefficients. One makes a similar argument for $\Theta(\beta_E)$. This proves the corollary. ☐

We now explain the proof of Corollary 1.2.3. To do so, we first construct a special $\beta_m \in V_{m\lambda_3}$. Thus let

- K be an imaginary quadratic field, so that $H \otimes K$ is split, such as $K = \mathbf{Q}(\sqrt{-1})$.
- $a_2 \in \Theta \otimes K$, with $a_2 \neq 0$ but $n(a_2) = 0$
- $a_3 \in \Theta \otimes K$ with $n(a_3) = -1$
- $a_2' \in \Theta \otimes K$ with $n(a_2') = 1$ and $(a_2', a_2) = 1$.

Then, as in section 3, we set

- $x = \begin{pmatrix} 1 & a_3 & a_2^* \\ a_3^* & -1 & a_3^* a_2^* \\ a_2 & a_2 a_3 & 0 \end{pmatrix}$
- $z = \begin{pmatrix} 0 & 0 & (a_2')^* \\ 0 & 1 & 0 \\ a_2' & 0 & 0 \end{pmatrix}$
- and $y = z \times x = \begin{pmatrix} 0 & (a_2')^*(a_2 a_3) & * \\ * & -1 & a_3^*(a_2')^* \\ a_2' - a_2 & * & 1 \end{pmatrix}$.

We set $\beta_{K,m} = (x \wedge y)^{\otimes m}$. It is proved in section 3 that $\beta_{K,m} \in V_{m\lambda_3}$.

We require the following lemma.

Lemma 5.0.4. *Let the notation be as above, with $m > 0$ even. Set $w_0 = u^2 v - uv^2$. Then*

$$\left(\sum_{w \in W_{J_R} : rk(w)=1, pr_I(w) = w_0} \langle P_m(w), \beta_{K,m} \rangle_I \right) = 6.$$

Proof. As explained in [Pol20b, proof of Corollary 2.5.2], it follows from [EG96, Proposition 5.5] that the set of $w \in W_{J_R}$, with $rk(w) = 1, pr_I(w) = w_0$ consists of the six elements $(0, e_{ii}, -e_{jj}, 0)$, $i \neq j$, where e_{kk} is the diagonal matrix in J with a 1 in the k^{th} place and 0's elsewhere. Now observe that

(1) $\langle e_{22} \wedge e_{33}, x \wedge y \rangle_I = -1$
(2) $\langle e_{33} \wedge e_{11}, x \wedge y \rangle_I = -1$
(3) $\langle e_{11} \wedge e_{22}, x \wedge y \rangle_I = -1$

The lemma follows directly. ☐

Proof of Corollary 1.2.3. The corollaries now follow immediately from Theorem 1.2.1 and Lemma 5.0.4. ☐

We now explain the proof of Corollary 1.2.4.

Proof of Corollary 1.2.4. First note that the cubic ring $\mathbf{Z} \times \mathbf{Z}[t]/(t^2 - pt - q)$ is associated to the binary cubic form $-y(x^2+pxy+qy^2)$. Indeed, setting $\omega = (1,0)$ and $\theta = (0,t)$ in $\mathbf{Z}\times\mathbf{Z}[t]/(t^2-pt-q)$, this is a good basis, and computing its multiplication table gives rise to the binary cubic form $-y(x^2 + pxy + qy^2)$. The ring $\mathbf{Z} \times \mathbf{Z}_D$ is of this form with $p = D$ and $q = \frac{D-D^2}{4}$.

To explicitly compute the Fourier coefficients of Δ_{G_2}, we now make a specific choice of x, z, y as in the proof of Corollary 1.2.3. Namely, we take $K = \mathbf{Q}(\sqrt{-1})$, $a_2 = e_2$, $a_3 = u_0$, and $a_2' = e_2 - e_2^*$. We obtain

$$x = \begin{pmatrix} 1 & x_3 & x_2^* \\ x_3^* & -1 & x_1 \\ x_2 & x_1^* & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & y_3 & y_2^* \\ y_3^* & -1 & y_1 \\ y_2 & y_1^* & 1 \end{pmatrix}$$

with

- $x_1 = \frac{1}{2}(0, 1 - \sqrt{-1}i)$
- $x_2 = \frac{1}{2}(0, 1 - \sqrt{-1}i)$
- $x_3 = -\sqrt{-1}(i, 0)$
- $y_1 = -\sqrt{-1}(0, i)$
- $y_2 = \frac{1}{2}(0, 1 + \sqrt{-1}i)$
- $y_3 = -\frac{1}{2}(1 + \sqrt{-1}i, 0)$.

Now, given the binary cubic $-y(x^2 + pxy + qy^2)$, in order to compute the associated Fourier coefficient of Δ_{G_2}, we must compute the set of $(0, T_1, T_2, q) \in W_{J_R}$ so that $\mathrm{tr}(T_2) = p$, $\mathrm{tr}(T_1) = -1$, and $(0, T_1, T_2, q)$ rank one. We assume $q \neq 0$. Then $(0, T_1, T_2, q)$ is rank one if and only if $n(T_2) = 0$ and $T_2^\# = qT_1$, which implies that T_1 is rank one. As mentioned above, the set of rank one T_1 in J_R with $\mathrm{tr}(T_1) = -1$ consists just of the three elements $-e_{11}, -e_{22}, -e_{33}$.

Suppose that $T_1 = -e_{11}$. Then $0 = (T_1, T_2)$ because $(T_2, T_2^\#) = 3n(T_2) = 0$. So, the $(1, 1)$ entry of $T_2 = 0$. It now follows easily, using that $T_2^\# = -qe_{11}$, that T_2 is of the form

$$T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_2 & x \\ 0 & x^* & c_3 \end{pmatrix}$$

with $c_2c_3 - n(x) = -q$ and $c_2 + c_3 = p$. Substituting $p = D$, $q = \frac{D-D^2}{4}$, $c_2 = \frac{1}{2}(v + D)$, $c_3 = \frac{1}{2}(-v+D)$, one finds that v is an integer and $v^2 + 4n(x) = D$. The contribution to the Fourier coefficient $a_{\Delta_{G_2}}(\mathbf{Z} \times \mathbf{Z}_D)$ for such a term is computed to be $(v + \sqrt{-1}(x, (0, i)))^2$.

One makes a completely similar calculation if $T_1 = -e_{22}$ or $T_1 = -e_{33}$. One obtains

$$a_{\Delta_{G_2}}(\mathbf{Z} \times \mathbf{Z}_D) = \sum [v + \sqrt{-1}(x, (0, i))]^2 + [v - \sqrt{-1}(x, (0, i))]^2 + [v + \sqrt{-1}(x, (-i, 0))]^2$$

where the sum is over pairs (v, x) in $\mathbf{Z} \oplus R_\Theta$ such that $v^2 + 4n(x) = D$.

But, this is clearly the Fourier coefficient of a harmonic theta function in $S_{13/2}(\Gamma_0(4))^+$, associated to the lattice $\langle 2\rangle \perp 2E_8$. This space has dimension 1, spanned by $\delta(z)$. By looking at the coefficient of $D = 1$, one obtains the corollary. □

5.1. **Theta lifts.** In this subsection, we complete the proof of Theorem 1.2.1. More specifically, we will now use our knowledge of the computation of $\{D^{2m}\Theta(g, 1), \beta\}$ for arbitrary β to compute $\{D^{2m}\Theta(g, \gamma_E), \beta'\}$.

Let $\Theta = \Theta_{Gan}$ and denote by Θ_w the w-Fourier coefficient of this automorphic form. Suppose $g_\infty \in G_J(\mathbf{R})$. To begin, observe that

$$\begin{aligned} a(w)(\gamma_E)W_{2\pi w}(g_\infty) &= \Theta_w(g_\infty \gamma_E) \\ &= \Theta_w(\gamma_E g_\infty) \\ &= \Theta_w(\delta_E^{\mathbf{Q}} \delta_E^{\mathbf{R},-1} \delta_E^{\widehat{\mathbf{Z}}} g_\infty) \\ &= \Theta_{\delta_E^{\mathbf{Q},-1} w}(\delta_E^{\mathbf{R},-1} g_\infty) \\ &= a(\delta_E^{\mathbf{Q},-1} w) W_{2\pi w \cdot \delta_E^{\mathbf{Q}}}(\delta_E^{\mathbf{R},-1} g_\infty) \\ &= a(\delta_E^{\mathbf{Q},-1} w) W_{2\pi w}(g_\infty). \end{aligned}$$

Now, if $g \in G_2(\mathbf{R})$, we prove

$$\{D^{2m} W_w(g), \beta\} = \langle P_m(w), \beta \rangle W_{w_0, 4+m}(g)$$

where w_0 is the binary cubic $pr_I(w) = au^3 + (b, I^\#)u^2v + (c, I)uv^2 + dv^3$ if $w = (a, b, c, d)$. Observe that $pr_I(\delta_Q w) = pr_E(w)$, where $pr_E(w) = au^3 + (b, E^\#)u^2v + (c, E)uv^2 + dv^3$. Combining these two facts, one obtains

$$\begin{aligned} \{D^{2m}\Theta(g, \gamma_E), \beta\} &= \sum_{rk(w)=1} a(\delta_E^{\mathbf{Q},-1} w) \langle P_m(w), \beta \rangle W_{pr_I(w)}(g) \\ &= \sum_{rk(w)=1} a(w) \langle P_m(\delta_E^{\mathbf{Q}} w), \beta \rangle W_{pr_E(w)}(g) \\ &= \sum_{rk(w)=1} a(w) \langle P_m(w), \delta_E^{\mathbf{Q},-1} \beta \rangle_E W_{pr_E(w)}(g). \end{aligned}$$

Here, if $b \in J, c \in J^\vee \simeq J$, $x, y \in J$ then

$$\langle b \wedge c, x \wedge y \rangle_E = (b, x)_E (c, y) - (b, y)_E (c, x)$$

and one extends to $(\wedge^2 J)^{\otimes m}$. We have used Lemma 3.2.2 to get

$$\langle P_w(\delta_E^{\mathbf{Q}} w), \beta \rangle_I = \langle P_m(w), \delta_E^{\mathbf{Q},-1} \beta \rangle_E.$$

Thus, putting everything together,

$$\begin{aligned} \Theta(\alpha)(g) &:= \int_{[F_4^I]} \{D^{2m}\Theta(g, h), \alpha(h)\}\, dh \\ &= \frac{1}{|\Gamma_I|}\{D^{2m}\Theta(g, 1), \alpha(1)\} + \frac{1}{|\Gamma_E|}\{D^{2m}\Theta(g, \gamma_E), \alpha(\gamma_E)\} \\ &= \frac{1}{|\Gamma_I|} \sum_{rk(w)=1} a(w) \langle P_m(w), \alpha_I \rangle_I W_{pr_I(w)}(g) + \frac{1}{|\Gamma_E|} \sum_{rk(w)=1} a(w) \langle P_m(w), \alpha_E \rangle_E W_{pr_E(w)}(g). \end{aligned}$$

This completes the proof of Theorem 1.2.1.

6. The exponential derivative

The purpose of this section is to prove Theorem 4.0.1, which we restate here:

Theorem 6.0.1. *Suppose $g \in \mathrm{Sp}_6(\mathbf{R})$ and $\beta \in W(k_1, k_2)$. Let $\ell \geq 0$ be an integer. There is a nonzero constant B_{k_1,k_2}, independent of g and T, so that*

$$j(g, i)^\ell \rho_{[k_1,k_2]}(J(g, i))\{D^{k_1+2k_2}(j(g, i)^{-\ell} e^{2\pi i (T, g\cdot i)}), \beta\} = B_{k_1,k_2}\{P_{k_1,k_2}(T), \beta\} e^{2\pi i (pr(T), g\cdot i)}.$$

Moreover, $\{P_{k_1,k_2}(T), \beta\}$ lies in the highest weight submodule $S^{[k_1,k_2]}$ of $S^{k_1}(V_3) \otimes S^{k_2}(\wedge^2 V_3)$.

Our proof of this theorem is to, essentially completely explicitly, calculate the derivatives

$$X_{\alpha_1}\cdots X_{\alpha_n}(j(g,i)^{-\ell}e^{2\pi i(T,g\cdot i)}). \tag{2}$$

To do this, we use the Iwasawa decomposition $\mathfrak{h}_J = n(J)+\mathfrak{m}_J+\mathfrak{k}_{E_7}$, write each X_α as a sum in terms of this decomposition, and calculate the derivatives for each piece.

6.1. **Preliminaries.**

Recall from above the Cayley transform $C_h \in H_J(\mathbf{C})$, which satisfies

(1) $C_h^{-1}n_L(J\otimes\mathbf{C})C_h = \mathfrak{p}_J^+$
(2) $C_h^{-1}n_L^\vee(J\otimes\mathbf{C})C_h = \mathfrak{p}_J^-$
(3) $C_h^{-1}(\mathfrak{m}_J\otimes\mathbf{C})C_h = \mathfrak{k}_{E_7}$.

Given $\phi\in\mathfrak{m}_J$, let $M(\phi)$ denote its action on W_J [Pol20a, section 3.4].

Proposition 6.1.1. *One has the following identities:*

(1) $C_h^{-1}n_L(x)C_h = n_L(x) - \frac{i}{2}M(\Phi_{1,x}) + \frac{1}{2}(n_L^\vee(x)-n_L(x))$
(2) $C_h^{-1}n_L^\vee(\gamma)C_h = n_L(\gamma) + \frac{i}{2}M(\Phi_{1,\gamma}) + \frac{1}{2}(n_L^\vee(\gamma)-n_L(\gamma))$
(3) If $\phi(1)=0$, *then* $C_h^{-1}M(\phi)C_h = M(\phi)$.
(4) If $\phi=\Phi_{1,z}$, *then* $C_h^{-1}M(\phi)C_h = -in_L(z)+in_L^\vee(z)$.

Proof. Some of the facts needed to prove this are in Lemma 3.4.1 of [Pol20a]. Also useful is the identity $\Phi_{Y,1}=\Phi_{1,Y}$ (see Theorem 4.0.10 in [Pol21]). To help with the proof of the second statement, one computes that

$$n_G^\vee(\delta)M(\phi)n_G^\vee(-\delta) = M(\phi) - n_L^\vee(\widetilde{\phi}(\delta)).$$

We omit the rest of the proof. □

6.2. **Some computations.**

For $E\in J\otimes\mathbf{C}$, define $X^+(E) = iC_h^{-1}n_L(E)C_h$. To warm up, we will compute $X^+(E)(j(g,i)^{-\ell}e^{2\pi i(T,g\cdot i)})$. Let h denote the function $h(g)=j(g,i)^{-\ell}e^{2\pi i(T,g\cdot i)}$. Let m_Y denote any element of M_J for which $m_Y\cdot i = Yi\in\mathcal{H}_J$.

Lemma 6.2.1. *One has the following computations:*

(1) $n_L(E)h(M(\delta,m)) = 2\pi i(T,m(E))h(M(\delta,m))$;
(2) $M(\Phi_{1,E})e^{-2\pi(T,m(1))} = -4\pi(T,m(E))e^{-2\pi(T,m(1))}$;
(3) $M(\Phi_{1,E})j(m_Y,i)^{-\ell} = \ell(1,E)n(Y)^{\ell/2} = \ell(1,E)j(m_Y,i)^{-\ell}$;
(4) if $k=n_L^\vee(E)-n_L(E)$, *then* $kj(g,i)^{-\ell} = -\ell i(1,E)j(g,i)^{-\ell}$.

Proof. The first statement follows from the identity $M(\delta,m)n_L(E)M(\delta,m)^{-1} = n_L(m(E))$. The second statement follows from the fact that $\Phi_{1,E}(E') = \{E,E'\}$, so that $\Phi_{1,E}(1)=2E$. For the third statement, we have

$$M(\Phi_{1,E})j(m_Y,i)^{-\ell} = \frac{d}{dt}|_{t=0}(\langle m_Y\exp(t\Phi_{1,E})r_0(i),f\rangle)^{-\ell}.$$

Now $M(\Phi_{1,E})r_1(i) = (-(1,E),\ldots)$ so $M(\Phi_{1,E})j(m_Y,i)^{-\ell}$ is the coefficient in t of $(n(Y)^{-1/2}(1-(1,E)t))^{-\ell}$, which is $n(Y)^{\ell/2}\cdot\ell(1,E)$. Thus $M(\Phi_{1,E})j(m_Y,i)^{-\ell} = \ell(1,E)n(Y)^{\ell/2}$ as claimed. For the fourth statement, observe that $n_L^\vee(E)r_1(i) = (i(1,E),\ldots)$, $n_L(E)r_1(i) = (0,\ldots)$, so $e^{tk}r_1(i) = (1+i(1,E)t,\ldots)+O(t^2)$. As $j(ge^{tk},i) = j(g,i)j(e^{tk},i)$, the statement follows. □

Putting everything together, from Lemma 6.2.1 and Proposition 6.1.1 one obtains

$$\begin{aligned}(C_h^{-1}n_L(E)C_h)h(m) &= 2\pi i(T,m(E))h(m) + (-i/2)(-4\pi(T,m(E)) + \ell(1,E))h(m)\\ &\quad + \frac{1}{2}(-\ell i(1,E))h(m)\\ &= (4\pi i(T,m(E)) - \ell i(1,E))h(m)\\ (C_h^{-1}n_L^\vee(E)C_h)h(m) &= 2\pi i(T,m(E))h(m) + (i/2)(-4\pi(T,m(E)) + \ell(1,E))h(m)\\ &\quad + \frac{1}{2}(-\ell i(1,E))h(m)\\ &= 0.\end{aligned}$$

Thus

$$X_E^+ h(m) = (-4\pi(T,m(E)) + \ell(1,E))h(m).$$

We now explain the computation of (2). To setup the result, define $P_k : J^{\otimes k} \to T(J)$ inductively as follows: $P_0 = 1$, and for $k \geq 0$,

$$\begin{aligned}P_{k+1}(E_1,\ldots,E_k,E_{k+1}) = P_k(E_1,\ldots E_k)\otimes E_{k+1} + \ell(1,E_{k+1})P_k(E_1,\ldots,E_k) + \frac{1}{2}\{E_{k+1},P_k(E_1,\ldots E_k)\}\\ + \frac{1}{2}P_k(\{E_{k+1},E_1,\ldots,E_k\}).\end{aligned}$$

Here we write

$$\{E, V_1\otimes\cdots\otimes V_r\} := \sum_{j=1}^r V_1\otimes\cdots\otimes\{E,V_j\}\otimes\cdots\otimes V_r$$

and we interpret $\{E,C\} = 0$ if $C \in T^0(J)$ is constant. Thus

(1) $P_1(E_1) = E_1 + \ell(1,E_1)$.
(2) $P_2(E_1,E_2) = E_1\otimes E_2 + \ell(1,E_2)E_1 + \ell(1,E_1)E_2 + \{E_1,E_2\} + \ell^2(1,E_1)(1,E_2) + \frac{\ell}{2}(1,\{E_1,E_2\})$.

Define now $w_{T,m} : T(J) \to \mathbf{C}$ as

$$w_{T,m}(V_1\otimes\cdots\otimes V_r) = (-4\pi)^r(T,m(V_1))\cdots(T,m(V_r))$$

and extending to $T(J)$ by linearity.

Proposition 6.2.2. *Let the notation be as above. Then $X_{E_k}\cdots X_{E_1}h(m) = w_{T,m}(P(E_1,\ldots,E_k))h(m)$.*

Proof. We proceed by induction. Observe that

(1) $n_L(E_{k+1})X_{E_k}\cdots X_{E_1}h(m) = 2\pi i(T,mE_{k+1})X_{E_k}\cdots X_{E_1}h(m)$.
(2) $M(\Phi_{1,E})w_{T,m}(V_1\otimes\cdots\otimes V_r) = w_{T,m}(\{E,V_1\otimes\cdots\otimes V_r\})$.
(3) $M(\Phi_{1,E})h(m) = (\ell(1,E) - 4\pi(T,m(E)))h(m)$.
(4) If $\mu \in K$, then $\mu X_{E_k}\cdots X_{E_1}h(m) = j(\mu,i)^{-\ell}X_{\mu\cdot E_k}\cdots X_{\mu\cdot E_1}h(m)$.

Now, if $k = n_L^\vee(E) - n_L(E)$, then

$$\begin{aligned}[k, C_h^{-1}n_L(E')C_h] &= [-iC_h^{-1}M(\Phi_{1,E})C_h, C_h^{-1}n_L(E')C_h]\\ &= -iC_h^{-1}[M(\Phi_{1,E}), n_L(E')]C_h\\ &= -iC_h^{-1}n_L(\{E,E'\})C_h.\end{aligned}$$

Thus $[ik/2, X_{E'}] = \frac{1}{2}X_{\{E,E'\}}$. Consequently,

$$(ik/2)(X_{E_k}\cdots X_{E_1})h(m) = \frac{\ell}{2}(1,E)(X_{E_k}\cdots X_{E_1})h(m) + \frac{1}{2}(\sum_{j=1}^k X_{E_k}\cdots X_{\{E,E_j\}}\cdots X_{E_1})h(m).$$

As

$$X_E = iC_h^{-1}n_L(E)C_h = in_L(E) + \frac{1}{2}M(\Phi_{1,E}) + \frac{i}{2}(n_L^\vee(E) - n_L(E)),$$

one can now easily verify the proposition. □

We now prove:

Lemma 6.2.3. *Let $P_n : J^{\otimes k} \to T(J)$ be as above and let $\beta \in W(k_1,k_2) \subseteq V_7^{\otimes k_1} \otimes (\wedge^2 V_7)^{\otimes k_2}$. Set $n = k_1 + 2k_2$. Then*

$$\sum_{\alpha_i} P_n(E_{\alpha_1},\ldots,E_{\alpha_n})\{E_{\alpha_1}^\vee \otimes \cdots \otimes E_{\alpha_n}^\vee, \beta\} = \sum_{\alpha_i} E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_n}\{E_{\alpha_1}^\vee \otimes \cdots \otimes E_{\alpha_n}^\vee, \beta\}. \quad (3)$$

In other words, only the leading term contributes.

Proof. Suppose $h \in G_2(\mathbf{R})$. Observe

$$\begin{aligned}\sum_{\alpha_i} P_n(E_{\alpha_1},\ldots,E_{\alpha_n})\{E_{\alpha_1}^\vee \otimes \cdots \otimes E_{\alpha_n}^\vee, h\cdot\beta\} &= \sum_{\alpha_i} P_n(E_{\alpha_1},\ldots,E_{\alpha_n})\{h^{-1}E_{\alpha_1}^\vee \otimes \cdots \otimes h^{-1}E_{\alpha_n}^\vee, \beta\} \\ &= \sum_{\alpha_i} P_n(hE_{\alpha_1},\ldots,hE_{\alpha_n})\{E_{\alpha_1}^\vee \otimes \cdots \otimes E_{\alpha_n}^\vee, \beta\} \\ &= \sum_{\alpha_i} h\cdot P_n(E_{\alpha_1},\ldots,E_{\alpha_n})\{E_{\alpha_1}^\vee \otimes \cdots \otimes E_{\alpha_n}^\vee, \beta\}\end{aligned}$$

where we have used that P_n is equivariant for the action of $F_4 \supseteq G_2$. (This follows, for example, from the recursive formula.)

Now, because of the above equivariance, we can assume $\beta = e_2^{k_1} \otimes (e_2 \wedge e_3^*)^{k_2}$. Indeed, the two sides of the equality to be proved are linear in β and $e_2^{k_1} \otimes (e_2 \wedge e_3^*)^{k_2} \in W(k_1,k_2)$.

We now compute in the basis $\{u_0, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*\}$ of Θ^0. More precisely, we take a basis of J that is a basis of $H_3(F)$ union a basis $v_i \otimes u_j$ of $V_3 \otimes \Theta^0$, where $u_j \in \{u_0, e_1, e_2, e_3, e_1^*, e_2^*, e_3^*\}$ and v_1, v_2, v_3 is a basis of V_3. We compute in this basis. Then the only terms that contribute to the left-hand side of (3) are those with $E_{\alpha_i} = v_{\alpha_i} \otimes u_{\alpha_i}$ for all i. Then, still in order for these terms to contribute in a nonzero way to (3), we must have $u_k = e_2$ for $1 \leq k \leq k_1$ and $\{u_{\alpha_{2j-1}}, u_{\alpha_{2j}}\} = \{e_2, e_3^*\}$ for $2j-1 > k_1$. Consequently, all the E_{α_i} that contribute to the sum in our basis satisfy $(1, E_{\alpha_i}) = 0$, $E_{\alpha_i}^\# = 0$, and $\{E_{\alpha_i}, E_{\alpha_j}\} = 0$. It now follows from our recursive formula for P_n that only the leading term contributes. □

It follows immediately from Lemma 6.2.3 that

$$\begin{aligned}\sum_{\alpha_i} w_{T,m}(P_n(E_{\alpha_1},\ldots,E_{\alpha_n}))\{E_{\alpha_1}^\vee \otimes \cdots \otimes E_{\alpha_n}^\vee, \beta\} &= \sum_{\alpha_i} w_{T,m}(E_{\alpha_1} \otimes \cdots \otimes E_{\alpha_n})\{E_{\alpha_1}^\vee \otimes \cdots \otimes E_{\alpha_n}^\vee, \beta\} \\ &= (-4\pi)^n\{\widetilde{m}^{-1}(T^{\otimes n}), \beta\}.\end{aligned}$$

Combining this with Proposition 6.2.2, we obtain

Proposition 6.2.4. *Suppose $\beta \in W(k_1,k_2)$, $n = k_1 + 2k_2$ and $m \in M_J$. Then*

$$\{D^n(j(m,i)^{-\ell}e^{-2\pi(T,m(1))}), \beta\} = B_{k_1,k_2} j(m,i)^{-\ell}\{\widetilde{m}^{-1}(T^{\otimes n}), \beta\}e^{-2\pi(T,m(1))}$$

for a nonzero constant B_{k_1,k_2}.

We can now prove Theorem 4.0.1.

Proof of Theorem 4.0.1. Observe that by appropriate K_{Sp_6}-equivariance, and by equivariance for the unipotent radical of the Siegel parabolic, it suffices to check the equality of the statement of Theorem 4.0.1 when $g \in M$, the Levi of the Siegel parabolic. But this follows from Proposition 6.2.4 and the definition of $\rho_{[k_1,k_2]}(J(m,i))$; see the proof of Lemma 2.2.5 for the action of $\widetilde{m}^{-1}$ on T. The proof of the theorem now follows from Lemma 6.2.5 below. □

Recall that $S^{[k_1,k_2]}$ is the kernel of the contraction

$$S^{k_1}(V_3) \otimes S^{k_2}(\wedge^2 V_3) \to S^{k_1-1}(V_3) \otimes S^{k_2-1}(\wedge^2 V_3) \otimes \det(V_3).$$

Lemma 6.2.5. *If $\beta \in W(k_1,k_2)$, then $\{P_{k_1,k_2}(T),\beta\} \in S^{[k_1,k_2]}$.*

Proof. By equivariance and linearity, it suffices to verify the claim of the lemma for $\beta = e_2^{\otimes k_1} \otimes (e_2 \wedge e_3^*)^{\otimes k_2}$. Now suppose $T = T_0 + x_1 \otimes v_1 + x_2 \otimes v_2 + x_3 \otimes v_3$. Then

$$(T,e_2) = (x_1,e_2)v_1 + (x_2,e_2)v_2 + (x_3,e_2)v_3$$

and

$$(T\otimes T, e_2\wedge e_3^*) = (x_2\wedge x_3, e_2\wedge e_3^*)v_2\wedge v_3 + (x_3\wedge x_1, e_2\wedge e_3^*)v_3\wedge v_1 + (x_1\wedge x_2, e_2\wedge e_3^*)v_1\wedge v_2.$$

Thus contracting yields the term

$$(x_1,e_2)(x_2\wedge x_3, e_2\wedge e_3^*) + (x_2,e_2)(x_3\wedge x_1, e_2\wedge e_3^*) + (x_3,e_2)(x_1\wedge x_2, e_2\wedge e_3^*).$$

This is $(x_1\wedge x_2\wedge x_3, e_2\wedge e_2\wedge e_3^*) = 0$. The lemma follows. □

7. The quaternionic Whittaker derivative

The goal of this section is to prove Theorem 5.0.1, which we restate here:

Theorem 7.0.1. *Suppose $\ell = 4$ and $\beta \in V_{m\lambda_3}$ with $m \geq 0$. Then there is a nonzero constant B_m so that for all $w \in W_J$ rank one and $g \in G_2(\mathbf{R})$, one has*

$$\{D^{2m}W_{w,\ell}(g),\beta\} = B_m\langle P_m(w),\beta\rangle_I W_{pr_I(w),\ell+m}(g).$$

We begin with the following proposition.

Proposition 7.0.2. *Let $\beta \in V_{m\lambda 3} \subseteq (\wedge^2 J^0)^{\otimes m}$. Suppose $\ell = 4$ and w is rank one. Then the function $F_{m,\beta} : G_2(\mathbf{R}) \to S^{2m+2\ell}(V_2)$ defined as $\{D^{2m}W_{w,\ell}(g),\beta\}$ is quaternionic.*

Proof. Fix a rank one $w \in W_J$. Let $\chi = \chi_w$ be the character of N_J given as $\chi(n) = e^{i\langle w,\overline{n}\rangle}$. Here $\overline{n}$ is the image of n in $W_J \simeq N_J^{ab}$, the abelianization of N_J. Let $L : \Pi_{min} \to \mathbf{C}$ be the unique (up to scalar multiple) moderate growth linear functional satisfying $L(nv) = \chi(n)L(v)$ for all $n \in N(\mathbf{R})$ and $v \in \Pi_{min}$. (Such an L exists by a global argument: The global minimal representation has nonzero Fourier coefficients, so there is an L_0 for some w_0. Now $L(v) := L_0(m_w v)$ for an appropriate m_w is the desired functional. The uniqueness of L follows from [Pol20a].)

Now let x_j be a basis of the minimal $K = (\mathrm{SU}(2)\times E_7)/\mu_2$-type of Π_{min} and $x_j^\vee$ in $S^8(V_2)$ the dual basis. Note that $W_w(g) = \sum_j L(gx_j)\otimes x_j^\vee$. Then $E = \sum_j x_j \otimes x_j^\vee$ is in $(V_{min}\otimes S^8(V_2))^K$. One obtains that $D^{2m}E \in (V_{min}\otimes S^8(V_2)\otimes \mathfrak{p}^{\otimes 2m})^K$. This latter space maps $K' := \mathrm{SU}(2)\times \mathrm{SU}(2)_s \times F_4^I(\mathbf{R})$-equivariantly to

$$S^{8+2m}(V_2)\otimes \det(V_2^s)^{\otimes m}\otimes (\wedge^2 J_0)^{\otimes m}.$$

Finally, mapping $(\wedge^2 J^0)^{\otimes m}$ to $V_{m\lambda_3}$, we obtain a K' invariant element E' in $V_{min}\otimes (S^{8+2m}\otimes \mathbf{1}\otimes V_{m\lambda_3})$. By Huang-Pandzic-Savin [HPS96], this E' is either 0 or the minimal type of $\pi_{m+4}\otimes V_{m\lambda_3}$. Contracting now against some $\beta \in V_{m\lambda_3}$, we obtain some (possibly 0) multiple of $S^{2m+8}(V_2) \subseteq \pi_{m+4}$. Applying $L(g\cdot)$, it follows that $F_{m,\beta}$ is quaternionic. □

7.1. **General strategy.** As before, let $W_{w,\ell}(g)$ be the generalized Whittaker function of weight ℓ associated to $w \in W_J$, which is positive semi-definite. Set $W_{w,\ell}^{-\ell}(g) = (x^{2\ell}, W_w(g))$ be the component multiplying $y^{2\ell}/(2\ell)!$. (See [Pol20a] for the definition of x, y.) Here $(\ ,\ )$ is an $\mathrm{SL}_2(\mathbf{C})$-equivariant pairing on $S^{\cdot}(V_2)$.

Consider the quantity $\{D^{2m}W_{w,\ell}(m),\beta\}$. Then

$$\{D^{2m}W_w(m), k\cdot\beta\} = \{k^{-1}D^{2m}W_w(m),\beta\} = \{D^{2m}W_w(mk),\beta\}.$$

Thus if we can compute $\{D^{2m}W_w(m),\beta\}$ for $\beta = (x\wedge y)^{\otimes m}$ where $E_{x,y}$ is assumed singular and isotropic, then we can compute this quantity for general β.

What we actually do is compute $(x^{2m+2\ell}, \{D^{2m}W_w(m), \beta\})$ for $\beta = (x \wedge y)^{\otimes m}$ where $E_{x,y}$ is assumed singular and isotropic. To setup the computation, we fix a basis $x_1 = x, x_2 = y, \ldots$ of J^0, fix a basis of $W_J = W_Q \oplus V_2^s \otimes J^0$ that is the union of bases of $W_{\mathbf{Q}}$ and of the tensor product basis $\{x_s, y_s\} \otimes \{x_1, x_2, \ldots\}$ of $V_2^s \otimes J^0$. Then we fix our basis of $\mathfrak{p} \simeq V_2^\ell \otimes W_J$ to be the tensor product basis of $\{x, y\}$ with the above fixed basis of W_J.

Now, it is clear that $(x^{2m+2\ell}, \{D^{2m}W_w(m), \beta\})$ only contains the terms in D^{2m} where the X_{α_i} equal one of

$$X_1 = y \otimes (x_s \otimes x), X_2 = y \otimes (x_s \otimes y), X_3 = y \otimes (y_s \otimes x), X_4 = y \otimes (y_s \otimes y).$$

Note also that because $E_{x,y}$ is isotropic, the above Lie algebra elements all commute. We obtain that

$$\begin{aligned}(x^{2m+\ell}, \{D^{2m}W_w(m), (x \wedge y)^{\otimes m}\}) &= 2^m \sum_{j=0}^{m} (-1)^j \binom{m}{j} (X_1X_4)^{m-j}(X_2X_3)^j W_{w,\ell}^{-\ell}(m) \\ &= 2^m (X_1X_4 - X_2X_3)^m W_{w,\ell}^{-\ell}(m).\end{aligned}$$

We now observe the following fact: if $k \in F_4^I(\mathbf{R})$, then

$$(x^{2m+2\ell}, \{D^{2m}W_{w,\ell}(m), k \cdot \beta\}) = (x^{2m+2\ell}, \{k^{-1}D^{2m} \cdot W_{w,\ell}(m), \beta\}) = (x^{2m+2\ell}, \{D^{2m}W_{w,\ell}(mk), \beta\}).$$

Thus, if we can compute the right hand side, then we can compute the left hand side.

To compute the quantity $(x^{2m+2\ell}, \{D^{2m}W_{w,\ell}(mk), \beta\})$, we will represent $W_{w,\ell}^{-\ell}(g)$ as an integral, and differentiate under the integral sign. This is inspired by the work of McGlade-Pollack [MP22]. More exactly, set

$$a_{\ell,v}(g) = \frac{(e_\ell, g^{-1}v)^\ell}{(pr(g^{-1}v), pr(g^{-1}v))^{\ell+\frac{1}{2}}}.$$

Here $pr : \mathfrak{g}(J) \otimes \mathbf{R} \to \mathfrak{su}_2$ is the projection onto the Lie algebra of the long root SU_2 and we write $(X, Y) = B(X, Y)$ for short.

We prove the following theorem. To setup the theorem, recall from [Pol20a] that if $z \in J$ with $\mathrm{tr}(z) = 0$, then

$$V(z) = (0, iz, -z, 0), V^*(z) = (0, -iz, -z, 0).$$

Moreover, let $\nu : M_J \to \mathrm{GL}_1$ be the similitude character on the Levi of the Heisenberg parabolic of G_J. There is an identification $M_J \simeq H_J$ (see [Pol20a, Lemma 4.3.1]), and ν is the similitude character of H_J via this identification. I.e., for $h \in H_J$, ν satisfies $\langle hv, hv' \rangle = \nu(h)\langle v, v' \rangle$ for all $v, v' \in W_J$ and $\langle \, , \rangle$ Freudenthal's symplectic form on W_J.

Theorem 7.1.1. *Let the notation be as above. Let $N_w \subseteq N$ consist of the n with $\langle w, \overline{n} \rangle = 0$. Suppose $g \in M_J(\mathbf{R})$, the Levi of the Heisenberg parabolic of $G_J(\mathbf{R})$ and $w \in W_J$ is positive semidefinite. Set $w_1 = g^{-1}w$. There is a nonzero constant $B'_{\ell,m}$, independent of $w \geq 0$ and independent of g so that the integral*

$$\int_{\mathbf{R}=N_w\backslash N} e^{-i\langle w, \overline{n}\rangle}(X_1X_4 - X_2X_3)^m a_{\ell,v}(ng)\, dn$$

is equal to

$$B'_{\ell,m}\nu(g)^m W_{w,\ell+m}^{-(\ell+m)}(g)(\langle V(x), w_1\rangle\langle V^*(y), w_1\rangle - \langle V(y), w_1\rangle\langle V^*(x), w_1\rangle)^m.$$

Note that the $m = 0$ case of Theorem 7.1.1 represents the function $W_{w,\ell}^{-\ell}(g)$ as a integral, and the cases $m > 0$ of this theorem compute (by exchaning the order of integration and differentiation) the derivatives $(X_1X_4 - X_2X_3)^m W_{w,\ell}^{-\ell}(g)$. We justify this exchange of integration and differentiation in subsection 7.6.

Corollary 7.1.2. *Suppose g is in the Levi of the Heisenberg parabolic of $G_2(\mathbf{R})$, and $\beta' \in V_{m\lambda_3}$. Then*
$$(x^{2m+2\ell}, \{D^{2m} W_{w,\ell}(g), \beta'\}) = B''_{m,\ell} W_{w,\ell+m}^{-(\ell+m)}(g) \langle P_m(w), \beta'\rangle_I.$$
for a nonzero constant $B''_{m,\ell}$, independent of w and g.

Proof. Suppose $w = (a,b,c,d)$. First consider the case $\beta' = \beta$. We must simplify the quantity
$$\langle V(x), w_1\rangle\langle V^*(y), w_1\rangle - \langle V(y), w_1\rangle\langle V^*(x), w_1\rangle.$$
If $w_1 = w_1' + (0, b_1, -c_1, 0)$ with $\mathrm{tr}(b_1) = \mathrm{tr}(c_1) = 0$ and $w_1' \in W_{\mathbf{Q}}$, then
$$\begin{aligned}\langle V(x), w_1\rangle\langle V^*(y), w_1\rangle - \langle V(y), w_1\rangle\langle V^*(x), w_1\rangle &= (i(x,c_1) - (x,b_1))(-i(y,c_1) - (y,b_1)) \\ &\quad - (-i(x,c_1) - (x,b_1))(i(y,c_1) - (y,b_1)) \\ &= 2i((x,b_1)(y,c_1) - (x,c_1)(y,b_1)) \\ &= 2i(x\wedge y, b_1 \wedge c_1).\end{aligned}$$
Consequently, if $g \in \mathrm{GL}_2^s$, the Heisenberg Levi on G_2, and $w = w' + (0, b', -c', 0)$, then $b_1 \wedge c_1 = \det(g)^{-1} b' \wedge c'$, so
$$\langle V(x), w_1\rangle\langle V^*(y), w_1\rangle - \langle V(y), w_1\rangle\langle V^*(x), w_1\rangle = (2i)\det(g)^{-1}(x\wedge y, b'\wedge c').$$
In general, if $\beta' = \sum_j \alpha_j k_j (x\wedge y)^{\otimes m}$, with $\alpha_j \in \mathbf{C}$ and $k_j \in F_4^I(\mathbf{R})$, one finds that
$$\sum_j \alpha_j (\langle V(x), k_j^{-1}g^{-1}w\rangle\langle V^*(y), k_j^{-1}g^{-1}w\rangle - \langle V(y), k_j^{-1}g^{-1}w\rangle\langle V^*(x), k_j^{-1}g^{-1}w\rangle)^m$$
is equal to
$$\begin{aligned}(2i)^m\det(g)^{-m}\sum_j \alpha_j (k_j(x\wedge y), b'\wedge c')^m &= (2i)^m \det(g)^{-m}\sum_j \alpha_j (k_j(x\wedge y)^{\otimes m}, (b'\wedge c')^{\otimes m}) \\ &= (2i)^m\det(g)^{-m}(\beta, (b'\wedge c')^{\otimes m}) \\ &= (-2i)^m \det(g)^{-m}(\beta, (b\wedge c)^{\otimes m})\end{aligned}$$
if $w = (a,b,c,d)$.

The $\det(g)^{-m}$ cancels the $\nu(g)^m$ from Theorem 7.1.1, giving the corollary. □

Theorem 5.0.1 follows easily from Corollary 7.1.2:

Proof of Theorem 5.0.1. Both sides of the desired equality transform on the left under $N(\mathbf{R})$ in the same way, and on the right under K_{G_2} in the same way. Moreover, for $\ell = 4$, they are both known to be quaternionic functions. Thus to prove their equality, it suffices to pair against $x^{2\ell}$, and evaluate on g in the Levi of the Heisenberg parabolic. But this is precisely what is done in Corollary 7.1.2, so the theorem is proved. □

7.2. **Some derivatives.** We now focus on proving Theorem 7.1.1. We must consider some derivatives of the function
$$a_{\ell,v}(g) = \frac{(e_\ell, g^{-1}v)^\ell}{(pr(g^{-1}v), pr(g^{-1}v))^{\ell+\frac{1}{2}}}.$$
Suppose $X = x\otimes w \in \mathfrak{p}$. Then $X \cdot e_\ell = 0$, and thus $X\{g \mapsto (e_\ell, g^{-1}v)\} = 0$. Consequently, the function $(e_\ell, g^{-1}v)^\ell$ can be considered constant for the purposes of differentiating with respect to $x \otimes W \subseteq \mathfrak{p}$. We therefore must just differentiate the function
$$b_{\ell,v}(g) = (pr(g^{-1}v), pr(g^{-1}v))^{-(\ell+\frac{1}{2})}.$$

We obtain

$$\begin{aligned} Xb_{\ell,v}(g) &= (-(\ell+\frac{1}{2}))\cdot(pr(g^{-1}v),pr(g^{-1}v))^{-(\ell+\frac{3}{2})}\cdot(2)\cdot(-pr([X,g^{-1}v]),pr(g^{-1}v)) \\ &= (2\ell+1)(pr([X,g^{-1}v]),pr(g^{-1}v))(pr(g^{-1}v),pr(g^{-1}v))^{-(\ell+\frac{3}{2})}. \end{aligned}$$

We write $\mathcal{C}_v(X,\cdot) = (pr([X,g^{-1}v]),pr(g^{-1}v))$ and $\mathcal{C}_v(X_1,X_2) = (pr([X_1,g^{-1}v]),pr([X_2,g^{-1}v]))$ and $\mathcal{C}_v = (pr(g^{-1}v),pr(g^{-1}v))$. Thus,

$$Xb_{\ell,v}(g) = (2\ell+1)\mathcal{C}_v(X,\cdot){\mathcal{C}_v}^{-(\ell+\frac{3}{2})}.$$

Now suppose $X_1 = X = x\otimes w_1$, and $X_2 = x\otimes w_2$ is such that $\mathrm{Span}\{w_1,w_2\}$ is isotropic and singular. Because it is isotropic, $[X_1,X_2]=0$. Because it is singular, $\mathrm{Span}\, X_1,X_2$ consists of rank one elements of $\mathfrak{g}(J)$. Recall that $X\in\mathfrak{g}(J)$ is rank one means $[X,[X,y]]+2B(X,y)X=0$ for all $y\in\mathfrak{g}(J)$, where $B(\,,\,)$ is the bilinear form proportional to the Killing form defined in [Pol20a, section 4.2.2]. Thus, by symmetrizing and using that X_1,X_2 commute, one arrives at $-[X_1,[X_2,y]] = B(X_1,y)X_2+B(X_2,y)X_1$. Thus, $pr([X_1,[X_2,y]])=0$. Using this, we differentiate $b_{\ell,v}(g)$ again to obtain

$$X_2X_1b_{\ell,v}(g) = -(2\ell+1)\mathcal{C}_v(X_1,X_2){\mathcal{C}_v}^{-(\ell+\frac{3}{2})} + (2\ell+1)(2\ell+3)\mathcal{C}_v(X_1,\cdot)\mathcal{C}_v(X_2,\cdot){\mathcal{C}_v}^{-(\ell+\frac{5}{2})}.$$

For $0\le k\le\lfloor n/2\rfloor$, define $\mathcal{C}_{v,n,k}$ to be the symmetric sum of terms of the form

$$\mathcal{C}_v(X_1,X_2)\cdots\mathcal{C}_v(X_{2k-1},X_{2k})\mathcal{C}_v(X_{2k+1},\cdot)\cdots\mathcal{C}_v(X_n,\cdot).$$

Then we have the following proposition.

Proposition 7.2.1. *Suppose $X_1,\dots,X_n$ are such that*

(1) $X_j = x\otimes w_j$ for all j with
(2) $\mathrm{Span}(w_1,\dots,w_n)$ singular and isotropic.

Then

$$X_n\cdots X_1b_{\ell,v}(g) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k2^{n-k}(\ell+\frac{1}{2})_{n-k}\mathcal{C}_{v,n,k}{\mathcal{C}_v}^{-(\ell+n-k+\frac{1}{2})}.$$

Proof. We proceed by induction, noting that the proposition is true for $n=1$ and $n=2$ as checked above. Note that

$$X_{n+1}{\mathcal{C}_v}^{-(\ell+n-k+\frac{1}{2})} = 2(\ell+n-k+\frac{1}{2})\mathcal{C}_v(X_{n+1},\cdot){\mathcal{C}_v}^{-(\ell+n+1-k+\frac{1}{2})}.$$

Thus, making the induction assumption, $X_{n+1}\cdots X_1b_{\ell,v}(g)$ is equal to

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k2^{n+1-k}(\ell+\frac{1}{2})_{n+1-k}\mathcal{C}_{v,n,k}\mathcal{C}_v(X_{n+1},\cdot){\mathcal{C}_v}^{-(\ell+n+1-k+\frac{1}{2})}$$

plus

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{k-1}2^{n-k}(\ell+\frac{1}{2})_{n-k}(-X_{n+1}\mathcal{C}_{v,n,k}){\mathcal{C}_v}^{-(\ell+n-k+\frac{1}{2})}.$$

But now note that

$$\mathcal{C}_{v,n,j}\mathcal{C}_v(X_{n+1},\cdot)+(-X_{n+1}\mathcal{C}_{v,n,j-1}) = \mathcal{C}_{v,n+1,j}.$$

The proposition follows. □

The next step is to calculate the $\mathcal{C}_v(X_j,\cdot)$ and $\mathcal{C}_v(X_i,X_j)$, then integrate.

7.3. **Some Lie algebra calculations.** We set $v = e \otimes w$ and $x = ng \in N_J(\mathbf{R})M_J(\mathbf{R})$.

Then if $n = \exp(e \otimes \overline{n})$, $n^{-1}v = e \otimes w + \langle w, \overline{n}\rangle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, so $x^{-1}v = e \otimes w_1 + z_1 \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ where $w_1 = g^{-1}w$ and $z_1 = \nu(g)^{-1}\langle w, \overline{n}\rangle$.

Now recall the long root $\mathfrak{sl}_2$ triple of $\mathfrak{g}(J)$ given by

- $e_\ell = \frac{1}{4}(ie + f) \otimes r_0(i)$,
- $f_\ell = \frac{1}{4}(ie - f) \otimes r_0(-i)$ and
- $h_\ell = \frac{i}{2}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) + n_L(-1) + n_L^\vee(1))$.

For $X \in \mathfrak{g}$ we have $pr(X) = B(X, f_\ell)e_\ell + \frac{1}{2}B(X, h_\ell)h_\ell + B(X, e_\ell)f_\ell$. Note that the exceptional Cayley transform C [Pol20a, section 5] takes e_ℓ, h_ℓ, f_ℓ to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. We obtain

$$pr(x^{-1}v) = \frac{1}{4}(\alpha^* e_\ell - iz_1 h_\ell - \alpha f_\ell)$$

where[4] $\alpha = \langle w_1, r_0(i)\rangle$.

We now compute $\mathcal{C}_v(X_j, \cdot) = (pr([X_j, x^{-1}v]), pr(x^{-1}v))$. We have

(1) $h_3 = \frac{1}{2}\left(\begin{smallmatrix} -1 & i \\ i & 1 \end{smallmatrix}\right)$, which under C goes to $-ie \otimes (1, 0, 0, 0)$.
(2) $h_1(X) = \frac{1}{2}(ie + f) \otimes V(X)$, which goes to $-ie \otimes (0, X, 0, 0)$
(3) $h_{-1}(Z) = \frac{i}{2}(n_L(Z) + n_L^\vee(Z)) + \frac{1}{2}M(\Phi_{1,Z})$, which goes to $-ie \otimes (0, 0, Z, 0)$
(4) $h_{-3} = \frac{1}{4}(ie - f) \otimes r_0(i)$, which goes to $-ie \otimes (0, 0, 0, 1)$.

Denote by $pr_{\mathfrak{p}}$ the projection from $\mathfrak{g}(J) \otimes \mathbf{R} \to \mathfrak{p}$. Let's suppose $C^{-1}(pr_{\mathfrak{p}}(x^{-1}v)) = e \otimes w_e + f \otimes w_f$. Write also $X_j = C(e \otimes w_j)$ with $w_j = (0, b_j, c_j, 0)$. Now

$$\begin{aligned} \mathcal{C}_v(X_j, \cdot) &= ([C^{-1}X_j, C^{-1}x^{-1}v], C^{-1}(pr(x^{-1}v))) \\ &= \frac{1}{4}([e \otimes w_j, e \otimes w_e + f \otimes w_f], \alpha^* E - iz_1 H - \alpha F) \\ &= \frac{1}{4}(\langle w_j, w_e\rangle E - \frac{1}{2}\langle w_j, w_f\rangle H, \alpha^* E - iz_1 H - \alpha F) \\ &= -\frac{\alpha}{4}\langle w_j, w_e\rangle + i\frac{z_1}{4}\langle w_j, w_f\rangle \end{aligned}$$

Here $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $H = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$.

We now compute

$$\begin{aligned} \langle w_j, w_f\rangle &= (-e \otimes w_j, f \otimes w_f) \\ &= (-e \otimes w_j, C^{-1}(pr_{\mathfrak{p}}(x^{-1}v))) \\ &= (-C(e \otimes w_j), x^{-1}v = e \otimes w_1 + z_1 E) \\ &= (-ih_1(b_j) - ih_{-1}(c_j), e \otimes w_1 + z_1 E) \\ &= -i(h_1(b_j), e \otimes w_1) \\ &= \frac{1}{2}((e - if) \otimes V(b_j), e \otimes w_1) \\ &= -\frac{i}{2}\langle V(b_j), w_1\rangle. \end{aligned}$$

[4]This is slightly different than how α is defined in [Pol19].

Similarly we compute

$$\begin{aligned}\langle w_j, w_e\rangle = (f\otimes w_j, e\otimes w_e)\\ &= (f\otimes w_j, C^{-1}(pr_{\mathfrak{p}}(x^{-1}v)))\\ &= (C(f\otimes w_j), x^{-1}v = e\otimes w_1 + z_1 E)\\ &= (i\overline{h_{-1}}(b_j) - i\overline{h_1}(c_j), e\otimes w_1 + z_1 E)\\ &= (-i\overline{h_1}(c_j), e\otimes w_1)\\ &= -i(\frac{1}{2}(-ie+f)\otimes V^*(c_j), e\otimes w_1)\\ &= -\frac{i}{2}\langle V^*(c_j), w_1\rangle.\end{aligned}$$

We therefore obtain

$$\mathcal{C}_v(X_j,\cdot) = \frac{i\alpha}{8}\langle V^*(c_j), w_1\rangle + \frac{z_1}{8}\langle V(b_j), w_1\rangle.$$

We now compute $\mathcal{C}_v(X_i,X_j)$. We assume $X_i = C(e\otimes w_i = e\otimes(0,b_i,c_i,0))$, $X_j = C(e\otimes w_j = e\otimes(0,b_j,c_j,0))$. Then for $k=i,j$, we have

$$\begin{aligned}pr([X_k, x^{-1}v]) &= pr([C(e\otimes w_k), pr_{\mathfrak{p}}(x^{-1}v)])\\ &= pr([C(e\otimes w_k), C(e\otimes w_e + f\otimes w_f)])\\ &= (pr\circ C)([e\otimes w_k, e\otimes w_e + f\otimes w_f])\\ &= (pr\circ C)(\langle w_k, w_e\rangle E - \frac{\langle w_k, w_f\rangle}{2}H)\\ &= \langle w_k, w_e\rangle e_\ell - \frac{\langle w_k, w_f\rangle}{2}h_\ell.\end{aligned}$$

Thus

$$\mathcal{C}_v(X_i,X_j) = \frac{1}{2}\langle w_i, w_f\rangle\langle w_j, w_f\rangle = -\frac{1}{8}\langle V(b_i), w_1\rangle\langle V(b_j), w_1\rangle.$$

We summarize what we've proved in a proposition.

Proposition 7.3.1. *Suppose* $X_i = C(e\otimes w_i = e\otimes(0,b_i,c_i,0))$, $X_j = C(e\otimes w_j = e\otimes(0,b_j,c_j,0))$. *Then one has*

(1) $(e_\ell, x^{-1}v) = -\frac{\alpha}{4}$;
(2) $\mathcal{C}_v = -\frac{1}{8}(|\alpha|^2 + z_1^2)$;
(3) $\mathcal{C}_v(X_j,\cdot) = \frac{i\alpha}{8}\langle V^*(c_j), w_1\rangle + \frac{z_1}{8}\langle V(b_j), w_1\rangle$;
(4) $\mathcal{C}_v(X_i,X_j) = -\frac{1}{8}\langle V(b_i), w_1\rangle\langle V(b_j), w_1\rangle$.

Here $x = ng \in N_J(\mathbf{R})M_J(\mathbf{R})$, $\alpha = \langle w_1, r_0(i)\rangle$, $w_1 = g^{-1}w$ *and* $z_1 = \nu(g)^{-1}\langle w, \overline{n}\rangle$.

Rewriting the above, we have

(1) $-\mathcal{C}_v(X_1,\cdot) = -\frac{z_1}{8}\langle V(x), w_1\rangle$
(2) $-\mathcal{C}_v(X_2,\cdot) = -\frac{z_1}{8}\langle V(y), w_1\rangle$
(3) $-\mathcal{C}_v(X_3,\cdot) = i\frac{\alpha}{8}\langle V^*(x), w_1\rangle$
(4) $-\mathcal{C}_v(X_4,\cdot) = i\frac{\alpha}{8}\langle V^*(y), w_1\rangle$
(5) $-\mathcal{C}_v(X_1,X_1) = \frac{1}{8}\langle V(x), w_1\rangle\langle V(x), w_1\rangle$
(6) $-\mathcal{C}_v(X_1,X_2) = \frac{1}{8}\langle V(x), w_1\rangle\langle V(y), w_1\rangle$
(7) $-\mathcal{C}_v(X_2,X_2) = \frac{1}{8}\langle V(y), w_1\rangle\langle V(y), w_1\rangle$

7.4. **More calculations.** Putting together the work we have done above, we obtain the following lemma. Define $\mathcal{C}_v{}^* = -\mathcal{C}_v$, $\mathcal{C}_v(X_j,\cdot)^* = -\mathcal{C}_v(X_j,\cdot)$, $\mathcal{C}_v(X_i,X_j)^* = -\mathcal{C}_v(X_i,X_j)$ and $\mathcal{C}^*_{v,n,k}(X_1\cdots X_n)$ to be the quantity one obtains by replacing every $\mathcal{C}_v(X_j,\cdot)$ with a $\mathcal{C}_v(X_j,\cdot)^*$ and every $\mathcal{C}_v(X_i,X_j)$ with a $\mathcal{C}_v(X_i,X_j)^*$.

Lemma 7.4.1. *Suppose $n=2m$. Consider*

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}\mathcal{C}^*_{v,2m,k}((X_1X_4)^{m-r}(X_2X_3)^r).$$

This quantity is

$$\left(\frac{1}{8}\right)^k\left(\frac{i\alpha}{8}\right)^m\left(\frac{-z_1}{8}\right)^{m-2k}\binom{m}{2k}\frac{(2k)!}{k!2^k}(\langle V(x),w_1\rangle\langle V^*(y),w_1\rangle-\langle V(y),w_1\rangle\langle V^*(x),w_1\rangle)^m.$$

(Interpret this formula to mean that it is 0 if $2k>m$.)

Proof. We evaluate $\mathcal{C}^*_{v,2m,k}((X_1X_4)^{m-r}(X_2X_3)^r)$. The point is that every nonzero term that goes into the definition of this sum is the same. One obtains that any such term is equal to

$$\left(\frac{1}{8}\right)^k\left(\frac{i\alpha}{8}\right)^m\left(\frac{-z_1}{8}\right)^{m-2k}(\langle V(x),w_1\rangle\langle V^*(y),w_1\rangle)^{m-r}(\langle V(y),w_1\rangle\langle V^*(x),w_1\rangle)^r.$$

There are

$$\frac{1}{k!}\binom{m}{2}\binom{m-2}{2}\binom{m-4}{2}\cdots\binom{m-2k+2}{2}=\binom{m}{2k}\frac{(2k)!}{k!2^k}$$

such nonzero terms. Thus

$$\begin{aligned}\mathcal{C}^*_{v,2m,k}((X_1X_4)^{m-r}(X_2X_3)^r)&=\binom{m}{2k}\frac{(2k)!}{k!2^k}\left(\frac{1}{8}\right)^k\left(\frac{i\alpha}{8}\right)^m\left(\frac{-z_1}{8}\right)^{m-2k}\\&\quad\times(\langle V(x),w_1\rangle\langle V^*(y),w_1\rangle)^{m-r}(\langle V(y),w_1\rangle\langle V^*(x),w_1\rangle)^r\end{aligned}$$

and the lemma follows. □

Putting everything together, we obtain the following proposition. Set

$$b^*_{\ell,v}(g)=(-(pr(g^{-1}v),pr(g^{-1}v)))^{-(\ell+\frac{1}{2})}.$$

Proposition 7.4.2. *One has*

$$\begin{aligned}(X_1X_4-X_2X_3)^m b^*_{\ell,v}(g)&=8^{\ell+\frac{1}{2}}(-i\alpha)^m(\langle V(x),w_1\rangle\langle V^*(y),w_1\rangle-\langle V(y),w_1\rangle\langle V^*(x),w_1\rangle)^m\\&\times\left(\sum_{k=0}^{\lfloor m/2\rfloor}(-1)^k2^{2m-k}z_1^{m-2k}\binom{m}{2k}\frac{(2k)!}{k!2^k}(\ell+\frac{1}{2})_{2m-k}(|\alpha|^2+z_1^2)^{-(\ell+2m-k+\frac{1}{2})}\right).\end{aligned}$$

For a positive real number β set

$$I_{\ell,m}(\beta)=\int_{\mathbf{R}}e^{-iz}\left(\sum_{k=0}^{\lfloor m/2\rfloor}\frac{C_{m,k}z^{m-2k}(\ell+\frac{1}{2})_{2m-k}}{(\beta^2+z^2)^{\ell+2m-k+\frac{1}{2}}}\right)dz$$

where

$$C_{m,k}=(-1)^k2^{2m-k}\binom{m}{2k}\frac{(2k)!}{k!2^k}.$$

Set $\alpha^*_{\ell,v}(g)=-(e_\ell,g^{-1}v)b^*_{\ell,v}(g)$. We have proved:

Proposition 7.4.3. *Set $z = \nu(g)z_1$. One has*

$$\int_{\mathbf{R}=N_w\backslash N} e^{-i\langle w,\overline{n}\rangle}(X_1X_4 - X_2X_3)^m a^*_{\ell,v}(ng)\,dn = (-i)^m 2^{\ell+\frac{3}{2}}(\nu(g)\alpha)^{\ell+m}|\nu(g)|\nu(g)^{\ell+2m} I_{\ell,m}(|\nu(g)\alpha|) \times (\langle V(x),w_1\rangle\langle V^*(y),w_1\rangle - \langle V(y),w_1\rangle\langle V^*(x),w_1\rangle)^m.$$

The following proposition now implies Theorem 7.1.1.

Proposition 7.4.4. *One has*

$$I_{\ell,m}(\beta) = \frac{2^{1-\ell}i^m}{(\frac{1}{2})_\ell}\beta^{-(\ell+m)}K_{\ell+m}(\beta).$$

7.5. **Evaluation of an integral.** In this subsection, we prove Proposition 7.4.4.

For $m = 0$, from equation (5) of [Wei], one obtains

$$I_{\ell,0}(\beta) = \frac{2}{(2\ell-1)(2\ell-3)\cdots(3)(1)}\beta^{-\ell}K_\ell(\beta).$$

Set, for $r > 0$, $v \in \mathbf{Z}_{>0}$,

$$I_{v,0}(r,\beta) = \int_{\mathbf{R}} e^{-irz}(\beta^2+z^2)^{-(v+\frac{1}{2})}\,dz.$$

Changing variables, one obtains

$$I_{v,0}(r,\beta) = D_v(r/\beta)^v K_v(r\beta).$$

where $D_v = 2^{-v+1}\frac{1}{(\frac{1}{2})_v}$. Differentiating under the integral sign, and using that

$$C_{m,k}(\ell+\frac{1}{2})_{2m-k}(-i)^{m-2k}D_{\ell+2m-k} = \frac{2^{1-\ell}(-i)^m}{(\frac{1}{2})_\ell},$$

we get

$$I_{\ell,m}(\beta) = \frac{2^{1-\ell}(-i)^m}{(\frac{1}{2})_\ell}\left(\sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}\frac{(2k)!}{k!2^k}\partial_r^{m-2k}((r/\beta)^{\ell+2m-k}K_{\ell+2m-k}(r\beta))|_{r=1}\right).$$

Finally, making the variable change $u = r\beta$, one gets

$$I_{\ell,m}(\beta) = \frac{2^{1-\ell}(-i)^m}{(\frac{1}{2})_\ell}\left(\sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}\frac{(2k)!}{k!2^k}\beta^{-2\ell-3m}\partial_u^{m-2k}(u^{\ell+2m-k}K_{\ell+2m-k}(u))|_{u=\beta}\right).$$

Set

$$c_n^j = \frac{n!}{j!(n-2j)!2^j} = \binom{n}{2j}\frac{(2j)!}{j!2^j}.$$

Lemma 7.5.1. *Suppose $b \geq n$. One has*

$$\partial_u^n(u^bK_b(u)) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{n-j}c_n^j u^{b-j}K_{b-n+j}(u).$$

Proof. The proof is by induction, using the recurrence $c_{n+1}^j = c_n^j + (n-2j+2)c_n^{j-1}$. See also [oIS]. ☐

From the immediately verified identity

$$\sum_{0\leq j+k=\text{ fixed}\leq\lfloor m/2\rfloor}(-1)^j c_m^k c_{m-2k}^j = 0$$

if $j + k > 0$, we obtain Proposition 7.4.4.

7.6. **Technical justification.** We still must justify our differentiation under the integral. In other words, we must justify the identity

$$\int_{N_w\backslash N\simeq \mathbf{R}} Z_1\cdots Z_r\left(e^{-i\langle w,\overline{n}\rangle}(a_{\ell,v}(ng)\right)\,dn = Z_1\cdots Z_r\left(\int_{N_w\backslash N\simeq\mathbf{R}} e^{-i\langle w,\overline{n}\rangle}a_{\ell,v}(ng)\,dn\right).$$

Here the Z_i are in $\mathfrak{p}$ and differentiate with respect to the g variable.

Write $n = n(z)$ if $n \mapsto z$ under the identification $N_w\backslash N \simeq \mathbf{R}$. To do this justification, it suffices to show (for all non-negative integers r) that there is a small neighborhood U of g, and a positive function $A_r(z)$ on $\mathbf{R}$, so that for all $x \in U$,

$$|Z_1\cdots Z_r a_{\ell,v}(n(z)x)| \leq A_r(z) \text{ and } \int_{\mathbf{R}} A_r(z)\,dz < \infty.$$

The function $A_r(z)$ can depend upon g and $Z_1,\ldots,Z_r$, but we drop them from the notation.

It is easy to see (e.g., by induction) that the derivative $Z_1\cdots Z_r(a_{\ell,v}(nx))$ is of the form

$$\frac{(e_\ell, x^{-1}n^{-1}v)^\ell}{(pr(x^{-1}n^{-1}v),pr(x^{-1}n^{-1}v))^{\ell+r+\frac{1}{2}}}P_{Z_1,\ldots,Z_r}(x,n)$$

where $P_{Z_1,\ldots,Z_r}(x,n)$ consists of sums of products of terms of the form $(pr([Z_j,x^{-1}n^{-1}v],pr(x^{-1}n^{-1}v)),$ $(pr([Z_j,x^{-1}n^{-1}v],pr([Z_k,x^{-1}n^{-1}v]))$ etc. The key point is that, if $n = n(z)$, then when written as a polynomial in z, $P_{Z_1,\ldots,Z_r}(n,z)$ has degree at most $2r$. The coefficients of this polynomial depend on x and $Z_1,\ldots,Z_r$, but are easily seen to be bounded for x in a small compact set around g.

To finish the proof, we now must bound

$$\left|\frac{(e_\ell, x^{-1}n^{-1}v)^\ell}{(pr(x^{-1}n^{-1}v),pr(x^{-1}n^{-1}v))^{\ell+r+\frac{1}{2}}}\right| \leq ||pr(x^{-1}n^{-1}v)||^{-(\ell+2r+1)}.$$

Here $||v|| = |B(v,\theta(v))|^{1/2}$ is the K-equivariant norm on $\mathfrak{g}(J)\otimes\mathbf{R}$, where θ is the Cartan involution. Write $g = n_g m_g k_g$ in the Iwasawa decomposition. We take small open neighborhoods around n_g, m_g and k_g, and let U be the product of these neighborhoods. Then, if $n = n(z)$, $||pr(x^{-1}n^{-1}v)||$ is bounded away from 0 for z small, independent of x, and is bounded below by $(|z|^2+|\alpha_0|^2)^{1/2}$ for z large, with α_0 independent of $x \in U$. The existence of $A_r(z)$ with the desired properties follows.

8. Arithmeticity of modular forms on G_2

The purpose of this section is to prove Theorem 1.0.1, which we recall here. Suppose φ is a cuspidal quaternionic modular form on G_2 of weight $\ell \geq 1$. Then

$$\varphi_Z(g_f g_\infty) = \sum_\chi a_\chi(g_f)W_\chi(g_\infty)$$

is its Fourier expansion. The locally constant functions $a_\chi : G_2(\mathbf{A}_f) \to \mathbf{C}$ are its Fourier coefficients. We say that φ has Fourier coefficients in a ring R if $a_\chi(g_f) \in R$ for all characters χ of $N(\mathbf{Q})\backslash N(\mathbf{A})$ and all $g_f \in G_2(\mathbf{A}_f)$. We write $S_\ell(G_2;R)$ for the space cuspidal quaternionic modular forms on G_2 of weight ℓ with Fourier coefficients in R.

Let $\mathbf{Q}_{cyc} = \mathbf{Q}(\mu_\infty)$ be the cyclotomic extension of $\mathbf{Q}$.

Theorem 8.0.1. *Suppose $\ell \geq 6$ is even. Then there is a basis of the cuspidal quaternionic modular forms of weight ℓ with all Fourier coefficients in $\mathbf{Q}_{cyc}$. In other words, $S_\ell(G_2,\mathbf{C}) = S_\ell(G_2,\mathbf{Q}_{cyc})\otimes_{\mathbf{Q}_{cyc}}\mathbf{C}$.*

The proof of this theorem has the following steps:

(1) Set $S_\ell(G_2)_\Theta$ the subspace of $S_\ell(G_2,\mathbf{C})$ consisting of theta lifts from algebraic modular forms on F_4^I. It is clear that it is a $G_2(\mathbf{A}_f)$ submodule. Moreover, as an application of Theorem 5.0.1, it is easy to see that $S_\ell(G_2)_\Theta$ is defined over $\mathbf{Q}_{cyc}$, i.e., that it has a basis consisting of elements with Fourier coefficients in $\mathbf{Q}_{cyc}$.
(2) To show that every element of $S_\ell(G_2;\mathbf{C})$ is a theta lift, one uses the Siegel-Weil theorem of [Pol23] and the Rankin-Selberg integral of [GS15, Seg17].
(3) For the previous step to go through, a certain archimedean Zeta integral must be shown to be non-vanishing. One shows the non-vanishing of this integral by a global method, using Corollary 1.2.3.

We will break the proof into various lemmas.

Lemma 8.0.2. *Suppose $\phi \in V_{min}$ is $\phi = \sum_j \mu_j g_j \cdot \phi_0$ with $\mu_j \in \mathbf{Q}$ and ϕ_0 the spherical vector. Let $\mathcal{A}(F_4^I, U; V_{m\lambda_3})$ be the algebraic modular forms on F_4^I of level $U \subseteq F_4^I(\mathbf{A}_f)$ and for the representation $V_{m\lambda_3}$. Then there is a lattice $\Lambda \subseteq \mathcal{A}(F_4^I, U; V_{m\lambda_3})$ so that if $\alpha \in \Lambda$, then $\Theta_\phi(\alpha) \in S_{\ell+m}(G_2;\mathbf{C})$ has Fourier coefficients in $\mathbf{Q}_{cyc}$.*

Proof. Write

$$\varphi(g) = \int_{[F_4^I]} \{D_{\mathfrak{p}}^{2m}\Theta_\phi(g,h), \alpha(h)\}\, dh. \tag{4}$$

Now

$$F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{R})F_4^I(\mathbf{A}_f)/U = \cup_{j=1}^N \Gamma_j\backslash F_4^I(\mathbf{R})\gamma_j$$

where $\gamma_j \in F_4^I(\mathbf{A}_f)$. In other words,

$$F_4^I(\mathbf{Q})\backslash F_4^I(\mathbf{A}) = \cup_{j=1}^N \Gamma_j\backslash F_4^I(\mathbf{R})\gamma_j U.$$

Thus the integral over $[F_4^I]$ is a finite sum of terms $c_j\{D_{\mathfrak{p}}^{2m}\Theta_\phi(g,1), \alpha(\gamma_j)\}$ where $c_j = \frac{meas(U)}{|\Gamma_j|}$ is rational. But now Theorem 5.0.1 implies that if the Fourier coefficients of $\Theta_\phi(g)$ are in some ring R, and $\alpha(\gamma_j)$ is in an appropriate lattice, then the Fourier coefficients of $\{D_{\mathfrak{p}}^{2m}\Theta_\phi(g,1), \alpha(\gamma_j)\}$ are in R. Thus we obtain the fact that the Fourier coefficients of theta lifts can all be made in $\mathbf{Q}_{cyc} = \mathbf{Q}(\mu_\infty)$, as soon as we prove the same result for the Fourier coefficients of $\Theta_\phi(g)$.

For the latter, simply observe the following identities: suppose $g_f \in G_J(\mathbf{A}_f)$. Then we can write $g_f = u_f m_f k_f$ with u_f in the N_J, $k_f \in G_J(\widehat{\mathbf{Z}}) = K_f$, and $m_f \in H_J(\mathbf{A}_f)$. Then we can further write $m_f = (m_{\mathbf{Q}} m_{\mathbf{R}}^{-1})k'$ with $m_{\mathbf{Q}} \in H_J(\mathbf{Q})$ and $k' \in K_f$. This follows from strong approximation on the simply connected group H_J^1. Thus, if $a(\omega)(g_f)$ denotes a Fourier coefficient of $\Theta_{Gan}(g)$ (level one), then

$$a(\omega)(g_f) = \psi(\langle \omega, u_f\rangle) a(\omega)(m_{\mathbf{Q}} m_{\mathbf{R}}^{-1}).$$

Additionally,

$$a(\omega)(m_{\mathbf{Q}} m_{\mathbf{R}}^{-1}) W_\omega(g_\infty) = \Theta_\omega(m_{\mathbf{Q}} m_{\mathbf{R}}^{-1} g_\infty) = \Theta_{\omega\cdot m_{\mathbf{Q}}}(m_{\mathbf{R}}^{-1} g_\infty) = a_{\omega\cdot m_{\mathbf{Q}}}(1) W_{\omega\cdot m_{\mathbf{Q}}}(m_{\mathbf{R}}^{-1} g_\infty).$$

This last term is $\det(m_{\mathbf{R}})^{-\ell}|\det(m_{\mathbf{R}})|^{-1} a_{\omega\cdot m_{\mathbf{Q}}}(1) W_\omega(g_\infty)$. Thus since all the $a_{\omega'}(1)$ are integral, all $a(\omega)(g_f) \in \mathbf{Q}_{cyc}$. It thus follows that if $\phi = \sum_j \mu_j g_j \cdot \phi_0$ with $\mu_j \in \mathbf{Q}$, then all the Fourier coefficients of Θ_ϕ are in $\mathbf{Q}_{cyc}$. This completes the argument. □

Write $S_\ell(G_2)_\Theta$ for the space of all lifts φ as in equation (4).

Lemma 8.0.3. *The subspace $S_\ell(G_2)_\Theta$ is a $G_2(\mathbf{A}_f)$-submodule.*

Proof. This is clear. □

Recall the projection $\mathfrak{p}^{\otimes m} \to S^{2m}(V_\ell) \otimes (\wedge^2 J^0)^{\otimes m}$. Let $W \subseteq (\wedge^2 J^0)^{\otimes m}$ be the set of all $w \in (\wedge^2 J^0)^{\otimes m}$ for which $\{w, v\} = 0$ for all $v \in V_{m\lambda_3}$. We set $V_m^* = (\wedge^2 J^0)^{\otimes m}/W$, and let $P : \mathfrak{p}^{\otimes m} \to S^{2m}(V_\ell) \otimes V_m^*$ be the composite projection.

To prove that $S_\ell(G_2)_\Theta = S_\ell(G_2;\mathbf{C})$ it suffices to show that if $\varphi \in S_\ell(G_2;\mathbf{C})$ generates an irreducible representation π, then

$$\int_{[G_2]} \{\varphi(g), P(D_{\mathfrak{p}}^{2m}\Theta_\phi(g,h))\}\, dg \neq 0 \tag{5}$$

for some $\varphi \in V_{min}$. Indeed, in this case, the submodule $S_\ell(G_2)_\Theta$ has orthocomplement equal to 0. Moreover, we can assume that φ is a pure tensor in π.

Suppose E is a totally real cubic étale extension of $\mathbf{Q}$, and let S_E be the group of type Spin_8 defined in terms of E that has $S_E(\mathbf{R})$ compact. See [Pol23] for a precise definition. To prove (5), it then further suffices to show that

$$\int_{[G_2]\times[S_E]} \{\varphi(g), P(D_{\mathfrak{p}}^{2m}\Theta_\phi(g,h))\}\, dg \neq 0 \tag{6}$$

for some such E.

The integral in (6) can be evaluated using the main theorem of [Pol23] and the Rankin-Selberg integral studied in [GS15, Seg17] (see also [Pol19]). To set up the result, following [Pol23], write G_E for a certain simply connected group of absolute Dynkin type D_4, defined in terms of E and split over $\mathbf{R}$.

We have

$$D_{\mathfrak{p}}^{2m} \otimes \Theta(g) = D_{\mathfrak{p}}^{2m}\left(\sum_v \Theta_{v_j}(g) \otimes [x^{4+j}][y^{4-j}]\right) = \sum_{\alpha,j} \Theta_{u_\alpha v_j}(g) \otimes v_j^\vee \otimes u_\alpha^\vee$$

for elements $u_\alpha \in U(\mathfrak{g}(J)\otimes \mathbf{C})$. Thus by Corollary 9.4.8 of [Pol23],

$$\int_{[G_2]\times[S_E]} \{\varphi(g), P(D_{\mathfrak{p}}^{2m}\Theta_\phi(g,h))\}\, dg\, dh = \int_{[G_2]} \{\varphi(g), \sum_{\alpha,j} E_1(\phi, u_\alpha v_j)(g, s=5) \otimes P(v_j^\vee \otimes u_\alpha^\vee)\}\, dg. \tag{7}$$

Here $E_1(\phi, u_\alpha v_j, s=5)$ is the Siegel-Weil Eisenstein series on the group G_E.

The integral of (7) can now be written as a partial L-function times some local Zeta integrals at bad finite places (including the archimedean place). Specifically, we have the following proposition. Moreover, these local zeta integrals at the finite places can be trivialized with Siegel-Weil inducing data for the Eisenstein series E_1 on G_E. Specifically, we have the following proposition.

Suppose $\chi : N(\mathbf{Q})\backslash N(\mathbf{A}) \to \mathbf{C}^\times$ is a unitary character. To setup the proposition, we define

$$I_{\infty,\chi}(s) = \int_{N_{0,\chi}\backslash G_2(\mathbf{R})} \{W_\chi(g), \sum_{\alpha,j} f_{u_\alpha v_j}(\gamma_{0,\chi} g, s) \otimes P(v_j^\vee \otimes u_\alpha^\vee)\}\, dg.$$

This is the local archimedean Zeta integral that comes from the Rankin-Selberg integral (7). The notation is from [Pol19, Theorem 5.2], which is a restatement of a Theorem of [GS15, Seg17]. Here also $f_{u_\alpha v_j}(g,s)$ is the Siegel-Weil inducing section from [Pol23]. It follows from Proposition 8.0.8 below that $I_{\infty,\chi}(s)$ converges absolutely for $Re(s) > 1$.

Proposition 8.0.4. *Let χ be a unitary character of $N(\mathbf{Q})\backslash N(\mathbf{A})$ for which $a_\chi(\varphi)(1) \neq 0$. Recall that from the theory of binary cubic forms one associates to φ is a rank three $\mathbf{Z}$-module R_χ in a cubic étale algebra E over $\mathbf{Q}$. Suppose S is a set of finite places of $\mathbf{Q}$ that satisfies the following conditions:*

(1) If $p \notin S$, then π_p is unramified and φ is spherical at p;
(2) If $p \notin S$, then $R_\chi \otimes \mathbf{Z}_p$ is a ring, and in fact the maximal order of $E \otimes \mathbf{Z}_p$;
(3) $S \supseteq \{2,3\}$.

Then the finite vector ϕ can be chosen so that ϕ is spherical outside S, and the integral (7) is equal to $L^S(\pi, Std, s=3) I_{\infty,E}(s=3)$. Moreover, if φ is unramified at all finite primes, and $R_\chi = \mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ in $\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}$, then S may be chosen to be empty.

Observe that if ϕ is level one and has $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ Fourier coefficient nonzero, then we may take S to be empty.

Proof. The fact that the global integral represents the partial L-function is from [GS15, Seg17], for a slightly larger set S. In [Pol19] the set S is shrunk to that in the statement of the proposition, except that [Pol19] includes $2,3\in S$ in all cases. Then, in [cDD+22], the case where φ is level one and R_χ is $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ is handled.

That the bad local integrals may be trivialized with some data is in [Seg17, Section 7]. What we state and use is slightly stronger. Specifically, we must verify that the bad local integrals can be trivialized for Siegel-Weil inducing sections. This follows simply because the Siegel-Weil inducing sections make up all of the induced representation $Ind_{P_E(\mathbf{Q}_p)}^{G_E(\mathbf{Q}_p)}(\delta_{P_E})$, where P_E is the Heisenberg parabolic of G_E. To see this, recall that $Ind_{P_E(\mathbf{Q}_p)}^{G_E(\mathbf{Q}_p)}(\delta_{P_E})$ is generated by any vector which is not annihilated by the long intertwining operator, which turns out to be given by an absolutely convergent integral. The restriction of the spherical vector from $G_J(\mathbf{Q}_p)$ is positively valued on $G_E(\mathbf{Q}_p)$, so it cannot be annihilated by the long intertwining operator.

Finally, what we have stated is that ϕ may be chosen to trivialize the integral, without changing the cusp form φ. This is slightly stronger from what is stated in [Seg17]. This claim follows from Lemma 8.0.7 below. □

It turns out the L-value $L^S(\pi, Std, s=3)$ is always nonzero. This is a direct consequence of the main theorem of [Mui97].

Theorem 8.0.5. *Let π be a cuspidal automorphic representation of G_2 over $\mathbf{Q}$. Then the Euler product defining the partial standard L-function $L^S(\pi, Std, s)$ converges absolutely for $Re(s)>2$.*

Proof. The local factors π_p are unitarizable. In [Mui97], the unitary dual of p-adic G_2 is completely and explicitly determined. In particular, when π_p is spherical, one has tight bounds on the Satake parameters of π_p. These bounds imply the absolute convergence statement of the theorem. □

Finally, it turns out that if $\ell\geq 6$ is even, the archimedean Zeta integral $I_{\infty,\chi}(s=3)$ is nonzero for all non-degenerate χ.

Proposition 8.0.6. *Suppose $\ell\geq 6$ is even. Then $I_{\infty,\chi}(s=3)$ is nonzero for all non-degenerate χ.*

Proof. A change of variables in the integral $I_{\infty,\chi}(s)$ shows that the non-vanishing of $I_{\infty,\chi}(s=3)$ is equivalent for all χ. So, we take χ with $R_\chi=\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$. We will prove that this integral is nonvanishing by a global argument, using Proposition 8.0.4, Corollary 1.2.3, and Theorem 8.0.5. Fix now $E=\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}$.

Let $\beta_{K,m}$ be as in the proof of Corollary 1.2.3. Set $\beta_0=\int_{S_E(\mathbf{R})}k\cdot\beta_{K,m}\,dk$. Then one sees that $\Theta(k\cdot\beta_{K,m})$ has $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ Fourier coefficient equal to 6, so $\Theta(\beta_0)$ does as well.

We can write $\Theta(\beta_0)$ as a finite sum of level one cuspidal eigenforms forms φ_j, with $\langle\varphi_j,\Theta(\beta_0)\rangle\neq 0$. Thus there is some such φ with $\mathbf{Z}\times\mathbf{Z}\times\mathbf{Z}$ Fourier coefficient nonzero; fix this φ.

Now
$$\int_{[G_2]\times[S_E]}\{\varphi(g),P(D_{\mathfrak{p}}^{2m}\Theta(g,h))\}\,dg\,dh=L(\pi_\varphi,Std,s=3)I_{\infty,\chi}(s=3)$$
from Proposition 8.0.4.

We have $\{\varphi,\Theta(\beta_0)\}\neq 0$. So
$$\int_{[G_2]}\{\varphi(g)\otimes\beta_0,P(D_{\mathfrak{p}}^{2m}\Theta(g,1))\}\,dg\neq 0.$$
Let $w(h)=\int_{[G_2]}\{\varphi(g),P(D_{\mathfrak{p}}^{2m}\Theta(g,h))\}\,dg$. Then
$$\int_{[S_E]}w(h)\,dh=|\Gamma_{S_E}|^{-1}\int_{S_E(\mathbf{R})}w(k)\,dk$$

where Γ_{S_E} is some finite group. Here we are using that $S_E(\mathbf{A}) = S_E(\mathbf{Q})S_E(\mathbf{R})S_E(\widehat{\mathbf{Z}})$.

But this latter integral is nonzero, because it is nonzero after pairing with β_0. Consequently, we have deduced that $I_{\infty,\chi}(s=3)$ is nonzero. ◻

Proof of Theorem 8.0.1. We have proved that the space of lifts $S_\ell(G_2)_\Theta$ has a $\mathbf{Q}_{cyc}$ structure, from the Fourier coefficients. We have also prove that if $\ell \geq 6$ is even, then $S_\ell(G_2)_\Theta = S_\ell(G_2;\mathbf{C})$. This proves the theorem. ◻

We end with some of the technical details that were used in the proofs above.

Lemma 8.0.7. *Let V_p denote the space of the representation π_p, and suppose $L : V_p \to \mathbf{C}$ is an (N,χ) functional. Given $v \in V_p$, there is a Schwartz-Bruhat function Φ on $\mathfrak{g}_E \otimes \mathbf{Q}_p$ so that*

$$I_p(\Phi, v, s) = \int_{N_{0,E}\backslash G_2(\mathbf{Q}_p)} L(gv)f(\gamma_{0,E}g,\Phi,s)\,dg$$

is equal to $L(v)$, independent of s.

Proof. Write $\gamma_{0,E}^{-1}E_{13} = e\otimes\omega$ in the notation of [Pol19]. We have

$$I_p(\Phi, v, s) = \int_{\mathrm{GL}_1\times N_{0,E}\backslash G_2} |t|^s\Phi(tg^{-1}e\otimes\omega)L(gv)\,dg\,dt.$$

The function Φ is on $\mathfrak{g}_E$, and we have $\mathfrak{g}_E = \mathfrak{g}_2 \oplus E^0 \otimes V_7$. In this decomposition, we can write $e\otimes\omega = (e\otimes\omega') + (\alpha_0 e_1 + \alpha_1 e_3^*)$, where α_0, α_1 are a basis of E^0, the trace 0 elements of E.

We take Φ a pure tensor, $\Phi = \Phi_{\mathfrak{g}_2}\otimes\Phi_{E\otimes V_7}$. We make $\Phi_{E^0\otimes V_7}$ be the characteristic function of a set very close to $\alpha_0 e_1 + \alpha_1 e_3^*$. Let Z_{GL_2} be the center of the GL_2 Levi of the Heisenberg parabolic on G_2. Then $\Phi_{E^0\otimes V_7}(tg^{-1}) \neq 0$ implies $g \in Z_{\mathrm{GL}_2}(t)N_{Heis}K_{G_2}(p^M)$ for some $M >> 0$. Here $K_{G_2}(p^M)$ is the elements of $G_2(\mathbf{Q}_p)$ that are 1 modulo p^M in the 7×7 matrix representation of G_2. Here we are using that if $h \in G_2$, with $h^{-1}e_1 = e_1 + \delta_1$ and $h^{-1}e_3^* = e_3^* + \delta_2$, with $\delta_j \in p^M V_7(\mathbf{Z}_p)$, then there is $k \in K_{G_2}(p^M)$ so that $(hk)^{-1}e_1 = e_1$ and $(hk)^{-1}e_3^* = e_3^*$. (Indeed, this latter fact can be proved by using $K_{\mathrm{GL}_2}(p^M)$ and also unipotent elements in $N(p^M\mathbf{Z}_p)$.)

Thus we must evaluate

$$\int_{\mathrm{GL}_1\times G_a} L(z(t)v)\Phi_{\mathfrak{g}_2}(t(e\otimes\omega' + zE_{13}))|t|^s\psi(z)\,dt\,dz.$$

We choose $\Phi_{\mathfrak{g}_2}$ to be a pure tensor in our root basis of $\mathfrak{g}_2$, so that $\Phi_{\mathfrak{g}_2}(t(e\otimes\omega' + zE_{13})) = \Phi'_{\mathfrak{g}_2}(t(e\otimes\omega'))\Phi_{E_{13}}(tz)$. But $\int_{G_a}\Phi_{E_{13}}(tz)\psi(z)\,dz = \widehat{\Phi_{E_{13}}}(t^{-1})|t|^{-1}$. By choosing $\Phi_{E_{13}}$ so that its Fourier transform is supported near $t = 1$, we see that we can trivialize the integral to a constant multiple of $L(v)$. This proves the lemma. ◻

We now prove the absolute convergence of the archimedean Zeta integral. The integral in question is

$$\int_{N_{0,E}\backslash G_2(\mathbf{R})} \{W_\chi(g), f(\gamma_0 g, s)\}\,dg. \tag{8}$$

Proposition 8.0.8. *The integral (8) converges absolutely for $Re(s) > 1$.*

Proof. Let Φ be a Sschwartz function on $\mathfrak{g}_{\mathbf{R}}$. We obtain an inducing section from Φ as

$$f(g,\Phi,s) = \int_{\mathrm{GL}_1(\mathbf{R})} |t|^s\Phi(tg^{-1}E_{13})\,dt.$$

We will check that every inducing section in $I(s)$ is of this form, and we will prove the proposition for these inducing sections.

For the first part, observe that if $f \in I(s)$, then restricting to K_∞ we obtain $f(k_1 k, s) = f(k,s)$ for all $k_1 \in K_\infty \cap M(\mathbf{R})$. These are the $k_1 \in K_\infty$ for which $k_1 E_{13} = \pm E_{13}$, and note that the negative sign does indeed occur. Now let β be an arbitrary even smooth function on $\mathfrak{g}_E \otimes \mathbf{R}$ and α a smooth compactly supported function on $\mathbf{R}_{>0}$. We set $\Phi(v) = \alpha(||v||)\beta\left(\frac{v}{||v||}\right)$; this is a Scwhartz function.

We have now

$$f(k,\Phi,s) = \int_{\mathbf{R}^\times} |t|^s \alpha(|t|)\beta(k^{-1}E_{13})\,dt = 2\beta(k^{-1}E_{13})\int_{\mathbf{R}_{>0}} t^s\alpha(t)\,dt.$$

We have used the evenness of β. But because $\beta(k^{-1}E_{13})$ gives an arbitrary function on $(K_\infty \cap M(\mathbf{R}))\backslash K_\infty$, we see that every inducing section in $I(s)$ is an $f(g,\Phi,s)$.

We thus now proceed to prove the absolute convergence of the double integral

$$\int_{\mathrm{GL}_1(\mathbf{R})}\int_{N_{0,E}\backslash G_2(\mathbf{R})} W_\chi(g)|t|^s\Phi(tg^{-1}e\otimes\omega)\,dg\,dt$$

when $Re(s) > 1$.

The bound we use below on $W_\chi(mk)$ is independent of k, so it suffices to integrate over $\mathrm{GL}_1 \times (\mathrm{G}_a \times \mathrm{GL}_2)$. Without loss of generality, we can assume that Φ is a pure tensor of the appropriate sort. We thus must bound the integral

$$\int_{\mathrm{GL}_1\times\mathrm{G}_a\times\mathrm{GL}_2} |t|^s\Phi_{13}(t\det(m)^{-1}z)\Phi_{\mathfrak{g}_2}(tm^{-1}\omega')|\det(m)|^{-3}\Phi_{M_2}(tm^{-1})W_\chi(m)\,dt\,dz\,dm.$$

Here Φ_{M_2} is a Schwartz function on the 2×2 matrices $M_2(\mathbf{R})$, and the rest of the notation is as in Lemma 8.0.7.

One has $\int_{\mathbf{R}} |\Phi_{13}(t\det(m)^{-1}z)|\,dz < C|det(m)||t|^{-1}$ and $\Phi_{\mathfrak{g}_2}$ is bounded above, so we must bound

$$\int_{\mathrm{GL}_1\times\mathrm{GL}_2} |t|^{s-1}|\det(m)|^{-2}\Phi_{M_2}(tm^{-1})W_\chi(m)\,dt\,dm.$$

Make the variable change $t \mapsto \det(m)t$, and set $m' = \det(m)m^{-1}$. Then we must bound

$$\int_{\mathrm{GL}_1\times\mathrm{GL}_2} |t|^{s-1}|\det(m)|^{s-3}\Phi_{M_2}(tm')W_\chi(m)\,dt\,dm.$$

Now we claim $\int_{\mathrm{GL}_1}|t|^{s-1}\Phi(tm')\,dt$ is, for $s>1$ (assumed real now) bounded by $C\frac{||m||^{s-1}}{s-1}$. Indeed, $\Phi(tm')$ is rapidly decreasing, so $|\Phi(tm')| < C\max\{1,|t|^{-N}||m||^{-N}\}$ for an N sufficiently large of our choosing. Thus

$$\int_{\mathrm{GL}_1}|t|^{s-1}|\Phi(tm')|\,dt < C\left(\int_0^{||m||}|t|^{s-1}\,\frac{dt}{t} + \int_{||m||}^\infty |t|^{s-1-N}||m||^{-N}\,\frac{dt}{t}\right).$$

Dropping terms of the form $\frac{1}{s-1}$ (since we are fixing s), both integrals above are bounded by $C||m||^{s-1}$. Thus we must bound

$$\int_{\mathrm{GL}_2}||m||^{s-1}|\det(m)|^{s-3}W_\chi(m)\,dm.$$

We break this integral into two pieces, one where $||m||\leq 1$ and the other with $||m||>1$. The first integral has a compact domain, so can be ignored. To show the convergence of the second integral, it suffices to show that $|W_\chi(m)| < \phi(||m||)$, where ϕ is a rapidly decreasing function. And since the K-Bessel function is rapidly decreasing, it suffices to show that $|\langle\omega', mr_0(i)\rangle| \geq C||m||$ for all $m \in \mathrm{GL}_2(\mathbf{R})$.

Both sides of the desired inequality scale linearly with the center of $\mathrm{GL}_2(\mathbf{R})$, so it suffices to check that

$$\frac{|\langle\omega', mr_0(i)\rangle|^2}{||m||^2}$$

is bounded away from 0 for $m \in \mathrm{SL}_2(\mathbf{R})$. Furthermore, by a change of variables again (now or in the initial integral), we can assume $\omega' = (0, 1, -1, 0)$. Then if

$$m = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} k,$$

we wish to bound below the quantity

$$y^{-3}|h(z)|((y^2 + x^2 + 1)/y)^{-1} = \frac{((x-1)^2 + y^2)(x^2 + y^2)}{(x^2 + y^2 + 1)y^2}$$

where $h(z) = z^2 - z = z(z-1)$.

Finally, to see that this rational function in x, y is bounded below for $y > 0$, we work in polar coordinates. We have

$$\frac{((x-1)^2 + y^2)(x^2 + y^2)}{(x^2 + y^2 + 1)y^2} = \left(1 - \frac{2x}{r^2+1}\right)\left(1 + \frac{x^2}{y^2}\right) = \left(1 + \frac{(x-1)^2}{y^2}\right)\left(1 - \frac{1}{r^2+1}\right).$$

If $|r| \geq 1/2$ then $r^2 \geq 1/4$ so $1 + r^2 \geq 5/4$ so $1/(1+r^2) \leq 4/5$ and then $1 - (1/(r^2+1)) \geq 1/5$, so the quantity is at least $1/5$. If $|r| \leq 1/2$, then

$$2x/(r^2+1) = 2r\cos(\theta)/(r^2+1) \leq 2r/(r^2+1) \leq 2(1/2)/((1/2)^2 + 1) = 4/5$$

because $f(r) = 2r/(r^2+1)$ is increasing on $(0, 1)$. This proves the claim, and thus the proposition. □

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