Mathematics 103B Practice problems
Exam 1

1. Let \( R = \mathbb{Z}[\sqrt{3}] := \{ a + b\sqrt{3} : a, b \in \mathbb{Z} \} \).
   (a) Prove that \( R \) is a ring.
   (b) Set \( A \subseteq R \) to be \( A = \{ a + b\sqrt{3} : a \in 4\mathbb{Z}, b \in \mathbb{Z} \} \). Is \( A \) an ideal? Be sure to prove your answer.
   (c) Set \( B \subseteq R \) to be \( B = \{ a + b\sqrt{3} : a \in 3\mathbb{Z}, b \in \mathbb{Z} \} \). Is \( B \) an ideal? Be sure to prove your answer.

2. Let \( R = M_2(\mathbb{Z}) \) and \( x = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \). Is \( x \) a unit of \( R \)? Be sure to prove your answer. **Hint:** Consider the determinant of \( x \).

3. Let \( R = M_2(\mathbb{Z}) \) and \( S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in R \right\} \). Is \( S \) a subring of \( R \)? Be sure to prove your answer.

4. Suppose \( R = \mathbb{Z}/2\mathbb{Z}[x] \), the polynomials in \( x \) with coefficients in \( \mathbb{Z}/2\mathbb{Z} \). Prove or disprove: 
   \((x + x^3 + x^5)^8 = x^8 + x^{24} + x^{40} \) in \( R \). **Hint:** This is closely related to a homework problem about rings of characteristic \( p \) a prime.

5. Prove that the only unit in \( \mathbb{Z}/2\mathbb{Z}[x] \) is 1.

6. Let \( R \) be the ring of (not necessarily continuous) functions \( f : \mathbb{R} \to \mathbb{R} \) under pointwise addition and multiplication, i.e., \((f + g)(x) = f(x) + g(x)\) and \((fg)(x) = f(x)g(x)\). Recall that an idempotent of the ring \( R \) is an element \( a \in R \) satisfying \( a^2 = a \). Does there exist an idempotent \( f \in R \) with \( f \neq 0 \) and \( f \neq 1 \)? Either prove that the only idempotents of \( R \) are 0 and 1 or give an example of an idempotent \( f \in R \) with \( f \neq 0, 1 \).

7. Is \( \mathbb{Z}/2\mathbb{Z}[x]/\langle x^2 + 1 \rangle \) an integral domain?

8. Let \( R = \mathbb{Z}[\sqrt{2}] = \{ a + b\sqrt{2} : a, b \in \mathbb{Z} \} \).
   (a) Prove that \( 3 + 2\sqrt{2} \) is a unit in \( R \).
   (b) Prove that the set of units of \( R \) is infinite.
   (c) Prove or disprove: \( R \) is a field.

9. Let \( R = \mathbb{Z}[i] = \{ a + bi : a, b \in \mathbb{Z} \} \) and let \( I = \langle 3 + 2i \rangle \). Prove that the characteristic of \( R/I \) is 13. **Hint:** First prove that \( 13 \in I \).

10. Suppose \( R_1 \) is a ring of characteristic 0 and \( R_2 \) is a ring of characteristic \( p \), where \( p > 0 \) is prime. What is the characteristic of the ring \( R_1 \times R_2 \)?