# THE RANKIN-SELBERG METHOD: A USER'S GUIDE

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ABSTRACT. We give an introduction to the Rankin-Selberg method. Instead of surveying the the landscape of Rankin-Selberg integrals, we focus on how one can compute a few of the simplest examples. We discuss two integrals for the standard $L$-function on $\mathrm{GL}_2$, one integral for the Spin $L$-function on $\mathrm{PGSp}_4$, and one integral for the standard $L$-function on $\mathrm{Sp}_4$. We describe how to evaluate these integrals in terms of the "New Way" method of Piatetski-Shapiro and Rallis. We end by discussing heuristics that inform the search and computation of Rankin-Selberg integrals.

## 1. INTRODUCTION

The goal of this article is to give an introduction to the so-called Rankin-Selberg method.

1.1. **The Rankin-Selberg method.** The Rankin-Selberg method can loosely be described as follows: One considers an automorphic cuspidal representation $\pi$ on a group $G$, say a $\mathbf{Q}$ reductive group. Let $V_\pi$ denote the space of $\pi$, as automorphic functions on $G(\mathbf{A})$. Then one considers a linear algebraic subgroup $H$ of $G$, possibly equal to $G$, and an automorphic function $E(h,s)$ on $H$ that depends on a complex parameter $s$. The function $E(h,s)$ might be an Eisenstein series on $H$; it might be an Eisenstein series on a larger group that contains $H$; it might be simply an automorphic character on $H$. From this data, one can construct the integral

$$I(\varphi, s) = \int_{H(\mathbf{Q})Z'(\mathbf{A})\backslash H(\mathbf{A})} E(h,s)\varphi(h)\,dh.$$

Here:

(1) $\varphi$ is a cusp form in the space $V_\pi$;
(2) $Z'$ is the intersection of $H$ and the center of $G$.

It is often relatively easy to prove that the integral $I(\varphi,s)$ has meromorphic continuation in $s$, and possibly a functional equation relating $s$ to $r-s$ for some real number $r$. Suppose one knows *a priori* that the integral $I(\varphi,s)$ does have these analytic properties. Then *in vary narrow circumstances*, it turns out that one can sometimes relate the integral $I(\varphi,s)$ with a partial Langlands $L$-function of $\pi$: $I(\varphi,s) \approx L^S(\pi,r,s)$. When one can do this, and when one knows that the integral $I(\bullet,s)$ has good analytic properties in $s$, one says that the linear functional $I(\bullet,s) : V_\pi \to \mathbf{C}$ is a Rankin-Selberg integral.

Key to our chosen definition of Rankin-Selberg integral is the idea that one knows *a priori* that the integral $I(\varphi,s)$ has good analytic properties in $s$. One can imagine abstract integrals that yield $L$-functions $L(\pi,r,s)$, but for which one does not know the analytic properties of the integral. For the purpose of this article, we do not consider such integrals to have the Rankin-Selberg property.

Rankin-Selberg integrals are somewhat mysterious mathematical objects. There are numerous of them (e.g., a few famous ones are [GPSR87],[CFGK19],[Gar87],[GS15]) but also they are rare gems. As of this writing, there is no general theory that classifies all the Rankin-Selberg integrals, or even a classification of which $L$-functions are attainable via this method.

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1.2. **What this article is about.** The point of this article is to give an introduction to the Rankin-Selberg method. We do this by explaining how to compute some of the simplest Rankin-Selberg integrals "from scratch". In that sense, our purpose is to give a sort of "user's guide" to the Rankin-Selberg method, explaining some techniques in detail so that the reader, if they so choose, can do their own research on $L$-functions. In particular, we try to highlight ideas and techniques that are less accessible in the literature. More specifically, we describe how to evaluate Rankin-Selberg integrals in terms of the "New Way" method of Piatetski-Shapiro and Rallis [PSR88], see also [BFG95].

What this article is not is a *survey* of the Rankin-Selberg method. For that, we urge the reader to see the surveys of Bump [Bum05, Bum89] and Cogdell [Cog08]. In fact, to get a complete picture, we recommend reading this article in conjunction with [Bum05, Bum89] and [Cog08].

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## 2. Hecke's integral

In this section we discuss the standard $L$-function of level one Hecke eigen cusp forms.

2.1. **Hecke's integral classically.** Suppose $f$ is a weight $\ell$, level one normalized Hecke eigenform. Then $f(z) = \sum_{n\geq 1} a_n q^n$ with $q = e^{2\pi i z}$ and $a_1 = 1$. The classical $L$-function of $f$ is defined as $L_{class}(f,s) = \sum_{n\geq 1} a_n n^{-s}$ and the completed classical $L$-function as $\Lambda_{class}(f,s) = \Gamma_{\mathbf{C}}(s) L_{class}(f,s)$ where $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

**Theorem 2.1.1** (Hecke)**.** *The classical completed $L$-function has an Euler product, analytic continuation in $s$, and functional equation:*

1. *Euler product:* $L_{class}(f,s) = \prod_p (1 - a_p p^{-s} + p^{\ell-1-2s})^{-1}$.
2. *Analytic continuation: Originally defined for $Re(s) >> 0$, the functions $L_{class}(f,s)$ and $\Lambda_{class}(f,s)$ have analytic continuation to the entire complex plane.*
3. *Functional equation: The completed classical $L$-function satisfies the functional equation* $\Lambda_{class}(f,s) = (-1)^{\ell/2}\Lambda_{class}(f,\ell - s)$.

Let's recall the proof of the analytic continuation and functional equation. To do so, consider the integral
$$I(f,s) = 2\int_0^\infty f(iy) y^s \frac{dy}{y}.$$
This will be our first Rankin-Selberg integral. Then, since $f(iy) = \sum_{n\geq 1} a_n e^{-2\pi n y}$, and $2\int_0^\infty e^{-2\pi n y} y^s \frac{dy}{y} = \Gamma_{\mathbf{C}}(s) n^{-s}$, one has $I(f,s) = \Lambda_{class}(f,s)$.

To understand the analytic properties of the integral, first observe that because $f$ is a modular form, $f(-1/z) = z^\ell f(z)$, and thus $f(iy^{-1}) = (-1)^{\ell/2} y^\ell f(iy)$. Consequently,
$$\begin{aligned}\int_0^1 y^s f(iy)\,\frac{dy}{y} &= \int_1^\infty y^{-s} f(iy^{-1})\,\frac{dy}{y} \\ &= (-1)^{\ell/2}\int_1^\infty y^{\ell-s} f(iy)\,\frac{dy}{y}.\end{aligned}$$
Therefore
$$I(f,s) = 2\int_1^\infty (y^s + (-1)^{\ell/2} y^{\ell-s}) f(iy)\,\frac{dy}{y}$$
has analytic continuation and functional equation.

What about the Euler product? This comes from the fact that $f$ was assumed to be a Hecke eigenform. We will prove the Euler product property below in a not-exactly-standard way.

2.2. **Transition to automorphic representations.** To prove the Euler product property, and in fact to define what it means for $f$ to be a Hecke eigenform, it is most natural to transition to adele groups.

Thus suppose $f$ is a weight $\ell$, level one modular form as above. We will define functions $\varphi'_f : \mathrm{GL}_2(\mathbf{R})^+ \to \mathbf{C}$ and then $\varphi_f : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ out of $f$. To do this, first define $j : \mathrm{GL}_2(\mathbf{R})^+ \times \mathfrak{h} \to \mathbf{C}$ as $j(g,z) = \det(g)^{-1/2}(cz+d)$ if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The factor of automorphy $j$ satisfies $j(g_1g_2,z) = j(g_1, g_2z)j(g_2,z)$ for all $g_1, g_2 \in \mathrm{GL}_2(\mathbf{R})^+$ and all $z \in \mathfrak{h}$.

Now, for $g \in \mathrm{GL}_2(\mathbf{R})^+$, set $\varphi'_f(g) = j(g,i)^{-\ell} f(g\cdot i)$. Note that $\varphi'_f$ is left-invariant under $\Gamma = \mathrm{GL}_2(\mathbf{Z})^+$. Indeed, writing $z = gi$,

$$\varphi'_f(\gamma g) = j(\gamma g, i)^{-\ell} f(\gamma z) = j(\gamma,z)^{\ell} j(g,i)^{-\ell} j(\gamma,z)^{\ell} f(z) = \varphi'_f(g).$$

To define the function $\varphi_f$ on $\mathrm{GL}_2(\mathbf{A})$, we will use the following proposition.

**Proposition 2.2.1.** *One has* $\mathrm{GL}_2(\mathbf{A}) = \mathrm{GL}_2(\mathbf{Q})\,\mathrm{GL}_2(\mathbf{R})^+\,\mathrm{GL}_2(\widehat{\mathbf{Z}})$. *Moreover, the natural map*

$$\mathrm{GL}_2(\mathbf{Z})^+ \backslash \mathrm{GL}_2(\mathbf{R})^+ \to \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}) / \mathrm{GL}_2(\widehat{\mathbf{Z}})$$

*is a bijection.*

It follows that we may lift $\varphi'_f$ to a function $\varphi_f : \mathrm{GL}_2(\mathbf{Q})\backslash \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ that is right invariant under $\mathrm{GL}_2(\widehat{\mathbf{Z}})$.

Using $\varphi_f$, we can define what it means for $f$ to be a Hecke eigenform. We first define the local Hecke algebra. Thus set

$$\mathcal{H}_p = \{\eta \in C_c^\infty(\mathrm{GL}_2(\mathbf{Q}_p)) : \eta(k_1gk_2) = \eta(g) \forall k_1,k_2 \in \mathrm{GL}_2(\mathbf{Z}_p) \text{ and all } g \in \mathrm{GL}_2(\mathbf{Q}_p)\}.$$

Here the condition of being in $C_c^\infty(\mathrm{GL}_2(\mathbf{Q}_p))$ means that $f$ is $\mathbf{C}$-valued, locally constant, and of compact support.

Suppose now $\eta \in \mathcal{H}_p$ and $\varphi_f$ is as above. Define

$$\eta * \varphi_f(g) = \int_{\mathrm{GL}_2(\mathbf{Q}_p)} \eta(h)\varphi_f(gh)\,dh.$$

This is actually a finite sum.

**Definition 2.2.2.** One says that $f$ is a Hecke eigenform is $\eta * \varphi_f = \lambda_\eta \varphi_f$ for some constant $\lambda_\eta$, for all $\eta \in \mathcal{H}_p$ and for all $p$.

Let $\pi$ be the $\mathrm{GL}_2(\mathbf{A})$-representation[1] generated by $\varphi_f$. Then it is a fact that $f$ is a Hecke eigenform if and only if $\pi$ is irreducible.

2.3. **Hecke's integral adelically I.** Suppose $f$ is a level one, weight $\ell$ Hecke eigenform and $\pi$ is the associated automorphic representation. Then $L(\pi, Std, s) = L_{class}(f, s+\frac{\ell-1}{2})$ and $\Lambda(\pi, Std, s) = \Lambda_{class}(f, s+\frac{\ell-1}{2})$. If the reader is unfamiliar with the definition of automorhpic $L$-functions, then they can take these equalities as definitions. The functional equation for $\Lambda_{class}(f,s)$ becomes

$$\Lambda(\pi, Std, 1-s) = \Lambda_{class}(f, \frac{\ell+1}{2} - s) = (-1)^{\ell/2}\Lambda_{class}(f, \ell - (\frac{\ell+1}{2} - s)) = (-1)^{\ell/2}\Lambda(\pi, Std, s).$$

[1]We really mean the $\mathrm{GL}_2(\mathbf{A}_f) \times (\mathfrak{gl}_2, O(2))$-module generated by $\varphi_f$.

We can express the classical Hecke integral group theoretically using $\varphi_f$. Indeed, consider the integral

$$I(\varphi_f, s) := \int_{\mathrm{GL}_1(\mathbf{Q})\backslash \mathrm{GL}_1(\mathbf{A})} \varphi_f\left(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)\right) |t|^s \, dt.$$

Here the absolute value is the adelic one: if $t = (t_v)_v$ then $|t| = \prod_v |t_v|_v$; note that if $\gamma \in \mathrm{GL}_1(\mathbf{Q})$ then $|\gamma| = 1$ by the product formula.

First, let's see how this integral reduces to the classical Hecke integral. Indeed, one has that the natural map $\mathbf{R}^\times_{>0} \to \mathrm{GL}_1(\mathbf{Q})\backslash \mathrm{GL}_1(\mathbf{A})/\mathrm{GL}_1(\widehat{\mathbf{Z}})$ is a bijection. So, integrating on the right over $\mathrm{GL}_1(\widehat{\mathbf{Z}})$ in $I(\varphi_f, s)$–using that $\varphi_f(g)$ is right invariant under this subgroup–we obtain an integral over $\mathbf{R}^\times_{>0}$. To understand the resulting integrand, we observe that if $t \in \mathbf{R}^\times_{>0}$, then

$$\varphi_f\left(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)\right) = j\left(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right), i\right)^{-\ell} f(ti) = t^{\ell/2} f(ti).$$

Hence

$$\begin{aligned} I(\varphi_f, s) &= \int_{\mathbf{R}^\times_{>0}} f(ti)|t|^{s+\ell/2} \, dt \\ &= \int_0^\infty f(iy) y^{s+\ell/2} \, \frac{dy}{y} \\ &= \Lambda_{classical}(f, s+\ell/2) = \Lambda(f, s+\frac{1}{2}). \end{aligned}$$

Here we have used that the Haar measure $dt$ on $\mathrm{GL}_1(\mathbf{R})$ is the measure $y^{-1}\, dy$, where $dy$ denotes the usual Lebesgue measure on $\mathbf{R}$.

2.4. **Unfolding the integral.** Our goal is to prove that the global integral $I(\varphi_f, s)$ is an Euler product, identified with the $L$-function of $\pi$. In order to do this, we begin by considering the Fourier expansion of $\varphi_f$. We will rewrite the integral $I(\varphi_f, s)$ in terms of Fourier coefficients of $\varphi$ and an integral whose domain is $\mathrm{GL}_1(\mathbf{A})$. This is called "unfolding" the integral.

To get us started, let $\psi : \mathbf{Q}\backslash\mathbf{A} \to \mathbf{C}^\times$ be the standard additive character. Thus $\psi = \prod_v \psi_v$ with $\psi_\infty(x) = e^{2\pi i x}$, and if $x \in \mathbf{Q}_p$ with $x = x_0 + x_1$, $x_0 \in \mathbf{Z}_p$ and $x_1 = \frac{m}{p^r}, m \in \mathbf{Z}$, then $\psi_p(x) = e^{-2\pi i x_1}$. Recall that the additive characters of $\mathbf{Q}\backslash\mathbf{A}$ are of the form $\psi_\mu(x) := \psi(\mu x)$ for $\mu \in \mathbf{Q}$.

We will now write the Fourier expansion of $\varphi_f$ adellically. We have

$$\varphi(g) = \sum_{\mu \in \mathbf{Q}} \varphi_{\psi_\mu}(g)$$

where for a character $\chi$ of $\mathbf{Q}\backslash\mathbf{A}$ we write $\varphi_\chi(g) = \int_{\mathbf{Q}\backslash\mathbf{A}} \chi^{-1}(x)\varphi_f\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g\right) dx$.

Define $W_\varphi(g) = \varphi_\psi(g)$; this is called the Whittaker coefficient of $\varphi$.

**Lemma 2.4.1.** *For $\mu \in \mathbf{Q}^\times$, one has $\varphi_{\psi_\mu}(g) = W_\varphi\left(\left(\begin{smallmatrix} \mu & \\ & 1 \end{smallmatrix}\right) g\right)$.*

*Proof.* This follows from changing variable in the integral:

$$\begin{aligned} W_\varphi\left(\left(\begin{smallmatrix} \mu & \\ & 1 \end{smallmatrix}\right) g\right) &= \int_{\mathbf{Q}\backslash\mathbf{A}} \psi^{-1}(x)\varphi\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \mu & \\ & 1 \end{smallmatrix}\right) g\right) dx \\ &= \int_{\mathbf{Q}\backslash\mathbf{A}} \psi^{-1}(x)\varphi\left(\left(\begin{smallmatrix} \mu^{-1} & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \mu & \\ & 1 \end{smallmatrix}\right) g\right) dx \\ &= \int_{\mathbf{Q}\backslash\mathbf{A}} \psi^{-1}(\mu x)\varphi\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) g\right) dx \\ &= \varphi_{\psi_\mu}(g). \end{aligned}$$

□

Now, because $\varphi$ is a cusp form, $\varphi_{\psi_0}(g) \equiv 0$. Thus, by the Lemma, we have the Fourier expansion

$$\varphi(g) = \sum_{\mu \in \mathbf{Q}^\times} W_\varphi\left(\left(\begin{smallmatrix} \mu & \\ & 1 \end{smallmatrix}\right) g\right).$$

With this Fourier expansion in hand, we can now unfold the integral $I(\varphi, s)$. We obtain

$$\begin{aligned} I(\varphi, s) &= \sum_{\mu \in \mathbf{Q}^\times} \int_{\mathrm{GL}_1(\mathbf{Q}) \backslash \mathrm{GL}_1(\mathbf{A})} W_\varphi\left(\left(\begin{smallmatrix} \mu & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)\right) |t|^s \, dt \\ &= \int_{\mathrm{GL}_1(\mathbf{A})} W_\varphi\left(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)\right) |t|^s \, dt. \end{aligned}$$

It's a theorem, which we won't prove, that the Whittaker function $W_\varphi$ is an Euler product. That is, for every place $v$, there is a function $W_v : \mathrm{GL}_2(\mathbf{Q}_v) \to \mathbf{C}$ with the following properties:

(1) If $p$ is finite, $W_p(1) = 1$;
(2) One has $W_\varphi(g) = \prod_v W_v(g_v)$ for every $g \in \mathrm{GL}_2(\mathbf{A})$, with all but finitely many factors being equal to 1.

It follows that $I(\varphi, s)$ is an Euler product: $I(\varphi, s) = \prod_v I_v(s)$ with $I_v(s) = \int_{\mathrm{GL}_1(\mathbf{Q}_v)} W_v\left(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right)\right) |t|^s \, dt$.

2.5. **Hecke's integral adelically II.** If we take for granted the fact that the Whittaker function $W_\varphi(g)$ is an Euler product, then we deduce that $I(\varphi, s)$ is an Euler product. We will, in fact, prove that the global integral $I(\varphi, s)$ is an Euler product without using that the Whittaker function has this property. The method we will use is often referred to as the "New way method" or "Method of non-unique models".

Consider the linear functional on $\pi$, $L_\alpha : V_\pi \to \mathbf{C}$ as $L_\alpha(\varphi) = \varphi_\alpha(1)$. Then

$$\begin{aligned} L_\alpha\left(\left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right) \cdot \varphi\right) &= \left(\left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right) \cdot \varphi\right)_\alpha (1) \\ &= \varphi_\alpha\left(\left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right)\right) \\ &= \psi(\alpha r)\varphi_\alpha(1) \\ &= \psi(\alpha r) L_\alpha(\varphi). \end{aligned}$$

Note that $W_\varphi(g) = \varphi_1(g) = L_1(g \cdot \varphi)$.

Thinking about the unfolded global integral $I(\varphi, s)$, we will consider local integrals of the form

$$I(\ell, s) = \int_{\mathrm{GL}_1(\mathbf{Q}_p)} |t|_p^s \ell\left(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) v_0\right) dt$$

where

(1) $v_0$ is a spherical vector in $V_p$
(2) $\ell : V_p \to \mathbf{C}$ is a linear functional satisfying $\ell\left(\left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right) v\right) = \psi(r)\ell(v)$ for all $r \in \mathbf{Q}_p$ and $v \in V_p$.

After we have considered and evaluated such integrals, we will go back and explain the relevance to proving that $I(\varphi, s)$ is an Euler product.

2.6. **The definition of the $L$-function.** For $g \in \mathrm{GL}_2(\mathbf{Q}_p)$, let $\Delta_p(g) = \mathrm{char}(g \in M_2(\mathbf{Z}_p))$, the characteristic function of $M_2(\mathbf{Z}_p)$. Set $T_{p^n} \in \mathcal{H}_p$ to be the characteristic function of those matrices in $M_2(\mathbf{Z}_p)$ for which the absolute value of their determinant is equal to $|p^n|$. Then $\Delta_p(g) = T_0 + T_p + T_{p^2} + \cdots$, and

$$\Delta_p(g) |\det(g)|^s = T_0 + T_p |p|^s + T_{p^2} |p|^{2s} + \cdots. \tag{1}$$

Now, consider the integral $\int_{\mathrm{GL}_2(\mathbf{Q}_p)} |\det(h)|^s \Delta_p(h) h \cdot v_0$. Because $v_0$ is an eigenvector of the Hecke operators, this integral is a power series in $p^{-s}$ with coefficients that are Hecke eigenvalues of $\pi_p$. For now, we define $L(\pi_p, s)$ via the equality

$$\int_{\mathrm{GL}_2(\mathbf{Q}_p)} |\det(h)|^s \Delta_p(h) h \cdot v_0 \, dh = L(\pi_p, s - \frac{1}{2}) v_0. \tag{2}$$

We define $L(\pi, s) = \prod_p L(\pi_p, s)$ and $\Lambda(\pi, s) = \Gamma_{\mathbf{C}}(s + \frac{\ell-1}{2}) L(\pi, s)$. With these definitions, we will proceed to sketch the proof that $I(\varphi_f, s) = \Lambda(\pi, s + \frac{1}{2})$.

2.7. **The local integrals.** From (2), we obtain

$$L(\pi_p, s - \frac{1}{2}) \ell(v_0) = \int_{\mathrm{GL}_2(\mathbf{Q}_p)} \Delta_p(g) |\det(g)|^s \ell(g v_0) \, dg. \tag{3}$$

To evaluate this integral, we apply the Iwasawa decomposition. Let $N = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\}$, $T = \{\left(\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right)\}$ and $K_p = \mathrm{GL}_2(\mathbf{Z}_p)$. Then the Iwasawa decomposition says that $\mathrm{GL}_2(\mathbf{Q}_p) = N(\mathbf{Q}_p) T(\mathbf{Q}_p) K_p$ and moreover

$$\int_{\mathrm{GL}_2(\mathbf{Q}_p)} f(g) \, dg = \int_{T(\mathbf{Q}_p)} \int_{N(\mathbf{Q}_p)} \int_{K_p} f(ntk) \delta_B(t)^{-1} \, dk \, dn \, dt$$

where if $t = \mathrm{diag}(t_1, t_2)$ then $\delta_B(t) = |t_1/t_2|$.

We apply this decomposition to the integral (3). We obtain

$$\begin{aligned} &= \int_T \int_N \delta_B^{-1}(t) \Delta_p(nt) |\det(t)|^{s+1} \ell(n t v_0) \, dn \, dt \\ &= \int_T |\det(t)|^{s+1} \delta_B^{-1}(t) \ell(t v_0) \left( \int_{\mathbf{Q}_p} \psi(x) \Delta_p\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) t\right) \, dx \right) dt. \end{aligned}$$

We now evaluate this inner integral.

**Proposition 2.7.1.** *Suppose $t = \mathrm{diag}(t_1, t_2)$. The inner integral above*

$$\int_{\mathbf{Q}_p} \psi(x) \Delta_p\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) t\right) \, dx = 0$$

*if $|t_2| \neq 1$ and is $\Delta\left(\left(\begin{smallmatrix} t_1 & \\ & 1 \end{smallmatrix}\right)\right)$ if $|t_2| = 1$.*

*Proof.* To see this, note that $\Delta_p\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) t\right) \neq 0$ implies $t_1, t_2 \in \mathbf{Z}_p$ and $x \in t_2^{-1} \mathbf{Z}_p$. But if $|t_2| < 1$ the character $\psi$ is nontrivial on $t_2^{-1} \mathbf{Z}_p$. The claim follows. $\square$

Therefore

$$\begin{aligned} L(\pi_p, s + \frac{1}{2}) \ell(v_0) &= \int_{\mathrm{GL}_2(\mathbf{Q}_p)} \Delta_p(g) |\det(g)|^{s+1} \ell(g v_0) \, dg \\ &= \int_{\mathrm{GL}_1(\mathbf{Q}_p)} \ell(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) v_0) |t|^s \, \mathrm{char}(t \in \mathbf{Z}_p) \, dt. \end{aligned}$$

Now, one has

**Lemma 2.7.2.** *If $t \in \mathrm{GL}_1(\mathbf{Q}_p)$ and $\ell(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) v_0) \neq 0$, then $t \in \mathbf{Z}_p$.*

From the lemma we deduce

**Proposition 2.7.3.** *One has*

$$\int_{\mathrm{GL}_1(\mathbf{Q}_p)} \ell(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) v_0) |t|^s \, dt = L(\pi_p, s + \frac{1}{2}) \ell(v_0).$$

*Proof of Lemma 2.7.2.* Because $v_0$ is right invariant under $\mathrm{GL}_2(\mathbf{Z}_p)$ one has for all $x \in \mathbf{Z}_p$

$$\ell(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) v_0) = \ell(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) v_0) = \psi(tx) \ell(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) v_0).$$

If $\ell(\left(\begin{smallmatrix} t & \\ & 1 \end{smallmatrix}\right) v_0) \neq 0$ then $\psi(tx) = 1$ for all $x \in \mathbf{Z}_p$, from which it follows that $t \in \mathbf{Z}_p$. $\square$

2.8. **Hecke's integral adelically III.** For a finite set $\Omega$ of places, define

$$I(\Omega,\varphi,s) = \int_{\mathrm{GL}_1(\mathbf{A}_\Omega)} |t|^s L_1(\left(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\right)\varphi)\, dt.$$

**Lemma 2.8.1.** *One has* $I(\{\infty\},\varphi,s) = \frac{1}{2}\Gamma_{\mathbf{C}}(s+\ell/2)a_f(1)$

*Proof.* If $g \in \mathrm{GL}_2(\mathbf{R})$, we have

$$\varphi_1(g) = \int_{\mathbf{Q}\backslash\mathbf{A}} \psi^{-1}(x)\varphi\left(\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right) g\right)\, dx = \int_{\mathbf{Z}\backslash\mathbf{R}} e^{-2\pi i x}\varphi\left(\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right) g\right)\, dx.$$

Now if $g = \left(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\right)$ with $t>0$ then

$$\varphi\left(\left(\begin{smallmatrix} 1 & x \\ & 1\end{smallmatrix}\right) g\right) = j\left(\left(\begin{smallmatrix} t & x \\ & 1\end{smallmatrix}\right), i\right)^{-\ell} f(x+ti) = t^{\ell/2} f(x+ti).$$

Thus $\varphi_1(g) = a_f(1)t^{\ell/2}e^{-2\pi t}$ from the usual Fourier expansion of $f$. If $t<0$ one can prove that $W_\infty\left(\left(\begin{smallmatrix} t & \\ & 1\end{smallmatrix}\right)\right) = 0$. Consequently,

$$I(\{\infty\},\varphi,s) = a_f(1)\int_{\mathbf{R}^\times_{>0}} t^{s+\ell/2}e^{-2\pi t}\, dt = a_f(1)(2\pi)^{-(s+\ell/2)}\Gamma(s+\ell/2) = \frac{a_f(1)}{2}\Gamma_{\mathbf{C}}(s+\ell/2).$$

□

**Proposition 2.8.2.** *One has* $I(\Omega\cup\{p\},\varphi,s) = L(\pi_p, s+\frac{1}{2})I(\Omega,\varphi,s)$.

*Proof.* Set Here $\ell_{t_\Omega}(v) = L_1(\mathrm{diag}(t_\Omega,1)v)$. Then

$$\begin{aligned} I(\Omega\cup\{p\},\varphi,s) &= \int_{\mathrm{GL}_1(\mathbf{A}_\Omega)} |t_\Omega|^s \int_{\mathrm{GL}_1(\mathbf{Q}_p)} |t_p|^s L_1(\mathrm{diag}(t_\Omega,1)\,\mathrm{diag}(t_p,1)\varphi)\, dt_p\, dt_\Omega \\ &= \int_{\mathrm{GL}_1(\mathbf{A}_\Omega)} |t_\Omega|^s \int_{\mathrm{GL}_1(\mathbf{Q}_p)} |t_p|^s \ell_{t_\Omega}(\mathrm{diag}(t_p,1)v_0)\, dt_p\, dt_\Omega \\ &= L(\pi_p, s+\frac{1}{2})\int_{\mathrm{GL}_1(\mathbf{A}_\Omega)} |t_\Omega|^s \ell_{t_\Omega}(v_0)\, dt_\Omega \\ &= L(\pi_p, s+\frac{1}{2})I(\Omega,\varphi,s). \end{aligned}$$

□

Combining the previous results with the fact that, by definition, $I(\varphi,s) = \lim_\Omega I(\Omega,\varphi,s)$ we obtain $I(\varphi,s) = \frac{a_f(1)}{2}\Lambda(\pi, s+\frac{1}{2})$.

2.9. **Summary of automorphic $L$-functions.** We now go back and summarize the definition of Langlands $L$-functions. Thus suppose $G$ is a reductive algebraic $\mathbf{Q}$-group, that we assume is split for simplicity. Let $\pi \subseteq \mathcal{A}_0(G)$ be a cuspidal automorphic representation. Then $\pi = \otimes'\pi_v$ is a restricted tensor product of representations $\pi_v$ of $G(\mathbf{Q}_v)$. The first fact to know is that $\pi_p$ is **unramified** for almost every $p$. Here unramified means that $V_p^{K_p} \neq 0$ for $K_p$ a hyperspecial maximal compact subgroup of $G(\mathbf{Q}_p)$. It is another fact that, in this case, $V_p^{K_p}$ is necessarily one-dimensional.

Associated to $\pi_p$ there is a conjugacy class $c_p$ in $\widehat{G}(\mathbf{C})$, the complex dual group of $G$. Automorphic $L$-functions are associated to finite dimensional algebraic representations $r : \widehat{G}(\mathbf{C}) \to \mathrm{GL}_N(\mathbf{C})$. Namely, for such an $r$, one defines

$$L^S(\pi,r,s) := \prod_{p\notin S} L(\pi_p,r,s) := \prod_{p\notin S}\det(1_N - r(c_p)p^{-s})^{-1}$$

where $S$ is a finite set of places outside of which $\pi_p$ is unramified. The Euler product defining $L^S(\pi, r, s)$ is known to converge for $Re(s) >> 0$. It is expected to have meromorphic continuation in $s$.

Suppose $\pi_p$ is unramified and $v_0$ a spherical vector for $\pi_p$. It is a fact, following from the Satake isomorphism, that there exists a unique function $\Delta_s^r(g) \in C^\infty(K_p \backslash G(\mathbf{Q}_p)/K_p)$ such that

$$L(\pi_p, r, s)v_0 = \int_{G(\mathbf{Q}_p)} \Delta_s^r(g) g \cdot v_0 \, dg.$$

Above, we used an explicit $\Delta_s^r$ for $G = \mathrm{GL}_2$ and $r = Std$ to help us evaluate Hecke's integral.

## 3. The standard $L$-function of modular forms

In this section, we present another integral that realizes the standard $L$-function of modular forms on $\mathrm{GL}_2$. The earliest reference I am aware of for this integral is [GKZ87, Chapter III.3]. One can think of $\mathrm{PGL}_2$ as $\mathrm{SO}(3)$, and then the standard $L$-function for cusp forms on $\mathrm{PGL}_2$ becomes the standard $L$-function on $\mathrm{SO}(3)$. From this point of view, the integral we discuss in this section generalizes from $\mathrm{SO}(3)$ to $\mathrm{SO}(n)$, $n \geq 3$ [MS94]. See also [Pol18] for some context.

3.1. **The global integral.** We begin by defining the global integral that will represent the standard $L$-function. Let $K$ be a quadratic field extension of $\mathbf{Q}$, and $H$ the algebraic $\mathbf{Q}$-group for which $H(\mathbf{Q})$ is the subgroup of $\mathrm{GL}_2(K)$ for which the determinant is in $\mathbf{Q}^\times$. We will define an Eisenstein series $E(h, \Phi, s)$ on $H(\mathbf{A})$ and consider the global integral

$$I(\varphi, \Phi, s) = \int_{\mathrm{GL}_2(\mathbf{Q})Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} E(g, \Phi, s)\varphi(g) \, dg$$

where $\pi$ is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ and $\varphi$ a cusp form in the space of $\pi$.

To define the Eisenstein series, let $\Phi$ be a Schwartz-Bruhat function on $K^2 \otimes_{\mathbf{Q}} \mathbf{A} = \mathbf{A}_K^2$. Set

$$f(h, \Phi, s) = |\det(h)|_{\mathbf{Q}}^s \int_{\mathrm{GL}_1(\mathbf{A}_K)} |t|_K^s \Phi(t(0, 1)h) \, dt.$$

Then $f(\left(\begin{smallmatrix} a & b \\ & d \end{smallmatrix}\right) h, \Phi, s) = |a/d|_K^{s/2} f(h, \Phi, s)$. Letting $B_H$ be the upper-triangular Borel of $H$, we set

$$E(h, \Phi, s) = \sum_{\gamma \in B_H(\mathbf{Q})\backslash H(\mathbf{Q})} f(\gamma h, \Phi, s).$$

This Eisenstein series has meromorphic continuation in $s$ and satisfies a functional equation relating $s$ to $2 - s$.

We can now unfold the global integral. We begin with the following lemma.

**Lemma 3.1.1.** *Suppose $K = \mathbf{Q}(\sqrt{D})$. The double coset $B_H(\mathbf{Q})\backslash H(\mathbf{Q})/\mathrm{GL}_2(\mathbf{Q})$ has two elements $\{1, \gamma_0\}$. The element $\gamma_0$ can be chosen so that $(0, 1)\gamma_0 = (\sqrt{D}, 1)$.*

*Proof.* One can identify $B_H(\mathbf{Q})\backslash H(\mathbf{Q})$ with the $K$-lines in $K^2$ via $\gamma \mapsto (0, 1)\gamma$. Writing a nonzero vector $v \in K^2$ as $v_1 + \sqrt{D}v_2$ with $v_1, v_2 \in \mathbf{Q}^2$, we see that there are two possibilities: either $v_1, v_2$ are linearly dependent, or $v_1, v_2$ are a basis of $\mathbf{Q}^2$. In the first case, one can use the action of $\mathrm{GL}_2(\mathbf{Q})$ to move the line $Kv$ to $K(0, 1)$. In the second case, one can use the action of $\mathrm{GL}_2(\mathbf{Q})$ to move the line $Kv$ to $K((0, 1) + \sqrt{D}(1, 0))$. The lemma follows. □

Let $R_K \simeq Res_{K/\mathbf{Q}}(\mathrm{GL}_1)$ be the subgroup of $\mathrm{GL}_2$ satisfying $(\sqrt{D}, 1)r \in K^\times(\sqrt{D}, 1)$. In matrices, $R_K$ consists of the $\left(\begin{smallmatrix} d & b \\ Db & d \end{smallmatrix}\right)$.

Using the lemma, one obtains

$$I(\varphi, \Phi, s) = \int_{B(\mathbf{Q})Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} f(g, \Phi, s)\varphi(g) \, dg + \int_{R_K(\mathbf{Q})Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} f(\gamma_0 g, \Phi, s)\varphi(g) \, dg.$$

The first integral vanishes by the cuspidality of $\varphi$.

Set
$$\varphi_K(g) = \int_{R_K(\mathbf{Q})Z(\mathbf{A})\backslash R_K(\mathbf{A})} \varphi(rg)\, dr.$$
The second integral is equal to
$$\int_{R_K(\mathbf{Q})Z(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} f(\gamma_0 g, \Phi, s)\varphi(g)\, dg = \int_{R_K(\mathbf{A})\backslash \mathrm{GL}_2(\mathbf{A})} f(\gamma_0 g, \Phi, s)\varphi_K(g)\, dg$$
$$= \int_{\mathrm{GL}_2(\mathbf{A})} |\det(g)|^s \Phi((\sqrt{D}, 1)g)\varphi_K(g)\, dg.$$
Thus we have proved

**Proposition 3.1.2.** *The integral $I(\varphi, \Phi, s)$ is equal to $\int_{\mathrm{GL}_2(\mathbf{A})} |\det(g)|^s \Phi((\sqrt{D}, 1)g)\varphi_K(g)\, dg$.*

3.2. **Local calculation.** By the method of non-unique models, to evaluate the integral $I(\varphi, \Phi, s)$, it suffices to consider integrals of the form
$$I(\ell, \Phi_p, s) = \int_{\mathrm{GL}_2(\mathbf{Q}_p)} |\det(g)|^s \Phi_p((\sqrt{D}, 1)g)\ell(gv_0)\, dg$$
where $\ell$ is a linear functional on $V_p$. We will do this when $p$ is finite and $\Phi_p$ is the characteristic function $\mathbf{Z}_p[\sqrt{D}]^2$. Under these assumptions, $\Phi_p((\sqrt{D}, 1)g) = \mathrm{char}(g \in M_2(\mathbf{Z}_p)) = \Delta_p(g)$. Consequently,
$$I(\ell, \Phi_p, s) = \int_{\mathrm{GL}_2(\mathbf{Q}_p)} |\det(g)|^s \Delta_p(g)\ell(gv_0)\, dg = L(\pi_p, Std, s - \frac{1}{2})\ell(v_0).$$
Therefore, as in the previous section, we obtain the following theorem.

**Theorem 3.2.1.** *Suppose $S$ is a finite set of places of $\mathbf{Q}$ containing $\infty$ and such that if $p \notin S$ then $\pi_p$ is unramified, $\varphi$ is spherical at $p$, and $\Phi_p$ is the characteristic function of $\mathbf{Z}_p[\sqrt{D}]^2$. Then*
$$I(\varphi, \Phi, s) = L^S(\pi, Std, s - \frac{1}{2}) \int_{\mathrm{GL}_2(\mathbf{A}_S)} |\det(g)|^s \Phi_S((\sqrt{D}, 1)g)\phi_K(g)\, dg.$$

We leave the analysis of the integral $\int_{\mathrm{GL}_2(\mathbf{A}_S)} |\det(g)|^s \Phi_S((\sqrt{D}, 1)g)\phi_K(g)\, dg$ to the energetic reader.

## 4. Andrianov's integral

In this section, we sketch an evaluation of Andrianov's integral [And74] for the Spin $L$-function of cusp forms on $\mathrm{PGSp}_4$. One can think of $\mathrm{PGSp}_4$ as $\mathrm{SO}(5)$, and the Spin $L$-function on $\mathrm{PGSp}_4$ as the standard $L$-function on $\mathrm{SO}(5)$. From this point of view, the integral we discuss in this section generalizes to produce the standard $L$-function on $\mathrm{SO}(n)$ for $n \geq 4$ [Sug85]. Again, one can see [Pol18] for some context.

4.1. **Siegel modular forms.** Siegel modular forms are one of the generalizations of classical modular forms. To define them, we first recall the symplectic similitude group. It is defined as
$$\mathrm{GSp}_{2n} = \left\{ (g, \nu(g)) \in \mathrm{GL}_{2n} \times \mathrm{GL}_1 : g \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} g^t = \nu(g) \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \right\}.$$
Note that $\mathrm{GSp}_2 = \mathrm{GL}_2$ because if $g \in \mathrm{GL}_2$ then $g \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g^t = \det(g) \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Another way of thinking of the group $\mathrm{GSp}_{2n}$ is as the group preserving a symplectic form on vector space, up to scaling. In more detail, if $R$ is a ring, set $W_{2n}(R) = R^{2n}$ with basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. Define a bilinear form $\langle\, , \,\rangle : W_{2n}(R) \to R$ via

(1) $\langle w, v\rangle = -\langle v, w\rangle$
(2) $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$
(3) $\langle e_i, f_j\rangle = \delta_{ij}$.

Then

$$\mathrm{GSp}_{2n}(R) = \{(g, \nu(g)) \in \mathrm{GL}_{2n}(R) \times \mathrm{GL}_1(R) : \langle gv, gw\rangle = \nu(g)\langle v, w\rangle \forall v, w \in W_{2n}(R)\}.$$

Define $\mathfrak{h}_n = \{Z \in M_n(\mathbf{C}) : Z^t = Z \text{ and } Im(Z) > 0\}$ where the second condition means that the imaginary part of $Z$ is positive-definite. Thus $Z = X + iY$ with $X, Y$ symmetric and $Y$ positive definite.

Define $\mathrm{GSp}_{2n}(\mathbf{R})^+$ to the subgroup of $\mathrm{GSp}_{2n}(\mathbf{R})$ consisting of matrices with $\nu(g) > 0$. Suppose that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GSp}_{2n}(\mathbf{R})$ in $n \times n$ block form. It turns out that if $Z \in \mathfrak{h}_n$ then

(1) $n \times n$ matrix $cZ + d$ is invertible
(2) the $n \times n$ matrix $(aZ + b)(cZ + d)^{-1}$ is in $\mathfrak{h}_n$
(3) The map $\mathrm{GSp}_{2n}(\mathbf{R})^+ \times \mathfrak{h}_n \to \mathfrak{h}_n$ given by $(g, Z) \mapsto (aZ + b)(cZ + d)^{-1}$ defines a transitive action.

Set $\Gamma = \mathrm{GSp}_{2n}(\mathbf{Z})^+ = \mathrm{Sp}_{2n}(\mathbf{Z})$, and $\ell > 0$ is an integer. A **Siegel modular form** for $\Gamma$ of weight $\ell$ is a holomorphic function $f : \mathfrak{h}_n \to \mathbf{C}$ satisfying

(1) $f(\gamma Z) = \det(cZ + d)^{\ell} f(Z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$
(2) the $\Gamma$-invariant function $|\det(Y)^{\ell/2} f(Z)|$ is of moderate growth, in a suitable sense.

The Siegel modular forms have a Fourier expansion. Note that

$$\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \\ X^t & 1 \end{pmatrix} = \begin{pmatrix} X^t - X & 1 \\ 1 & 0 \end{pmatrix}$$

so $\left(\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right) \in \mathrm{GSp}_{2n}$ if and only if $X = X^t$. Thus if $X \in M_n(\mathbf{Z})$ and is symmetric, and if $f$ is a Siegel modular form, then $f(Z + X) = f(Z)$.

Denote by $S_n$ the $n \times n$ symmetric matrices. It follows that

$$f(Z) = \sum_{T \in S_n(\mathbf{Z})^\vee} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)}$$

where the Fourier coefficients $a_f(T) \in \mathbf{C}$ and

$$\begin{aligned} S_n(\mathbf{Z})^\vee &= \{T \in S_n(\mathbf{Q}) : \mathrm{tr}(TX) \in \mathbf{Z} \forall X \in S_n(\mathbf{Z})\} \\ &= \{T \in S_n(\mathbf{Q}) : T_{ii} \in \mathbf{Z} \text{ and } T_{ij} \in \frac{1}{2}\mathbf{Z}\}. \end{aligned}$$

Just like in the case of classical holomorphic modular forms, the moderate growth condition implies that the Fourier expansion has a restricted form. It turns out that the sum is only over the positive semi-definite matrices $T$, i.e., $a_f(T) \neq 0$ implies $T \geq 0$. The Siegel modular form $f$ is said to be a *cusp form* if it satisfies the condition $a_f(T) \neq 0$ implies $T > 0$ (positive definite), i.e., the only nonzero Fourier coefficients correspond to positive definite $T$.

4.2. **Definition of eigenforms.** Thus suppose $f$ is a Siegel modular form of level $\Gamma = \mathrm{Sp}_{2n}(\mathbf{Z})$ and weight $\ell$. We can associate to $f$ a function $\varphi_f$ on $\mathrm{GSp}_{2n}(\mathbf{Q})\backslash \mathrm{GSp}_{2n}(\mathbf{A})$ as follows.

First, for $g \in \mathrm{GSp}_{2n}(\mathbf{R})^+$ (the subgroup where the similitude is positive) and $Z \in \mathcal{H}_n$, set $j(g, Z) = \nu(g)^{-n/2} \det(cZ + d)$, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Define $\varphi'_f : \mathrm{GSp}^+_{2n}(\mathbf{R}) \to \mathbf{C}$ as $\varphi'_f(g) = j(g, i)^{-\ell} f(g \cdot i)$. Then because $j$ is a factor of automorphy, $\varphi'_f$ is left-invariant under $\Gamma$.

As we did for $\mathrm{GL}_2$, we have the following proposition.

**Proposition 4.2.1.** *One has* $\mathrm{GSp}_{2n}(\mathbf{A}) = \mathrm{GSp}_{2n}(\mathbf{Q}) \,\mathrm{GSp}^+_{2n}(\mathbf{R}) \,\mathrm{GSp}_{2n}(\widehat{\mathbf{Z}})$. *Moreover, the natural map*

$$\Gamma\backslash \mathrm{GSp}^+_{2n}(\mathbf{R}) \to \mathrm{GSp}_{2n}(\mathbf{Q})\backslash \mathrm{GSp}_{2n}(\mathbf{A}) / \mathrm{GSp}_{2n}(\widehat{\mathbf{Z}})$$

*is a bijection.*

We denote by $\varphi_f$ the function on $\mathrm{GSp}_{2n}(\mathbf{A})$ associated to $\varphi'_f$ via this proposition. By its definition, $\varphi_f$ is right invariant under $\mathrm{GSp}_{2n}(\mathbf{Z}_p)$ for every prime $p$. We can now define Hecke eigenforms.

Let

$$\mathcal{H}_p = \{\eta \in C_c^\infty(\mathrm{GSp}_{2n}(\mathbf{Q}_p)) : \eta(k_1 g k_2) = \eta(g) \forall k_1, k_2 \in \mathrm{GSp}_{2n}(\mathbf{Z}_p) \text{ and all } g \in \mathrm{GSp}_{2n}(\mathbf{Q}_p)\}.$$

We define $\eta * \varphi(g) = \int_{\mathrm{GSp}_{2n}(\mathbf{Q}_p)} \eta(h)\varphi(gh)\, dh$.

**Definition 4.2.2.** The Siegel modular form $f$ is said to be a Hecke eigneform if $\eta * \varphi_f = \lambda_\eta \varphi$ for some $\lambda_\eta \in \mathbf{C}$ for all $\eta \in \mathcal{H}_p$ and all $p$.

4.3. **The definition of the $L$-function.** Recall that the dual group of $\mathrm{PGSp}_{2n}$ is $\mathrm{Spin}_{2n+1}(\mathbf{C})$, which has a $2^n$-dimensional Spin $L$-function. Thus one can consider the Langlands $L$-functions $L(\pi, Spin, s)$ of Siegel modular eigenforms.

The following is a very special case of the conjectural Langlands program.

**Conjecture 4.3.1.** *Suppose $f$ is a level one cuspidal Siegel modular Hecke eigenform for $\mathrm{Sp}_{2n}(\mathbf{Z})$, and $\pi$ is it associated cuspidal automorphic representation. Then the $L$-function $L(\pi, Spin, s)$ can be completed to $\Lambda(\pi, Spin, s)$ which has meromorphic continuation with finitely many poles and satisfies a functional equation $\Lambda(\pi, Spin, s) = \pm\Lambda(\pi, Spin, 1 - s)$.*

The case $n = 2$ of this conjecture is due to Andrianov, and we will sketch a proof of this case: Define the completed $L$-function of $f$ is $\Lambda(\pi, Spin, s) = \Gamma_{\mathbf{C}}(s + \frac{1}{2})\Gamma_{\mathbf{C}}(s + \ell - \frac{3}{2})L(\pi, Spin, s)$.

**Theorem 4.3.2** (Andrianov)**.** *Suppose $f$ is a cuspidal Siegel eigenform. Then the completed $L$-function satisfies the functional equation $\Lambda(\pi, Spin, s) = \Lambda(\pi, Spin, 1-s)$, has analytic continuation and finitely many poles.*

The case $n = 3$ of Conjecture 4.3.1 was proved in [Pol17]. The cases $n > 3$ are open.

We will now state a result concerning $\Delta_s^{Spin}$ on $\mathrm{GSp}_4(\mathbf{Q}_p)$, which we will need in our proof of Theorem 4.3.2. So, for $g \in \mathrm{GSp}_4(\mathbf{Q}_p)$, define $\Delta_{Spin,p}(g) = \mathrm{char}(g \in M_4(\mathbf{Z}_p))$. Note that $\Delta_{Spin,p}(g)$ is bi-invariant under $\mathrm{GSp}_4(\mathbf{Z}_p)$.

One has

**Proposition 4.3.3.** *Suppose $\pi_p$ is an unramified irreducible admissible representation of* $\mathrm{PGSp}_4(\mathbf{Q}_p)$*, and $v_0$ is a spherical vector for $\pi_p$. Then*

$$\int_{\mathrm{GSp}_4(\mathbf{Q}_p)} \Delta_{Spin,p}(g)|\nu(g)|^s g \cdot v_0\, dg = \frac{L_p(\pi_p, s - \frac{3}{2})}{\zeta_p(2s-2)} v_0.$$

Here $\zeta_p(s) = (1 - p^{-s})^{-1}$. Also, we are using that $\pi_p$ has trivial central character. For a general central character $\omega_p$, the denominator in this formula should be $L(\omega_p, 2s-2) = (1-\omega_p(p)|p|^{2s-2})^{-1}$.

4.4. **A subgroup of** $\mathrm{GSp}_4$**.** Before sketching the proof of the theorem, we need a certain subgroup $H$ of $\mathrm{GSp}_4$. In fact, our group $H$ will be the same group that appeared in the previous section, but we review some aspects of this group and prove that it can be embedded in $\mathrm{GSp}_4$.

Fix a quadratic imaginary field $K$. Let $\langle\, , \rangle_K$ be the usual $K$-valued symplectic form on $K^2$, i.e., $\langle (a, b), (c, d)\rangle_K = ad - bc$. Now, supposing that $K = \mathbf{Q}(\sqrt{D})$ define $\langle\, , \rangle = \mathrm{tr}_{K/\mathbf{Q}} \circ \sqrt{D}^{-1} \langle\, , \rangle_K$ on $K^2 = W$ so that $\langle (a, b), (c, d)\rangle = \mathrm{tr}_{K/\mathbf{Q}}(\frac{1}{\sqrt{D}}(ad - bc))$.

We define an algebraic group $H$ over $\mathbf{Q}$ as follows: For a $\mathbf{Q}$-algebra $R$, one has $H(R) = \{(g, \lambda) \in \mathrm{GL}_2(K \otimes R) \times \mathrm{GL}_1(R) : \det(g) = \lambda\}$. In other words, $H(\mathbf{Q})$ is the subgroup of $\mathrm{GL}_2(K)$ with determinant in $\mathbf{Q}^\times \subseteq K^\times$.

Note that $H \subseteq \mathrm{GSp}(W, \langle\, ,\, \rangle) = \mathrm{GSp}_4$ through its action on $K^2 = W$. Indeed, say at the level of $\mathbf{Q}$-points, if $h \in H(\mathbf{Q})$ and $v, w \in K^2$, then

$$\langle hv, hw\rangle = \mathrm{tr}_{K/\mathbf{Q}}(\frac{1}{2\sqrt{D}}\langle hv, hw\rangle_K) = \mathrm{tr}_{K/\mathbf{Q}}((2\sqrt{D})^{-1}\det(h)\langle v, w\rangle_K) = \lambda\langle v, w\rangle$$

because $\det(h) = \lambda$ passes through the trace because it is in the ground field.

4.5. **The global integral.** Suppose $\pi$ is a cuspidal automorphic representation of $\mathrm{PGSp}_4$ and $\varphi$ is a cusp form in the space of $\pi$. The integral we will consider is

$$I(\varphi, \Phi, s) = \int_{H(\mathbf{Q})Z(\mathbf{A})\backslash H(\mathbf{A})} E(h, \Phi, s)\varphi(h)\, dh.$$

We will sketch a proof that this integral represents the completed $L$-function $\Lambda(\pi, Spin, s - \frac{1}{2})$. Then, from the fact that $E(h, \Phi, s)$ has finitely many poles and satisfies a functional equation relating $s$ to $2 - s$, we obtain that $\Lambda(\pi, Spin, s)$ has finitely many poles and satisfies a functional equation relating $s$ to $1 - s$.

First, we unfold this integral. We have

$$\begin{aligned} I(\varphi, \Phi, s) &= \int_{B_H(\mathbf{Q})Z(\mathbf{A})\backslash H(\mathbf{A})} f(h, \Phi, s)\varphi(h)\, dh \\ &= \int_{T_H(\mathbf{Q})N_H(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f(h, \Phi, s)\varphi_{0,K}(h)\, dh \end{aligned}$$

where

$$\varphi_{0,K}(g) = \int_{N_H(\mathbf{Q})\backslash N_H(\mathbf{A})} \varphi(ng)\, dn.$$

To further unfold, we will now produce a Fourier expansion of $\varphi_{0,K}$. For simplicity, assume $\mathcal{O}_K = \mathbf{Z}[\sqrt{D}]$. Then $e_1 = (1, 0)$, $e_2 = (\sqrt{D}, 0)$, $f_1 = (0, \sqrt{D})$, $f_2 = (0, 1)$ is a symplectic basis of $W$, and we obtain an embedding of $H$ into $\mathrm{GSp}_4$ in terms of matrices. Specifically, if $n_H(x) = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in H$ with $x \in K$, then the image of $n_H(x)$ in $\mathrm{GSp}_4$ is $n_G(S(x)) = \left(\begin{smallmatrix} 1 & S(x) \\ 0 & 1 \end{smallmatrix}\right)$ where if $x = u + v\sqrt{D}$ then $S(x) = \left(\begin{smallmatrix} v & u \\ u & Dv \end{smallmatrix}\right)$.

For $T \in S = S^2(\mathbf{Q}^2)$, define

$$\varphi_T(g) = \int_{S(\mathbf{Q})\backslash S(\mathbf{A})} \psi(\mathrm{tr}(TX))\varphi(n_G(X)g)\, dX.$$

Set $T_K = \left(\begin{smallmatrix} -D & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then one obtains easily that

$$\varphi_{0,K}(g) = \sum_{\mu \in \mathbf{Q}^\times} \varphi_{\mu T_K}(g) = \sum_{\mu \in \mathbf{Q}^\times} \varphi_{T_K}(\mathrm{diag}(\mu, \mu, 1, 1)g).$$

We now unfold further. Set $T'_H = \{\mathrm{diag}(\overline{z}, z) : z \in K^\times\}$. Then

$$\begin{aligned} I(\varphi, \Phi, s) &= \int_{T_H(\mathbf{Q})N_H(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f(h, \Phi, s)\varphi_{0,K}(h)\, dh \\ &= \int_{T'_H(\mathbf{Q})N_H(\mathbf{A})Z(\mathbf{A})\backslash H(\mathbf{A})} f(h, \Phi, s)\varphi_{T_K}(h)\, dh \\ &= \int_{T'_H(\mathbf{A})N_H(\mathbf{A})\backslash H(\mathbf{A})} f(h, \Phi, s)\varphi_K(h)\, dh \end{aligned}$$

where

$$\varphi_K(g) = \int_{T'_H(\mathbf{Q})Z(\mathbf{A})\backslash T'_H(\mathbf{A})} \varphi_{T_K}(rg)\, dr.$$

Plugging in the definition of $f(h,\Phi,s)$, we arrive at the following proposition.

**Proposition 4.5.1.** *One has*

$$I(\varphi,\Phi,s)=\int_{N_H(\mathbf{A})\backslash H(\mathbf{A})}|\det(h)|_{\mathbf{Q}}^{s}\varphi_K(h)\Phi((0,1)h)\,dh.$$

Like with the Whittaker function on $\mathrm{GL}_2$, the function $\varphi_K(g)$ on $\mathrm{GSp}_4$ is an Euler product, and thus the global integral $I(\varphi,\Phi,s)$ is an Euler product. We will evaluate $I(\varphi,\Phi,s)$ in terms of $L$-functions without using the result that $\varphi_K$ is an Euler product, similar to what was done in the prior two sections.

4.6. **The local integral.** Suppose $\pi_p$ is unramified, and $\ell: V_p\to\mathbf{C}$ is a linear functional satisfying $\ell(n_G(X)v)=\psi(\mathrm{tr}(T_KX))\ell(v)$ for all $X\in S(\mathbf{Q}_p)$ and all $v\in V_p$. Consider the integral

$$I(\ell,\Phi_p,s)=\int_{N_H(\mathbf{Q}_p)\backslash H(\mathbf{Q}_p)}|\det(h)|^{s}\Phi_p((0,1)h)\ell(h\cdot v_0)\,dh$$

where $v_0$ is a spherical vector in $V_p$. The goal of this section is to evaluate this integral. More precisely, our goal is to prove the following theorem.

**Theorem 4.6.1.** *Suppose $\Phi_p$ is the characteristic function of $\mathcal{O}_{K,p}$. Then $I(\ell,\Phi,s)=L(\pi_p,Spin,s-\frac{1}{2})\ell(v_0)$.*

We begin by applying the Iwasawa decomposition to obtain

$$I(\ell,\Phi_p,s)=\int_{T_H(\mathbf{Q}_p)}\delta_B^{-1}(t)|\det(t)|^{s}\,\mathrm{char}(t_2\in\mathcal{O}_{K,p})\ell(t\cdot v_0)\,dt$$

where $t=\mathrm{diag}(t_1,t_2)$.

On the other hand,

$$\begin{aligned}\frac{L(\pi_p,Spin,s-\frac{1}{2})}{\zeta_p(2s)}\ell(v_0)&=\int_{\mathrm{GSp}_4(\mathbf{Q}_p)}\Delta_p(g)|\nu(g)|^{s+1}\ell(gv_0)\,dg\\&=\int_{M(\mathbf{Q}_p)}\delta_P^{-1}(m)|\nu(m)|^{s+1}\Delta_{T_K}(m)\ell(mv_0)\,dm\end{aligned}$$

where

- $M=\left(\begin{smallmatrix}*&0\\0&*\end{smallmatrix}\right)\subseteq\mathrm{GSp}_4$ is the Siegel Levi, so that $M\simeq\mathrm{GL}_1\times\mathrm{GL}_2$
- $\Delta_{T_K}(m)=\int_{S(\mathbf{Q}_p)}\Delta(n_G(x)m)\psi(\mathrm{tr}(T_Kx))\,dx$.

The function $\Delta_{T_K}(m)$ is evaluated in the following proposition.

**Proposition 4.6.2.** *Suppose $m=\mathrm{diag}(a,d)$. Then $\Delta_{T_K}(m)=|\det(d)|^{-1}\,\mathrm{char}(a,d,p\nmid d)\,\mathrm{char}(T_K\cdot d:=\det(d)d^{-1}T_K\,{}^td^{-1}\in S(\mathbf{Z}_p)$. Moreover, $d$ satisfies this latter condition if and only if $\mathrm{diag}(a,d)\in T_H(\mathbf{Q}_p)M(\mathbf{Z}_p)$.*

*Proof.* We indicate some aspects of the proof. First, the conditions $\mathrm{char}(a,d)$ are clear. Moreover, it is easy to see that the integral defining $\Delta_{T_K}(m)$ vanishes if $p|d$, so the condition $p\nmid d$ is also clear. To see that $T_K\cdot d\in S(\mathbf{Z}_p)$, we make a change of variables in the integral: $x\mapsto\det(d)\,{}^td^{-1}xd^{-1}$ so that

$$\Delta_{T_K}(m)=\int_{S(\mathbf{Q}_p)}\mathrm{char}(a,d,xd)\psi(\mathrm{tr}(T_Kx))\,dx=C(d)\int_{S(\mathbf{Q}_p)}\mathrm{char}(a,d,\det(d)\,{}^td^{-1}x)\psi(T_K\cdot dx)\,dx.$$

For this integral to be nonvanishing, we thus must have $T_K\cdot d\in S(\mathbf{Z}_p)$.

Now suppose $T_K \cdot d$ is integral and $p \nmid d$. We claim that $xd$ integral implies $\mathrm{tr}(T_K x) \in \mathbf{Z}_p$, so that the integral defining $\Delta_{T_K}(m)$ is nonzero. Set $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then

$$\begin{aligned}\mathrm{tr}(T_K x) d &= (T_K x + J x T_K J^{-1}) d \\ &= T_K(xd) + JxdT_K \cdot d \det(d)^{-1} d^t J d^{-1} \\ &= T_K(xd) + JxdT_K \cdot dJ^{-1}\end{aligned}$$

is integral. Because $p \nmid d$, $\mathrm{tr}(T_K x)$ is integral.

We leave the rest of the proof to the energetic reader. See also [PS17, Lemma 4.3]. □

From the proposition, and using the fact that $\delta_P(\mathrm{diag}(t_1, t_2)) = |t_1/t_2|^{3/2}$, we have

$$L(\pi_p, Spin, s - \frac{1}{2})\ell(v_0) = \zeta_p(2s) \int_{T_H(\mathbf{Q}_p)} |t_1|_K^{s/2-1} |t_2|_K^{s/2+1} \,\mathrm{char}(t_1, t_2, p \nmid t_2) \ell(t v_0)\, dt.$$

Theorem 4.6.1 follows without much more work.

Applying Theorem 4.6.1 and the method of non-unique models, we obtain

**Corollary 4.6.3.** *For the global integral $I(\varphi, \Phi, s)$ one has the identity*

$$I(\varphi, \Phi, s) = L^S(\pi, Spin, s - \frac{1}{2}) \int_{N_H(\mathbf{A}_S)\backslash H(\mathbf{A}_S)} |\det(h)|^s \Phi_S((0,1)h) \varphi_K(h)\, dh.$$

*Remark* 4.6.4. The reader who is paying close attention will observe that we have now indicated two[2] different integrals that represent the standard $L$-function of cusp forms on $\mathrm{SO}(n)$. One integral involves an Eisenstein series on $\mathrm{SO}(n-1)$ and the other involves an Eisenstein series on $\mathrm{SO}(n+1)$. Thus one may ask: Why bother producing both integrals, if they represent the same $L$-function? The short answer is that different integral representations can have different applications and tell you different facts about $L$-functions. For example, the integral involving $\mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$ works even when the $\mathrm{SO}(n)$ is anisotropic, while the integral involving $\mathrm{SO}(n-1) \subseteq \mathrm{SO}(n)$ requires the $\mathrm{SO}(n)$ to have Witt rank at least 1. On the other hand, this latter integral can be used to produce a Dirichlet series for the standard $L$-function of holomorphic cusp forms on orthogonal groups in terms of their Fourier coefficients, and when $n = 5$ can be used to relate poles of the Standard $L$-function to periods of the cusp form.

## 5. The standard $L$-function on $\mathrm{Sp}_4$

In this section, we present a further integral that represents $L$-functions of Siegel modular forms on $\mathrm{Sp}_4$. It is an integral due to Andrianov [And76, And78] and Andrianov-Kalinin [AK78] for the Standard $L$-function on $\mathrm{Sp}_{2n}$, specialized to $n = 2$. It was reinterpreted by Piatetski-Shapiro and Rallis [PSR88]. For this integral, we do not show in detail how to evaluate it in terms of $L$-functions. Instead, we present it for two reasons:

(1) So that the reader may see a different sort of constructions that can be made to produce $L$-functions: The integral of Andrianov involves an integrand with *three* functions, instead of two: An Eisenstein series, a cusp form, and a theta function.
(2) So that the reader can see how to use a Rankin-Selberg integral to produce a Dirichlet series for an $L$-function, in terms of classical Fourier coefficients.

[2]In fact, there are even more.

5.1. **Andrianov's integral.** The dual group of $\mathrm{Sp}_{2n}$ is $\mathrm{SO}_{2n+1}(\mathbf{C})$, which has a standard representation into $\mathrm{GL}_{2n+1}(\mathbf{C})$. If $\pi$ is a cuspidal automorphic representation on $\mathrm{Sp}_{2n}$, we let $L(\pi, Std, s)$ denote its associated standard $L$-function. The integral of Andrianov for Siegel modular forms on $\mathrm{Sp}_{2n}$, which was generalized by Piatetski-Shapiro and Rallis to an integral for all cusp forms on $\mathrm{Sp}_{2n}$, produces this standard $L$-function. We will explain the integral in the case of Siegel modular forms.

To explain the integral, let $T$ be a two-by-two half-integral symmetric matrix, and let $V_T$ be the associated two-dimensional positive-definite quadratic space. That is, $V_T$ has a basis $v_1, v_2$ and a positive-definite quadratic form on it $(\ ,\ )$ so that

$$T = \frac{1}{2}\begin{pmatrix} (v_1, v_1) & (v_1, v_2) \\ (v_2, v_1) & (v_2, v_2) \end{pmatrix}.$$

Out of $T$, we can create a Siegel modular theta-function $\Theta_T$ on $\mathrm{Sp}_4$. Let $W = X \oplus Y$ be the standard Lagrangian decomposition of $W$, and let $\Phi$ be a Schwartz-Bruhat function on $(X \otimes V_T)(\mathbf{A})$. In group-theoretic terms, for $g \in \mathrm{Sp}_4(\mathbf{A})$,

$$\Theta_T(g) = \Theta_T^{\Phi}(g) = \sum_{\xi \in (X \otimes V_T)(\mathbf{Q})} \omega_\psi(g)\phi(\xi)$$

where $\omega_\psi$ denotes the Weil representation of $\mathrm{Sp}_4(\mathbf{A})$ on $S(X \otimes V_T(\mathbf{A}))$. In classical terms, $\Phi$ can be chosen so that $\Theta_T$ corresponds to a Siegel modular form $\Theta_T'$ of weight 1 with Fourier expansion

$$\begin{aligned} \Theta_T'(Z) &= \sum_{u_1, u_2 \in \Lambda_T} e^{\pi i \operatorname{tr}(((u_i, u_j))Z)} \\ &= \sum_{m \in M_2(\mathbf{Z})} e^{2\pi i \operatorname{tr}({}^t m T m Z)} \end{aligned}$$

where $\Lambda_T = \mathbf{Z}v_1 \oplus \mathbf{Z}v_T$.

We now define an Eisenstein series on $\mathrm{Sp}_4$. Let $P$ be the Siegel parabolic of $\mathrm{Sp}_4$. Let $\mu$ denote the character of $P(\mathbf{A})$ given by $\mu(\left(\begin{smallmatrix} m & * \\ & {}^t m^{-1} \end{smallmatrix}\right)) = \det(m)$, and let $\chi_T$ denote the character of $\mathrm{GL}_1(\mathbf{Q}) \backslash \mathrm{GL}_1(\mathbf{A})$ given by $\chi_T(r) = (r, \operatorname{disc}(V_T))_2 = (r, -\det(T))_2$ where $(\ ,\ )_2$ denotes the Hilbert symbol. We associate an Eisenstein series to the induced representation $Ind_{P(\mathbf{A})}\mathrm{Sp}_4(\mathbf{A})((\chi_T \circ \mu)|\mu|^s)$. That is, we define $E(g, f, s) = \sum_{\gamma \in P(\mathbf{Q}) \backslash \mathrm{Sp}_4(\mathbf{Q})} f(\gamma g, s)$ where $f(g, s) \in Ind_{P(\mathbf{A})}\mathrm{Sp}_4(\mathbf{A})((\chi_T \circ \mu)|\mu|^s)$ so that it satisfies $f(\left(\begin{smallmatrix} m & * \\ & {}^t m^{-1} \end{smallmatrix}\right) g, s) = \chi_T(\det(m))|\det(m)|^s f(g, s)$.

A normalized Eisenstein series can be defined as $E^*(g, s) = \zeta(2s-2)L(\chi_T, s)E(g, s)$. This normalized Eisenstein series satisfies a functional equation relating $s$ to $3 - s$.

Suppose now $\varphi$ is a Siegel modular cusp form on $\mathrm{Sp}_4$. A Rankin-Selberg integral can now be defined as

$$I(\varphi, s) = \int_{\mathrm{Sp}_4(\mathbf{Q}) \backslash \mathrm{Sp}_4(\mathbf{A})} \varphi(g) E^*(g, s) \overline{\Theta_T(g)}\, dg.$$

To make this integral be nonvanishing for Siegel modular forms of some fixed weight, one must choose the archimedean part of $f(g, s)$ in a special way, depending upon the weight. The integral will represent the partial $L$-function $L^S(\pi, Std, s-1)$.

The integral unfolds as

$$\begin{aligned}\frac{I(\varphi,s)}{\zeta(2s-2)L(\chi_T,s)} &= \int_{\mathrm{Sp}_4(\mathbf{Q})\backslash \mathrm{Sp}_4(\mathbf{A})} \varphi(g)E(g,s)\overline{\Theta_T(g)}\,dg \\ &= \int_{P(\mathbf{Q})\backslash \mathrm{Sp}_4(\mathbf{Q})} \varphi(g)f(g,s)\overline{\Theta_T(g)}\,dg \\ &= \int_{N_P(\mathbf{Q})\backslash \mathrm{Sp}_4(\mathbf{A})} \varphi(g)f(g,s)\overline{\omega(g)\Phi(e_1\otimes v_1+e_2\otimes v_2)}\,dg \\ &= \int_{N_P(\mathbf{A})\backslash \mathrm{Sp}_4(\mathbf{A})} \varphi_T(g)f(g,s)\overline{\omega(g)\Phi(e_1\otimes v_1+e_2\otimes v_2)}\,dg.\end{aligned}$$

One can analyze this integral using the "new way method". In fact, the method of Piatetski-Shapiro and Rallis was first used to analyze the generalization of this integral to $\mathrm{Sp}_{4n}$.

5.2. **Andrianov's Dirichlet series.** We now explain the relation between the above integrals and Dirichlet series for the standard $L$-function of $\varphi$.

Suppose $S$ is a set of bad places for the above integral, and $\mathbf{A}^S$ denotes the adeles away from $S$. Let $V^S$ be the space underlying the $\mathrm{Sp}_4(\mathbf{A}^S)$-representation $\pi^S = \otimes'_{p\notin S}\pi_p$ and let $L: V^S \to \mathbf{C}$ be a linear functional satisfying $L(\left(\begin{smallmatrix}1 & X\\ & 1\end{smallmatrix}\right) v) = \psi((\mathrm{tr}(TX)))L(v)$ for all $v \in V^S$ and $X \in S^2(\mathbf{A}^S)$. Then in the course of analyzing the integral $I(\varphi,s)$ one proves

$$L(v_0)\frac{L^S(\pi, Std, s-1)}{\zeta^S(2s-2)L^S(\chi_T,s)} = \int_{N_P(\mathbf{A}^S)\backslash \mathrm{Sp}_4(\mathbf{A}^S)} L(gv_0)f(g,s)\overline{\omega(g)\Phi^S(e_1\otimes v_1+e_2\otimes v_2)}\,dg.$$

Applying the Iwasawa decomposition, this latter integral is equal to

$$\int_{\mathrm{GL}_2(\mathbf{A}^S)} L(m(g)v_0)|\det(g)|^{s-2}\,\mathrm{char}(g\in M_2(\mathbf{Z}^S))\,dg.$$

Here, for $g \in \mathrm{GL}_2$, $m(g) \in \mathrm{Sp}_4$ is $m(g) = \left(\begin{smallmatrix} g & \\ & {}^tg^{-1}\end{smallmatrix}\right)$. Moreover, we have used that $\delta_P(m(g)) = |\det(g)|^3$, $f(m(g),s) = \chi_T(\det(g))|\det(g)|^s$, and $\omega(m(g))\Phi(e_1\otimes v_1+e_2\otimes v_2) = \chi_T(\det(g))|\det(g)|\,\mathrm{char}(g \in M_2(\mathbf{Z}^S))$.

To see the relation between these integrals and Dirichlet series, suppose that $L$ is the functional $L(v) = \alpha(v\otimes v_S)_T(1)$, where:

(1) $v_S \in \pi_S = \otimes_{v\in S}\pi_v$ is a fixed vector
(2) $\alpha: \pi = \pi^S \otimes \pi_S \to \mathcal{A}_0(\mathrm{Sp}_4)$ is the automorphic embedding
(3) $\alpha(v\otimes v_S)_T(1)$ is the value at 1 of the $T^{th}$ Fourier coefficient of $\alpha(v\otimes v_S)$.

Let's assume furthermore that $\varphi = \alpha(v_0\otimes v_S)$ is the automorphic form corresponding to a level one Siegel modular form $f(Z) = \sum_{R>0} a(R)q^R$ of weight $\ell$. So, we're assuming that $\pi$ is unramified at every finite prime, so that we only have to throw out bad primes coming from where the character $\chi_T$ is ramified.

Now, the integral above is equal to

$$\sum_{g\in \mathrm{GL}_2(\mathbf{A}^S)/\mathrm{GL}_2(\widehat{\mathbf{Z}}^S)} \varphi_T(m(g))|\det(g)|^{s-2}\,\mathrm{char}(g\in M_2(\widehat{\mathbf{Z}}^S)).$$

This sum can be rewritten in terms of the usual Fourier coefficients of $\varphi$. To do this, observe that $\mathrm{GL}_2(\mathbf{A}^S) \subseteq \mathrm{GL}_2(\mathbf{A}_f) = \mathrm{GL}_2(\mathbf{Q})\,\mathrm{GL}_2(\widehat{\mathbf{Z}})$, so that every $g_S \in \mathrm{GL}_2(\mathbf{A}_S)$ can be written as $g_S = g_{\mathbf{Q}}kg_{\mathbf{R}}$ with $g_{\mathbf{Q}} \in \mathrm{GL}_2(\mathbf{Q})$, $k \in \mathrm{GL}_2(\widehat{\mathbf{Z}})$ and $g_{\mathbf{R}} \in \mathrm{GL}_2(\mathbf{R})$. Set $M_2(\mathbf{Z})^S$ to be subset of

$M_2(\mathbf{Z})$ consisting of elements whose determinant is nonzero and only divisible by primes away from $S$. Then the sum above becomes

$$\sum_{g_{\mathbf{Q}} \in M_2(\mathbf{Z})^S / \mathrm{GL}_2(\mathbf{Z})} \varphi_T(m(g_{\mathbf{Q}} k g_{\mathbf{R}})) |\det(g_{\mathbf{Q}})|^{2-s}.$$

Now $\varphi_R(m(g_{\mathbf{R}})) = \det(g_{\mathbf{R}})^\ell a_R e^{-2\pi \operatorname{tr}(R g_{\mathbf{R}}\, {}^t g_{\mathbf{R}})}$, so that

$$\begin{aligned} \varphi_T(m(g_{\mathbf{Q}} k g_{\mathbf{R}})) &= \varphi_T(m(g_{\mathbf{Q}} g_{\mathbf{R}})) \\ &= \varphi_{T \cdot g_{\mathbf{Q}}}(m(g_{\mathbf{R}})) \\ &= a(T \cdot g_{\mathbf{Q}}) |\det(g_{\mathbf{Q}})|^{-\ell} e^{-2\pi \operatorname{tr}(T)} \end{aligned}$$

Putting everything together, we obtain:

$$a(T) \frac{L^S(\pi, Std, s)}{\zeta^S(2s) L^S(\chi_T, s+1)} = \sum_{g \in M_2(\mathbf{Z})^S / \mathrm{GL}_2(\mathbf{Z})} \frac{a({}^t g T g)}{|\det(g)|^{s+\ell-1}}.$$

This is a Dirichlet series for the partial standard $L$-function of a level one Siegel modular form on $\mathrm{Sp}_4$ in terms of its Fourier coefficients.

## 6. Heuristics

Coming up with Rankin-Selberg method integrals is as much of an art as it is a science. There is not, at present, any explanation for

(1) which $L$-functions can be represented by Rankin-Selberg integrals or
(2) whether or nor a given integral represents an $L$-function.

That being said, there are still some heuristics employed by practitioners that make it easier to come up with Rankin-Selberg integrals. In this final section, we explain a few of these heuristics.

6.1. **Normalizing the Eisenstein series.** Let's say you have some hypothetical Rankin-Selberg integral $I(\varphi, s) = \int_{H(\mathbf{Q})Z'(\mathbf{A})\backslash H(\mathbf{A})} E(h,s)\varphi(h)\, dh$. And let's say you're reasonable sure it represents the $L$-function $L(\pi, r, As+B)$, divided by some Dirichlet $L$-functions, for certain unknown constants $A, B$. How can you determine $A, B$ and these Dirichlet $L$-functions?

Here is the trick. Suppose the Eisenstein series that appears in your integral is a degenerate Eisenstein for a parabolic $P \subseteq G$. Assume for simplicity that $G$ is semisimple; for example, it might be an adjoint group. Then the character group of $P$ is one dimensional, say $X^*(P) = \mathbf{Z}\nu$. There are two choices for the generator $\nu$. Pin down $\nu$ so that the modulus character $\delta_P(p) = |\nu(p)|^{n_0}$ for a positive (as opposed to negative) integer $n_0$. Now let's say your Eisenstein series $E(g,s) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f(\gamma g, s)$ for a flat section $f(g,s) \in Ind_{P(\mathbf{A})}^{G(\mathbf{A})}(|\nu|^s)$.

The sum will converge absolutely for $Re(s) > n_0$ (this is theorem, not just a heuristic). By doing interwining operator calculations (I will omit explaining this aspect), one can find a product $\xi(s)$ of zeta functions so that $E^*(g,s) = \xi(s)E(g,s)$ satisfies a functional equation relating $s$ to $n_0 - s$. In other words, if $G$ is split and $f(g,s)$ is the flat spherical section, then $E^*(g,s) = E^*(g, n_0 - s)$.

Now, with this normalization, here is the heuristic: The Rankin integral $I(\varphi, s)$ will represent $\frac{L(\pi, r, s + \frac{1-n_0}{2})}{\xi(s)}$. So, by defining the Eisenstein series from a section in $Ind_P^G(|\nu|^s)$, the constant $A = 1$.

Also note that this expression is consistent with the expected functional equation of the $L$-function:

$$\begin{aligned} L(\pi, r, s + (1-n_0)/2) &= \xi(s) I(\varphi, s) \\ &= \int_{H(\mathbf{Q})Z'(\mathbf{A})\backslash H(\mathbf{A})} \varphi(h) E^*(h, s)\, dh \\ &= \int_{H(\mathbf{Q})Z'(\mathbf{A})\backslash H(\mathbf{A})} \varphi(h) E^*(h, n_0 - s)\, dh \\ &= L(\pi, r, n_0 - s + (1-n_0)/2) \\ &= L(\pi, r, 1 - (s + (1-n_0)/2)). \end{aligned}$$

6.2. **Counting the unfolding.** A powerful heuristic for helping to understand if a global integral unfolds is the following. One can assign a non-negative integer to each term in a Rankin-Selberg integral $\int_{H(\mathbf{Q})Z'(\mathbf{A})\backslash H(\mathbf{A})} \varphi(h)E(h,s)\theta(h)\, dh$, as follows:

(1) If the Eisenstein series $E(g,s) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f(\gamma g, s)$, then one assigns to this term the integer $\dim N_P$, the dimension of the unipotent radical of $P$. Because the variety $P\backslash G$ has dimension $\dim N_P$, one can think of this integer as counting "how big" the sum defining the Eisenstein series is. Call this integer $\dim(Eis)$.
(2) If $\theta(g) = \sum_{\gamma \in X(\mathbf{Q})} \xi(\gamma g)$ for some variety $X$ and some function $\xi$, then one assigns to $\theta$ the integer $\dim X$, again thinking of this integer as how big the sum defining $\theta$ is. Call this integer $\dim(\theta)$.
(3) Suppose $\varphi$ is a cusp form on a group $G'$, $R' \subseteq G'$ is an algebraic subgroup, and $\chi : R'(\mathbf{Q})\backslash R'(\mathbf{A}) \to \mathbf{C}^\times$ is a unitary character. The $\chi$ Fourier coefficient of $\varphi$ is defined as $\varphi_\chi(g) = \int_{R'(\mathbf{Q})\backslash R'(\mathbf{A})} \chi^{-1}(r)\varphi(rg)\, dr$. Rankin-Selberg integrals frequently unfold to $\chi$-Fourier coefficients. Let $\dim(\chi)$, the dimension of the Fourier coefficient, denote the dimension of the algebraic groups $R'$.
(4) Finally, set $\dim(H/\mathbf{Z}')$ to be the dimension of the algebraic group $Z'\backslash H'$.

With the above definitions, here is the Heuristic: If the global integral $I(\varphi, s)$ unfolds to the $\chi$ Fourier coefficient of $\varphi$, then

$$\dim(H/Z') = \dim(\chi) + \dim(Eis) + \dim(\theta).$$

A write-up of this heuristic appears in work of Friedberg-Ginzburg [FG21].

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