Cliques and sunflowers under bounded VC-dimension

Andrew Suk (UC San Diego)

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Set system $\mathcal{F} \subset 2^V$.

**Definition**

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$.

**Definition**

The **VC-dimension** of $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.
$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$. 

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**Definition**

The VC-dimension of $G$ is the VC-dimension of $\mathcal{F}$. 

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Examples of graphs with bounded VC-dimension
Intersection graphs of segments in the plane.
1. Intersection graphs of segments in the plane.

2. Unit distance graph of points in $\mathbb{R}^d$. 
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   - $V =$ points in $\mathbb{R}^d$
   - $E =$ defined by bounded degree polynomial inequalities.
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**Problem**

*Can we substantially improve some of the classical theorems in extremal graph theory for graphs with bounded VC-dimension?*
1 **Ramsey’s Theorem.** Every graph on $n$ vertices contains a clique or independent set of size $c \log n$.

2 **Turán’s Theorem.** Every $K_{2,2}$-free graph on $n$ vertices has at most $cn^{3/2}$ edges.

3 **Szemerédi’s regularity lemma.**
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**Problem**

*Can we improve these classical results for graphs with bounded VC-dimension?*
An application of the Milnor-Thom theorem:

**Theorem**

There are at most \(2^{cn \log n}\) semi-algebraic graphs on \(n\) vertices and with complexity at most \(d\), where \(c = c(d)\).

**Theorem (Anthony, Brightwell, Cooper 1995)**

There are at least \(2^{n^{2-\varepsilon}}\) graphs with VC-dimension at most \(d\) on \(n\) vertices, where \(\varepsilon = \varepsilon(d)\).
2 Turán’s Theorem. Every $K_{2,2}$-free graph on $n$ vertices has at most $cn^{3/2}$ edges.

Semi-algebraic graphs: Improve to $O(n^{3/2 - \epsilon})$. 
Turán’s Theorem. Every $K_{2,2}$-free graph on $n$ vertices has at most $cn^{3/2}$ edges.

- **Semi-algebraic graphs.** Improve to $O(n^{3/2-\varepsilon})$.

- **Bounded VC-dimension.** No improvement. There are $K_{2,2}$-free graphs on $n$ vertices with $\Omega(n^{3/2})$ edges.
First main result

In joint work with Jacob Fox and János Pach

- We establish tight bounds for multicolor Ramsey numbers for graphs with bounded VC-dimension.
Definition

For $m \geq 2$, The multicolor Ramsey number

$$r(3, \ldots, 3)$$

$m$ times

is the minimum integer $N$ such that for any $m$-coloring of the edges of $K_N$ contains a monochromatic copy of $K_3$. 
Known results

\[ r(3, 3) = 6 \]
\[ r(3, 3, 3) = 17 \]
\[ 51 \leq r(3, 3, 3, 3) \leq 62 \]
\[ 162 \leq r(3, 3, 3, 3, 3) \leq 307 \]

\[ 2^m < r(3, \ldots, 3) < m! \]

\[ m \text{ times} \]
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\[ N/m \]

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Best known bounds

**Lower bound:** Fredricksen-Sweet, Abbot-Moser.

**Upper bound:** Schur.

\[(3.199)^m < r(3, \ldots, 3) < 2^{O(m \log m)}\]

\[m \text{ times}\]
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Conjecture (Schur-Erdős)

\[r(3, \ldots, 3) = 2^{\Theta(m)}.\]
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$r(3, \ldots, 3) = 2^{\Theta(m)}$.

**Theorem (Fox-Pach-S., 2020)**

*The conjecture of true for semi-algebraic colorings with bounded complexity.*
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**Lower bound:** Fredricksen-Sweet, Abbot-Moser.

**Upper bound:** Schur.

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\[m \text{ times}\]

**Conjecture (Schur-Erdős)**

\[r(3, \ldots, 3) = 2^{\Theta(m)} \]

\[m \text{ times}\]

**Theorem (Fox-Pach-S., 2021)**

*The conjecture holds is true for colorings with bounded VC-dimension.*
Bounded VC-dimension setting

Color all edges of $K_N$ with $m$ colors.

$K_N =$
Color all edges of $K_N$ with $m$ colors.

$v$

$K_N = v$
Bounded VC-dimension setting

Color all edges of $K_N$ with $m$ colors.

![Graph](image)

**Notation:** $N_i(v) = \{u \in V : \chi(uv) = i\}$. 
If we insist that the $m$-coloring has bounded VC-dimension:

$$\mathcal{F} = \{ N_i(v) : v \in V, i \in [m] \}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

**Theorem (Fox-Pach-S. 2021)**

For $m \geq 2$,

$$r_d(3, \ldots, 3)^m = 2^{\Theta(m)}.$$
If we insist that the $m$-coloring has bounded VC-dimension:

$$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

**Theorem (Fox-Pach-S. 2021)**

For fixed $p \geq 3$ and $m \geq 2$,

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\( F \) has VC-dimension at most \( d = O(1) \).

**Theorem (Fox-Pach-S. 2021)**

For fixed \( p \geq 3 \) and \( m \geq 2 \),

\[ r_d(p, \ldots, p) = 2^{\Theta(m)} \]  
\( m \) times
Sketch of the proof:

\[ r_d(3, \ldots, 3) \leq 2^{cm}, \quad c = c(d) \]

**Idea:** We will use induction on \( m \). Set \( N = 2^{cm} \) and let \( V \) be an \( N \)-element vertex set.

\[ \chi : \binom{V}{2} \to \{1, 2, \ldots, m\} \text{ and } \mathcal{F} = \{ N_i(v) : v \in V, i \in [m] \}. \]
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\[ K_N = \]

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Goal: \( \exists v \in V \) such that \( |N_i(v)| \geq \epsilon N \) for some \( i \).

\[ |N_i(v)| \geq \epsilon N > 2^{c(m-1)}. \]
**Goal:** \( \exists v \in V \text{ such that } |N_i(v)| \geq \epsilon N \text{ for some } i. \)

\[
|N_i(v)| \geq \epsilon N > 2^{c(m-1)}.
\]

**Not true:** We can only assume \( |N_i(v)| \geq N/m \) by pigeonhole.
\[ \mathcal{F} = \{ N_i(v) : v \in V, i \in [m] \}. \]

**Crossing:** Let \( A \in \mathcal{F} \) and \( u, v \in V \). Then \( A \) crosses \( \{u, v\} \) if it contains one but not the other.
\[ \mathcal{F} = \{ N_i(v) : v \in V, i \in [m] \}. \]

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\[ K_N = \]

\begin{tabular}{c}
- \( u \cdot \) \\
- \( v \cdot \)
\end{tabular}
Crossing pairs of vertices

\[ \mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}. \]

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\[ K_N = \]

\[ A \]

\[ u \cdot \]

\[ v \cdot \]

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\( \mathcal{F} = \{ N_i(v) : v \in V, i \in [m] \} \), dual VC-dimension \( d \).

**Lemma**

For any \( \delta \) satisfying \( 1 \leq \delta \leq |\mathcal{F}| \), there is an equipartition \( V = S_1 \cup \cdots \cup S_r \) of \( V \) into \( r \leq c(\frac{|\mathcal{F}|}{\delta})^d \) parts, such that any pair of vertices from the same part \( S_t \) is crossed by at most \( 2\delta \) members of \( \mathcal{F} \).
Partition Lemma

$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$, dual VC-dimension $d$.

Lemma

For any $\delta$ satisfying $1 \leq \delta \leq |\mathcal{F}|$, there is an equipartition $V = S_1 \cup \cdots \cup S_r$ of $V$ into $r \leq c(|\mathcal{F}|/\delta)^d$ parts, such that any pair of vertices from the same part $S_t$ is crossed by at most $2\delta$ members of $\mathcal{F}$.

$K_N =$
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![Graph](image-url)
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\[ K_N = \]

\[
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
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Key observation:

\[ K_N = \]
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Case 1. If a part is missing many colors, we are done by induction.
Case 2. Each part has many distinct colors

\[ K_N = \]

\[
\begin{array}{ccc}
\text{many} & \text{many} & \text{many} \\
\text{many} & \text{many} & \text{many} \\
\text{many} & \text{many} & \text{many} \\
\end{array}
\]
**Key observation:** Otherwise, many vertices have small neighborhoods with respect to some colors.
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**Goal:** Find a vertex with large degree with respect to one color class.
Theorem (Fox-Pach-S. 2021)

For \( d = O(1) \),

\[
rd(3, \ldots, 3) = 2^{\Theta(m)}.
\]

Theorem (Fredricksen-Sweet and Abbot-Moser, Schur)

\[
(3.199)^m < r(3, \ldots, 3) < 2^{O(m \log m)}.
\]

Question. For what other classes of graphs can we improve the \( 2^{O(m \log m)} \) upper bound?
Improvement: Intersection size of sets

\( \mathcal{F} \subset 2^X, \ m\text{-uniform.} \)

1. **Vertices:** \( V = \mathcal{F} \).

2. **Edge coloring:** For \( A, B \in \mathcal{F} \), color \((A, B)\) with color \( i \in \{0, 1, \ldots, m - 1\} \) if \( |A \cap B| = i \).
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**Question:** How large does \( |\mathcal{F}| \) have to be in order to guarantee a monochromatic \( K_3 \)?
\( \mathcal{F} \subset \binom{X}{m}, \) \( m \)-uniform.

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1. **Schur:** \( 2^{cm \log m} \).

2. **Alweiss-Lovett-Wu-Zhang:** \( 2^{cm \log \log m} \).
\[ V = \text{ground set} \]
\[ \mathcal{F} \subset \binom{V}{m}, \text{m-uniform}. \]

\[ A_1, \ldots, A_p \in \mathcal{F} \text{ for a } p\text{-sunflower if } A_i \cap A_j = A_k \cap A_\ell \]
\( V = \text{ground set} \)

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\( A_1, A_2, A_3 \in \mathcal{F} \) for a \textbf{3-sunflower} if \( A_i \cap A_j = A_k \cap A_\ell \)
$V$ = ground set

$\mathcal{F} \subset \binom{V}{m}$, $m$-uniform.

$A_1, A_2, A_3 \in \mathcal{F}$ for a 3-sunflower if $A_i \cap A_j = A_k \cap A_\ell$
\( V = \text{ground set} \)

\( \mathcal{F} \subset \binom{V}{m}, \ m\text{-uniform}. \)

\( A_1, A_2, A_3 \in \mathcal{F} \) for a 3-\textbf{sunflower} if \( A_i \cap A_j = A_k \cap A_\ell \)
Theorem (Erdős-Rado)

Let $\mathcal{F} \subset \binom{V}{m}$ that does not contain a 3-sunflower. Then

$$|\mathcal{F}| \leq m!2^m = 2^{O(m \log m)}.$$
Erdős-Rado sunflower conjecture

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\( d(v) \geq |\mathcal{F}|/(2m) \).
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Conjecture (Erdős-Rado)

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Let $\mathcal{F} \subset \binom{V}{m}$ that does not contain a 3-sunflower. Then

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Question: What if $\mathcal{F}$ is defined geometrically?
Theorem (Alweiss-Lovett-Wu-Zhang)

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Conjecture (Erdős-Rado)

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Question: What if $\mathcal{F}$ has bounded VC-dimension?
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$\mathcal{F} = 2^m$
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$\mathcal{F} = 2^{m-1}$, VC-dimension 1, no 3-sunflower.
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$\mathcal{F} = 2^{m-1}$, VC-dimension 1, no 3-sunflower.

---

**Can be realized geometrically:** $V =$ points in the plane, $\mathcal{F} =$ disks with $m$ points inside.
Second main result

Theorem (Fox-Pach-S. 2021)

Let $\mathcal{F} \subset \binom{V}{m}$, such that $\mathcal{F}$ has VC-dimension $d = O(1)$ and no 3-sunflower. Then

$$|\mathcal{F}| \leq 2^{O(m(2d)^2 \log^* m)}.$$
Theorem (Fox-Pach-S. 2021)

Let $\mathcal{F} \subset \binom{V}{m}$, such that $\mathcal{F}$ has VC-dimension $d = O(1)$ and no 3-sunflower. Then

$$|\mathcal{F}| \leq 2^{O(m(2d)^2 \log^* m)}.$$ 

Sketch of Proof. Induction on $m$.

Let $\mathcal{F} \subset \binom{V}{m}$ with VC-dimension at most $d$ and no 3-sunflower.

$$f_d(m) = 2^{cm(2d)^2 \log^* m}.$$ 

$$|\mathcal{F}| \leq f_d(m).$$
$\mathcal{F} \subset \binom{V}{m}$, VC-dimension $d$, no 3-sunflower, $|\mathcal{F}| > f_d(m)$.
\[ \mathcal{F} \subset \binom{V}{m}, \text{ VC-dimension } d, \text{ no 3-sunflower, } |\mathcal{F}| > f_d(m). \]

\[ \mathcal{F} = \]

\[
S = s \text{ highest degree vertices, } s = 100m^2(f_d(\log m))^2.
\]
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\[
\mathcal{F} = S
\]

\[
\sum_{v \in S} d(v) \leq m|\mathcal{F}|
\]

\[
d(u) \leq (m/s)|\mathcal{F}|
\]
\[ d(u) \leq (m/s)|\mathcal{F}| \]

\[ \mathcal{F} = \]

A intersects at most \((m^2/s)|\mathcal{F}|\) outside of \(S\).
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\[ d(u) \leq (m/s)|\mathcal{F}| \]

At least \((1 - \frac{6m^2}{s})(|\mathcal{F}|^3)\) triples are pairwise disjoint outside of \(S\).
Case 1: For at least $|\mathcal{F}|/2$ sets $A \in \mathcal{F}$, $|A \cap S| \leq \log m$. 

$\mathcal{F} =$ 

$S$
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\[
\mathcal{F} = \{ A \cap S : |A \cap S| \leq \log m \}
\]
Case 1: For at least $|\mathcal{F}|/2$ sets $A \in \mathcal{F}$, $|A \cap S| \leq \log m$.

$\mathcal{F} = \{ A \cap S : |A \cap S| \leq \log m \}$

1. $\mathcal{F}'$ is a multiset system
2. $|\mathcal{F}'| \geq |\mathcal{F}|/2$
By induction: $|\mathcal{F}'| > 2^{cm(2d)^2 \log^* m}$, sets of size at most $\log m$.

Lemma

There are at least

$$\frac{1}{(f_d(\log m))^2 \left( \frac{|\mathcal{F}'|}{3} \right)}$$

triples that form a 3-sunflower in $S$. 

Andrew Suk (UC San Diego)  Cliques and sunflowers under bounded VC-dimension
By induction: $|F'| > 2^{cm(2d)^2 \log^* m}$, sets of size at most $\log m$.

$F = \begin{array}{c}
\end{array}$

Lemma

There are at least

$$\frac{1}{8(f_d(\log m))^2 \binom{|F|}{3}}$$

triples that form a 3-sunflower in $S$. 
By induction: $|\mathcal{F}'| > 2^{cm(2d)^2 \log^* m}$, sets of size at most $\log m$.

\[
\mathcal{F} =
\]

**Lemma**

There are at least

\[
\frac{1}{8(f_d(\log m))^2} \binom{|\mathcal{F}|}{3}
\]

triples that form a 3-sunflower in $S$. 
\[ s = 100m^2(f_d(\log m))^2. \]

\[ \mathcal{F} = \]

At least \( \frac{1}{8(f_d(\log m))^2}(|\mathcal{F}|) \) 3-sunflowers in \( S \).

At least \( (1 - \frac{6m^2}{s})(|\mathcal{F}|) \) triples are pairwise disjoint outside of \( S \).
Case 2: For at least $|\mathcal{F}|/2$ sets $A \in \mathcal{F}$, $|A \cap S| > \log m$. 

\[
\mathcal{F} = \quad S
\]
Case 2: For at least $|\mathcal{F}|/2$ sets $A \in \mathcal{F}$, $|A \cap S| > \log m$. 
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\[ \mathcal{F} = \]

\textbf{Sauer-Shelah:} At most $s^d$ distinct intersections with $S$. 
Case 2: For at least \(|\mathcal{F}|/2\) sets \(A \in \mathcal{F}\), \(|A \cap S| > \log m\).

\[ \mathcal{F} = \]

Sauer-Shelah: At most \(s^d\) distinct intersections with \(S\).

\(\exists S' \subset S, |S'| > \log m\).
Case 2: For at least $|\mathcal{F}|/2$ sets $A \in \mathcal{F}$, $|A \cap S| > \log m$.

Sauer-Shelah: at least $|\mathcal{F}|/(2s^d)$ sets $A$, $A \cap S = S'$. 
**Sauer-Shelah:** at least $|\mathcal{F}|/(2s^d)$ sets $A$, $A \cap S = S'$. 

\[
\mathcal{F} = \frac{|\mathcal{F}|}{2s^d} \leq f_d(m - \log m) \leq 2^{c(m - \log m)(2d)^2 \log^* m}
\]
Sauer-Shelah: at least $|\mathcal{F}|/(2s^d)$ sets $A$, $A \cap S = S'$. 

$$\mathcal{F} =$$

$$|\mathcal{F}| \leq 2^{cm(2d)^2 \log^* m} = f_d(m).$$
**Sauer-Shelah:** at least $|\mathcal{F}|/(2s^d)$ sets $A$, $A \cap S = S'$.

\[ f_d(m) < |\mathcal{F}| \leq 2^{cm(2d)^2 \log^* m} = f_d(m). \]

□
Open problems

Theorem (Fox-Pach-S. 2021)

Let $\mathcal{F} \subset {V \choose m}$, such that $\mathcal{F}$ has VC-dimension $d = O(1)$ and no 3-sunflower. Then

$$|\mathcal{F}| \leq 2^{O(m(2d)^2 \log^* m)}.$$

Questions

1. Semi-algebraic setting? I.e., points in spheres in $\mathbb{R}^d$.
2. (Weak delta-system) What about 3 sets that pairwise intersect with the same size?
3. Multicolor Ramsey numbers: What if each color class has bounded VC-dimension?
Thank you!
\[ s = 100m^2 (f_d(\log m))^2 = (100m^2) 2^{2c \log m (2d)^2 (\log^* m - 1)} . \]

\[ |\mathcal{F}| \leq 2s^d 2^{cm(2d)^2 \log^* m - c \log m (2d)^2 \log^* m} \]

\[ \leq 2^{cm(2d)^2 \log^* m} . \]