Ramsey results for graphs with bounded VC-dimension

Andrew Suk (UC San Diego)

February 6, 2020
Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

Definition

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$. 

$\mathcal{F} =$
Definition: VC-dimension

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$\mathcal{F} =$ [Diagram of shattered set system]
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Set system $\mathcal{F} \subseteq 2^V$, $|V| = n$.

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Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

**Definition**

A set $S \subset V$ is **shattered** by $\mathcal{F}$ if for all $X \subset S$, there is an $A \in \mathcal{F}$ such that $S \cap A = X$. 

$$\mathcal{F} = \{\text{sets in } V\text{, different sizes and shapes}\}$$
Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

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Definition: VC-dimension

Set system $\mathcal{F} \subset 2^V$, $|V| = n$.

**Definition**

The **VC-dimension of** $\mathcal{F}$ is the size of the largest subset $S \subset V$ that is shattered by $\mathcal{F}$.
\[ G = (V, E), \text{ let } \mathcal{F} \subset 2^V \text{ such that } \mathcal{F} = \{ N(v) : v \in V \}. \]

\[ |V| = |\mathcal{F}| = n \]
$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$. 
$|V| = |\mathcal{F}| = n$
$G = (V, E)$, let $\mathcal{F} \subset 2^V$ such that $\mathcal{F} = \{N(v) : v \in V\}$. $|V| = |\mathcal{F}| = n$
VC-dimension of a graph

Let $G = (V, E)$ be a graph, and let $\mathcal{F} \subseteq 2^V$ such that $\mathcal{F} = \{ N(v) : v \in V \}$. Then $|V| = |\mathcal{F}| = n$.

**Definition**

The VC-dimension of $G$ is the VC-dimension of $\mathcal{F}$. 

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Intersection graphs of segments in the plane.
Examples of graphs with bounded VC-dimension

1. Intersection graphs of segments in the plane.

2. Unit distance graph of points in $\mathbb{R}^d$. 
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3. **Semi-algebraic graphs** with bounded complexity.
   - $V =$ points in $\mathbb{R}^d$
   - $E =$ defined by bounded degree polynomial inequalities.
Examples of graphs with bounded VC-dimension

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   - $V = \text{points in } \mathbb{R}^d$
   - $E = \text{defined by bounded degree polynomial inequalities.}$

4. Intersection graph of pseudo-segments in the plane.
Examples of graphs with bounded VC-dimension

1. Intersection graphs of segments in the plane.

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3. **Semi-algebraic graphs** with bounded complexity.
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**Problem**

*Can we substantially improve some of the classical theorems in extremal graph theory for graphs with bounded VC-dimension?*
1. **Ramsey’s Theorem.** Every graph on \( n \) vertices contains a clique or independent set of size \( c \log n \).

2. **Turán’s Theorem.** Every \( K_{2,2} \)-free graph on \( n \) vertices has at most \( cn^{3/2} \) edges.

3. **Szemerédi’s regularity lemma.**
1 **Ramsey’s Theorem.** Every graph on \( n \) vertices contains a clique or independent set of size \( c \log n \).

- **Semi-algebraic graphs:** Improve to \( n^c \).

2 **Turán’s Theorem.** Every \( K_{2,2} \)-free graph on \( n \) vertices has at most \( cn^{3/2} \) edges.

- **Semi-algebraic graphs:** Improve to \( O(n^{3/2-\varepsilon}) \).

3 **Szemerédi’s regularity lemma.**

- **Semi-algebraic graphs:** Quantitative and qualitative improvements.
An application of the Milnor-Thom theorem:

**Theorem**

There are at most $2^{cn \log n}$ semi-algebraic graphs on $n$ vertices and with complexity at most $d$, where $c = c(d)$.

**Theorem (Anthony, Brightwell, Cooper 1995)**

There are at least $2^{n^{2-\varepsilon}}$ graphs with VC-dimension at most $d$ on $n$ vertices, where $\varepsilon = \varepsilon(d)$. 

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Turán’s Theorem. Every $K_{2,2}$-free graph on $n$ vertices has at most $cn^{3/2}$ edges.

- **Semi-algebraic graphs:** Improve to $O(n^{3/2-\varepsilon})$. 
Turán’s Theorem. Every $K_{2,2}$-free graph on $n$ vertices has at most $cn^{3/2}$ edges.

- Semi-algebraic graphs. Improve to $O(n^{3/2-\varepsilon})$.

- Bounded VC-dimension. No improvement. There are $K_{2,2}$-free graphs on $n$ vertices with $\Omega(n^{3/2})$ edges.
Two Ramsey-type results

In joint work with Jacob Fox and János Pach

1. We find large cliques or independent sets in graphs with bounded VC-dimension.

2. We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension.
Two Ramsey-type results

In joint work with Jacob Fox and János Pach

1. We find large cliques or independent sets in graphs with bounded VC-dimension (Erdős-Hajnal conjecture).

2. We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension.
The Erdős-Szekeres 1935

Every graph on $n$-vertices contains a clique or an independent set of size $\Omega(\log n) \approx e^{\log \log n}$.

The Erdős-Hajnal 1989

For every (fixed) graph $H$, there is a constant $c = c(H)$ such that the following holds. Every graph on $n$-vertices that does not contain $H$ as an induced subgraph contains a clique or an independent set of size $e^{c\sqrt{\log n}}$.

Conjecture (Erdős-Hajnal): Improve this to $n^c$, where $c = c(H)$. 

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\( H = \) forbidden induced graphs. The Erdős-Hajnal conjecture holds for

1. \( H \) with at most 3 vertices.
2. \( H \) has 4 vertices (Gyárfás 1997).
3. "blow ups" (Alon-Pach-Solymosi 2001)
4. \( H \) is a bull (5 vertices, 5 edges, Chudnovsky-Safra 2008)

**OPEN:** \( H = C_5 \). Recently improved to \( e^{c\sqrt{\log n \log \log n}} \) by Chudnovsky-Fox-Scott-Seymour-Spirkl.
Let $G = (V, E)$, $|V| = n$, with VC-dimension less than $d$. 

$G = \begin{array}{c}
\end{array}$
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$G =$

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Let $G = (V, E)$, $|V| = n$, with VC-dimension less than $d$. 

$G = \begin{array}{c}
\begin{tikzpicture}
  \draw [thick, red] (0,0) -- (1,1);
  \draw [thick, red] (0,0) -- (1,-1);
  \draw [thick, red] (1,1) -- (1,-1);
  \draw [thick, red] (0,0) -- (0,2);
  \draw [thick, red] (1,1) -- (1,2);
  \draw [thick, red] (1,-1) -- (1,-2);

  \draw [thick, red] (2,0) -- (3,1);
  \draw [thick, red] (2,0) -- (3,-1);
  \draw [thick, red] (3,1) -- (3,-1);
  \draw [thick, red] (2,0) -- (2,2);
  \draw [thick, red] (3,1) -- (3,2);
  \draw [thick, red] (3,-1) -- (3,-2);

  \node at (0,0) [circle,fill,inner sep=2pt] {}; 
  \node at (1,0) [circle,fill,inner sep=2pt] {}; 
  \node at (2,0) [circle,fill,inner sep=2pt] {}; 
  \node at (3,0) [circle,fill,inner sep=2pt] {}; 

  \node at (0,1) [circle,fill,inner sep=2pt] {}; 
  \node at (1,1) [circle,fill,inner sep=2pt] {}; 

  \node at (0,-1) [circle,fill,inner sep=2pt] {}; 
  \node at (1,-1) [circle,fill,inner sep=2pt] {}; 

  \node at (0,2) [circle,fill,inner sep=2pt] {}; 
  \node at (1,2) [circle,fill,inner sep=2pt] {}; 

  \node at (2,2) [circle,fill,inner sep=2pt] {}; 
  \node at (3,2) [circle,fill,inner sep=2pt] {}; 

  \node at (2,-2) [circle,fill,inner sep=2pt] {}; 
  \node at (3,-2) [circle,fill,inner sep=2pt] {}; 
\end{tikzpicture}
\end{array}$
Let $G = (V, E)$, $|V| = n$, with VC-dimension less than $d$. 

$G =$
Let $G = (V, E)$, $|V| = n$, with VC-dimension less than $d$.

There is a graph $H$ on $d + 2^d$ vertices such that $H$ is not an induced subgraph of $G$. 
Theorem (Erdős-Hajnal 1989)

For every (fixed) graph $H$, there is a constant $c = c(H)$ such that the following holds. Every graph on $n$-vertices that does not contain $H$ as an induced subgraph contains a clique or an independent set of size $e^{c\sqrt{\log n}}$.

Corollary

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{c\sqrt{\log n}}$, where $c = c(d)$.

Conjecture: Improve the result above to $n^c = e^{c\log n}$, where $c = c(d)$. 
**First Ramsey-type result**

Theorem (Erdős-Hajnal 1989)

For every (fixed) graph $H$, there is a constant $c = c(H)$ such that the following holds. Every graph on $n$-vertices that does not contain $H$ as an induced subgraph contains a clique or an independent set of size $e^{c\sqrt{\log n}}$.

Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Here, $o(1) = c\frac{\log d}{\log \log n}$. 

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Ramsey results for graphs with bounded VC-dimension
Let $H$ be a bipartite graph, $|V(H)| = k$.

$$H =$$

**Corollary (Fox-Pach-S. 2019)**

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H$ as an induced bipartite graph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof:** $G$ has VC-dimension at most $d = d(H)$. 

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**Note:** We are forbidding $2^{O(k^2)}$ induced subgraphs.
Let $H$ be a bipartite graph, define $H_1, H_2, H_3, H_4$ as follows.

$$H =$$

Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$. 

Andrew Suk (UC San Diego) Ramsey results for graphs with bounded VC-dimension
Let $H$ be a bipartite graph, define $H_1, H_2, H_3, H_4$ as follows.

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Let $H$ be a bipartite graph, define $H_1, H_2, H_3, H_4$ as follows.

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Let $H$ be a bipartite graph, define $H_1, H_2, H_3, H_4$ as follows.

$$H_3 =$$

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Let $H$ be a bipartite graph, define $H_1, H_2, H_3, H_4$ as follows.

$$H_4 = \quad \begin{array}{c}
\quad \end{array}$$

**Corollary (Fox-Pach-S. 2019)**

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$. 
Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof: $G$ has VC-dimension less than $d = d(H)$. 
Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof:** For contradiction, suppose VC-dimension is at least $d$. 

\[
G = 
\]
Corollary (Fox-Pach-S. 2019)

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![Graph Image]
Corollary (Fox-Pach-S. 2019)

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Proof: For contradiction, suppose VC-dimension is at least $d$. 

\[
G = \begin{array}{c}
\begin{array}{c}
\text{d}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{2}^d
\end{array}
\end{array}
\]
Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

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![Diagram of graph $G = d \times 2^d$]
Corollary (Fox-Pach-S. 2019)

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$$G = \begin{array}{c} d \\ \vdots \end{array}$$

$$2^d$$
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Proof: By the Erdős-Hajnal theorem:

$$G = \begin{array}{c} d \\ \vdots \\ 2^d \end{array}$$
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Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof:** By the Erdős-Hajnal theorem:

\[
G = \begin{array}{cc}
\bullet & \bullet \\
\bullet & \bullet \\
\bullet & \bullet \\
\end{array} \quad \begin{array}{c}
2^d \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

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Corollary (Fox-Pach-S. 2019)

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$$G = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{d}
\end{array}
\end{array}
\end{array}$$

$$2^{c\sqrt{d}}$$
Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

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Proof: Sauer-Shelah: $|\mathcal{F}| > c|V|^k$ implies $k$-set shattered.
Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

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Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof:** $c' \sqrt{d}$ vertices are shattered.

$$G = c'\sqrt{d}$$
Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{\left(\log n\right)^{1-o(1)}}$.

**Proof:** By Erdős-Hajnal (again)

\[ G = \begin{array}{c}
\vdots \\
\hline
\end{array} \]

\[ 2^{c\sqrt{d}} \]
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Corollary (Fox-Pach-S. 2019)

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**Proof:** The $2^{c'' \log d}$ set is shattered.

\[
G = 2^{c'' \sqrt{\log d}} \quad \text{and} \quad 2^{c \sqrt{d}}
\]
Corollary (Fox-Pach-S. 2019)

Let $H$ be a fixed bipartite graph. Then every $n$-vertex graph that does not contain $H_1, H_2, H_3, H_4$ as an induced subgraph, contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

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\[ G = \]

\[ 2^{c'' \sqrt{\log d}} \]

\[ 2^{c \sqrt{d}} \]
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**Proof:** For large $d = d(H)$, we obtain $H_1, H_2, H_3, \text{ or } H_4$. 

\[ G = \begin{array}{c|c}
2^{c'' \sqrt{\log d}} & \end{array} \]

\[ 2^{c\sqrt{d}} \]
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$. 
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) $G$ is $H$ induced free. Use induction to find a large perfect subgraph.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) Lemma: $\exists A, B \subseteq V$

$G = \begin{array}{c}
A \\
B
\end{array}$
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** (Erdős-Hajnal) Lemma: $\exists A, B \subset V$

$$G = \begin{array}{cc}
A & \varepsilon n \\
\varepsilon n & B
\end{array}$$
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) Lemma: $\exists A, B \subset V$
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) $v \in A$, $|N(v) \cap B| > (1 - \varepsilon)|B|$ or $|N(v) \cap B| < \varepsilon|B|$
Theorem (Fox-Pach-S. 2019)

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**Proof idea.** (Erdős-Hajnal) Apply induction in $A$. 

![Diagram of a graph $G = A \cup B$]
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) Apply induction in $A$. 

\[ G = \]
Theorem (Fox-Pach-S. 2019)

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$G = A \cup B$
Theorem (Fox-Pach-S. 2019)

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**Proof idea.** (Erdős-Hajnal) Apply induction in $B$. 

\[ G = \]

\[ A \]

\[ B \]

Andrew Suk (UC San Diego)  Ramsey results for graphs with bounded VC-dimension
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. (Erdős-Hajnal) Apply induction in $B$. Combine to get a large perfect graph.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** $G$ has bounded VC-dimension.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Apply a strong regularity lemma.
Theorem (Fox-Pach-S. 2019)

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Proof idea. Apply a strong regularity lemma.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. At least $(1 - \varepsilon)|V_i||V_j|$ edges
**Theorem (Fox-Pach-S. 2019)**

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** Less than $\varepsilon |V_i||V_j|$ edges
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** True for all but an $\varepsilon$-fraction pairs of parts.
Theorem (Fox-Pach-S. 2019)

For every \( n \)-vertex graph with VC-dimension at most \( d \) contains a clique or independent set of size \( e^{(\log n)^{1-o(1)}} \).

**Proof idea.** Apply Turan’s theorem

\[
G = \begin{array}{c}
V_1 \\
V_2 \\
V_3 \\
V_4 
\end{array}
\]
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** Apply Turan’s theorem
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Follow the Erdős-Hajnal argument with many parts.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Apply induction inside one of the parts.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** Complete or empty bipartite graphs.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

Proof idea. Complete or empty bipartite graphs.

\[ G = \]

\begin{figure}
\centering
\begin{tikzpicture}
\draw (0,0) rectangle (2,2);
\draw (0.5,0.5) rectangle (1.5,1.5);
\draw (0.5,1.5) rectangle (1.5,0.5);
\draw (1.5,0.5) rectangle (2.5,1.5);
\draw (1.5,1.5) rectangle (2.5,0.5);
\end{tikzpicture}
\end{figure}
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** Complete or empty bipartite graphs.
Theorem (Fox-Pach-S. 2019)

For every $n$-vertex graph with VC-dimension at most $d$ contains a clique or independent set of size $e^{(\log n)^{1-o(1)}}$.

**Proof idea.** Repeat the argument.
Two Ramsey-type results

In joint work with Jacob Fox and János Pach

• We find large cliques or independent sets in graphs with bounded VC-dimension (Erdős-Hajnal conjecture).

• We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension.
Two Ramsey-type results

In joint work with Jacob Fox and János Pach

1. We find large cliques or independent sets in graphs with bounded VC-dimension (Erdős-Hajnal conjecture).

2. We find large monochromatic cliques in multi-colored graphs with bounded VC-dimension (Schur-Erdős conjecture).
Multicolor Ramsey numbers

Color all edges of $K_N$ with $m$ colors.

**Question:** How large does $N$ have to be to guarantee a monochromatic $K_3$?

$$K_N =$$
Multicolor Ramsey numbers

Color all edges of $K_N$ with $m$ colors.

**Question:** How large does $N$ have to be to guarantee a monochromatic $K_3$?
Color all edges of $K_N$ with $m$ colors.

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Color all edges of $K_N$ with $m$ colors.

**Question:** How large does $N$ have to be to guarantee a monochromatic $K_3$?

**Notation:**

$$N_i(v) = \{u \in V : \chi(uv) = i\}.$$
Definition

For $m \geq 2$, The multicolor Ramsey number

$$r(3, \ldots, 3)$$

$m$ times

is the minimum integer $N$ such that for any $m$-coloring of the edges of $K_N$ contains a monochromatic copy of $K_3$. 
Known results

\[ r(3, 3) = 6 \quad r(3, 3, 3) = 17 \quad 51 \leq r(3, 3, 3, 3) \leq 62 \]

\[ 162 \leq r(3, 3, 3, 3, 3) \leq 307 \]

\[ 2^m < r(3, \ldots, 3) < m! \]

\[ m \text{ times} \]
Known results

\[ r(3, 3) = 6 \quad r(3, 3, 3) = 17 \quad 51 \leq r(3, 3, 3, 3) \leq 62 \]

\[ 162 \leq r(3, 3, 3, 3, 3) \leq 307 \]

\[ 2^m < r(3, \ldots, 3) < m! \]

\[ m \text{ times} \]

\[ 2^{m-1} \quad 2^{m-1} \]
Known results

\[ r(3, 3) = 6 \quad r(3, 3, 3) = 17 \quad 51 \leq r(3, 3, 3, 3) \leq 62 \]

\[ 162 \leq r(3, 3, 3, 3, 3) \leq 307 \]

\[ 2^m < r(3, \ldots, 3) < m! \]

\[ \underbrace{m \times \ldots \times m}_{m \text{ times}} \]
Known results

\[ r(3, 3) = 6 \]
\[ r(3, 3, 3) = 17 \]
\[ 51 \leq r(3, 3, 3, 3) \leq 62 \]
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\[ 2^m < r(3, \ldots, 3) < m! \]

\( m \) times

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Best known bounds

**Lower bound:** Fredricksen-Sweet, Abbot-Moser.

**Upper bound:** Schur.

\[(3.199)^m < r(3, \ldots, 3) < 2^{O(m \log m)}\]
Erdős prize problems

Problem ($100$)

Is the limit below finite or infinite?

\[ \lim_{m \to \infty} \left( \binom{r(3, \ldots, 3)}{m \text{ times}} \right)^{1/m} \]

Problem ($250$)

Determine

\[ \lim_{m \to \infty} \left( \binom{r(3, \ldots, 3)}{m \text{ times}} \right)^{1/m} \]
If we insist that the $m$-coloring has bounded VC-dimension:

$$\mathcal{F} = \{N_i(v) : v \in V, i \in [m]\}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

**Theorem (Fox-Pach-S. 2020)**

For $m \geq 2$,

$$r_d(3, \ldots, 3) = 2^{\Theta(m)}.$$
If we insist that the $m$-coloring has bounded VC-dimension:

$$\mathcal{F} = \{ N_i(v) : v \in V, i \in [m] \}$$

$\mathcal{F}$ has VC-dimension at most $d = O(1)$.

**Theorem (Fox-Pach-S. 2020)**

*For fixed $p \geq 3$ and $m \geq 2$,*

$$r_d(p, \ldots, p) = 2^{\Theta(m)}.$$
If we insist that the \( m \)-coloring has bounded VC-dimension:

\[
\mathcal{F} = \{ N_i(v) : v \in V, i \in [m] \}
\]

\( \mathcal{F} \) has VC-dimension at most \( d = O(1) \).

**Theorem (Fox-Pach-S. 2020)**

For fixed \( p \geq 3 \) and \( m \geq 2 \),

\[
r_d(p, \ldots, p) = 2^{\Theta(m)}.
\]

**Proof idea:** Use a different partition result based on Haussler’s packing lemma.
Open Problems

1. Is the Erdős-Hajnal conjecture true for string graphs.
2. Find more results for graphs with bounded VC dimension.
Thank you!