The obvious analogue of the Large Ramsey theorem does not translate to Van der Waerden

Andy Parrish

1 Introduction

Consider a set $A \subseteq N$ to be *large* if $|A|$ is at least as large as the smallest element of A. A theorem of Paris and Harrington says that, for all $m \in \mathbb{N}$, there is a N so that any 2-coloring of the edges of $K_{\{m,\ldots,N\}}$ yields a large homogeneous set of vertices A.

Proof: Ramsey's theorem tells us that any 2-coloring of the edges of $K_{\{m,m+1,\ldots\}}$ yields an infinite homogeneous set $A \subseteq \mathbb{N}$. As an infinite set, A is clearly large.

Suppose by way of contradiction that, for each N, there is a red/blue-coloring χ_N of the edges of $K_{\{m,\ldots,N\}}$ so that no large set is homogeneous. We will see how to use these to color the infinite complete graph without a large homogeneous set. Order the countably many edges of $K_{\{m,m+1,\ldots\}}$ as $\{e_1, e_2, \ldots\}$. We have infinitely many colorings χ_N which avoid large homogeneous sets, so one of the two colors — say red — occurs infinitely often for e_1. Color e_1 red, and throw out those colorings which color e_1 blue.

We still have infinitely many colorings, so one color occurs infinitely often for e_2 — say it's blue. Color e_2 blue, and again throw out those colorings which disagree with this. Continue for each edge. At each stage we maintain an infinite list of colorings which avoid large homogeneous sets and extend our current partial coloring, so we will be able to continue indefinitely. Therefore, this process colors all edges of $K_{\{m,m+1,\ldots\}}$, so it yields an infinite homogeneous set A. Truncate this set to A' finite but still large — if $k \in A$ is the minimum element, A' should be the first k elements of A. Let $L \in A'$ be the largest element. By construction, we know some coloring χ_N (in fact infinitely many) agrees with this coloring of $K_{\{m,\ldots,L\}}$, and hence yields the same large homogeneous set A'. This contradicts our initial assumption that χ_N admitted no such set. Thus, there must be some N with the property that any 2-coloring of the edges of $K_{\{m,\ldots,N\}}$ yields a large homogeneous set. ∎

The key to this proof is that the finite case follows from the infinite. This is no accident: Paris' and Harrington's motivation for this theorem is that, in fact, it is independent from Peano arithmetic, so only an infinitary proof like the above will suffice.

2 Large van der Waerden

Van der Waerden's theorem tells us that, for every $k \in \mathbb{N}$, there is an N so that any 2-coloring of $[N]$ yields a monochromatic k-term arithmetic progression (a k-AP). Unlike Ramsey's theorem, this fails in the infinite case — it is easy to 2-color $\mathbb{N}$ without an infinite arithmetic progression. It is natural to ask, then, whether a "large" form of van der Waerden's theorem holds. The Paris-Harrington result seems to suggest it will not. Indeed, the obvious analogue of the Large Ramsey theorem fails.

Theorem 2.1 *The numbers $\{3, 4, \ldots\}$ may be 2-colored without a large monochromatic arithmetic progression.*

Proof: First we present the coloring. We color by solid blocks, of length 1, 1, 2, 4, 8, 16, etc, to look like this (using 0 and 1 as our colors):

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Color:    0100111100000000111111111111111110...
          ||| |   |       |               |
Position: 345 7   11      19              35
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Explicitly, we color the number 3 with 0, and 4 with 1. From then on, for each $n \geq 0$, we have a block of length $2 \cdot 4^n$ of color 0 beginning at $3 + 2 \cdot 4^n$, and a block of length 4^{n+1} of color 1 beginning at $3 + 4^{n+1}$. This coloring has one key property:

$$\text{The length of each block is the total number of points preceding it.} \tag{1}$$

Now that we have a coloring, suppose that $A = \{a, a+d, \ldots, a+(k-1)d\}$ is a monochromatic k-AP. That this AP is not large will be seen from the following facts:

Fact 1 If $a = 3$, then $k \leq 2$
Proof: The first three terms would be $3, 3+d, 3+2d$, all colored 0. For $3+d$ to be in a block of 0s, we must have some n such that $3 + 2 \cdot 4^n \leq 3 + d < 3 + 4^{n+1}$. This lets us bound $3 + 2d$ by doubling all parts and subtracting 3. This says $3 + 4^{n+1} \leq 3 + 2d < 3 + 2 \cdot 4^{n+1}$, which says the third entry is in a block of 1s, so it is not a monochromatic AP.

Fact 2 If $k \geq a$ (ie if A is large), then A is not contained in a single block.
Proof: The block beginning at 3 is too short by 2. All the rest are too short by 3.

Fact 3 $\{a+d, \ldots, a+(k-1)d\}$ are all in the same block.
Proof: Pick any two terms a and $a+d$ of the same color to begin the AP. This fixes the common difference as d. By property (1), d is smaller than the length of the next block, which has a different color from a and $a+d$. This means that the next term, and indeed all remaining terms must be in the same block as $a+d$, as there is no way to "jump over" the next block.

Fact 4 If a and $a+d$ are in different blocks, then $k \leq 3$.
Proof: The length of the block before $a+d$ is half the length of the one holding $a+d$, so d is at least this large. This means at most one more term will fit in this block. Fact 3 shows that this ends the AP.

Putting these facts together, fact 1 says a large monochromatic AP could not begin at 3. Moreover, from facts 2 and 3, any such AP must have a in one block, and $a+d, \ldots, a+(k-1)d$ all in one other. Finally, fact 4 tells us a monochromatic AP of this form must be of length 3, so it is not large afterall.

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The remaining question is whether a theorem along these lines does hold, for a weaker definition of "large."