Math 20A Summer Bridge 2021: Homework 2

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Due Tuesday, August 17, 11:59 pm.

Remark. Problems written as "Exercise X.Y.Z" are from the textbook, section X.Y exercise Z. For example, Exercise 2.2.8 denotes Exercise 8 of section 2.2. For problems referring to a figure, find the question in the textbook for the corresponding figure.

Remark. You can apply any theorem or rule (as long as the problem does not say explicitly to not use that theorem or rule), but make sure to show that the assumptions of said theorem or rule apply.

Problem 1 Exercise 2.6.8

Evaluate using the squeeze theorem:

$$\lim_{x\to 0} \left[\tan(x)\cos(\sin(1/x))\right].$$

(Make sure to show that the assumptions of the squeeze theorem hold, and to prove the limit value of the functions that you are using to bound from above and below).

Hint: With $f(x) = \tan(x)\cos(\sin(1/x))$, one has $|f(x)| = |\tan(x)\cos(\sin(1/x))| = |\tan(x)||\cos(\sin(1/x))|$; then, use that $|\cos(\theta)| \leq 1$ for any θ.

Problem 2 Existence of roots by the IVT

Using the intermediate value theorem (IVT), prove the following: Any positive number $c > 0$ has an n^{th} root for any positive integer n.

Hint: We are trying to prove that there exists some $d \in \mathbb{R}$ such that $d^n = c$. Consider the (continuous) function $f(x) = x^n$. Show that there is a value a such that $f(a) < c$ and a value b such that $c < f(b)$. The conclusion then follows from the IVT. To find such an a, find a number such that when you raise it to the n^{th} power, it is less than any positive number c. To find such a b, use the fact that $\lim_{x\to\infty} f(x) = \infty$ and the definition of a limit being infinite.

Problem 3 Exercise 2.8.14

Using the IVT, show that $\cos(x) = \cos^{-1}(x)$ has a solution in $(0, 1)$.

Hint: Consider the function $f(x) = \cos(x) - \cos^{-1}(x)$. Check that f is a continuous function on $[0, 1]$ so that the IVT applies.

Problem 4 Using the IVT to show surjectivity

Consider $f : [-1, 1] \to [-1, 1]$ given by

$$f(x) = \begin{cases} x, & x < 0 \\ x^2, & x \geq 0 \end{cases}.$$

Using the IVT (and showing that the assumptions hold), show that this function is surjective.

(**Not graded**: in fact, this function is injective and so by the above, it is bijective. Try to also prove that this function is injective. Then, since it is bijective, it is invertible. Write a piecewise formula for the function $f^{-1}(y)$. This second part is not graded, but it might be good to do as preparation for the exam).

Problem 5 Exercise 3.1.30

Use the limit definition to compute $f'(a)$ and find an equation of the tangent line at $(a, f(a))$.

$$f(x) = 4 - x^2, \ a = -1.$$

Note: This problem asks you to compute the derivative using the limit definition directly. Do not use any differentiation laws such as the power law or sum law.

Problem 6 Exercise 3.2.10

Use the power rule to compute the derivative:

$$\frac{d}{dt}\left(t^{-2/5}\right)\Big|_{t=1}.$$

Problem 7 Exercise 3.2.18

Compute $f'(x)$ and find an equation of the tangent line to the graph at $x = a$:

$$f(x) = \sqrt[3]{x}, \ a = 8.$$

Problem 8 Exercise 3.2.46

Of the two functions f and g in Figure 12, which is the derivative of the other? Justify your answer.

Problem 9 Practice with the product rule

Using the product rule, evaluate the derivative

$$\frac{d}{dx}\Big[(x^4 + 2x^3 + 3x^2 + 2x + 1)(x^5 + 4x^3 + x^2 + x + 2)\Big].$$

Problem 10 Product of multiple functions

The product rule gives us a rule for differentiating a product of differentiable functions f and g. In fact, we can write a product rule for differentiating the product of any number of differentiable functions. In this problem, we will consider the product of three functions.

(a) Consider three differentiable functions f, g, h. Using the product rule for two functions, derive a formula for the derivative of the product of three functions:

$$\frac{d}{dx}(fgh).$$

To do this, first consider f as one function and the product gh as another function. Apply the product rule for two functions once, and then apply it again.

(b) Using your formula for the product rule for three functions, compute

$$\frac{d}{dx}(fgh),$$

where $f(x) = e^x,\ g(x) = x^3 + x^2 + 2x + 1,\ h(x) = x^4 + 3x^2 + x + 2.$

Problem 11 Exercise 3.3.8

Using the quotient rule, calculate df/dx where

$$f(x) = \frac{x+4}{x^2 + x + 1}.$$

Problem 12 More practice with the quotient rule

Compute $(dg/dt)|_{t=1}$ where

$$g(t) = \frac{t^4 + t^2}{\sqrt{t} + t + e^t}.$$

Problem 13 Rate of Change for Projectile Motion

The height h of a projectile moving vertically under the influence of gravity is given by

$$h(t) = h_0 + v_0 t + \frac{1}{2} g t^2,$$

where $h_0 = 0$ is the initial height ($h = 0$ is the height of the ground), $v_0 = 2$ is the initial velocity (directed upwards), and $g = -9.8$ is the acceleration by gravity.

(a) Find the maximum height that the object reaches, which occurs when the rate of change of the height (the velocity) is zero; i.e., it is the time t for which $h'(t) = 0$.

(b) What is the rate of change of the height, $h'(T)$, at the time T when the projectile hits the ground again.

(c) Graph both $h(t)$ and $h'(t)$ on the interval $[0, T]$ where T is the time when the projectile hits the ground again. Check that parts (a) and (b) agree with these graphs.

Problem 14 Exercise 3.5.10

Calculate y'' and y''' for

$$y(t) = 5t^{-3} + 7t^{-8/3}$$

(on the domain $t > 0$).

Problem 15 Exercise 3.5.40

The second derivative f'' is shown in Figure 7. Which of (A) or (B) is the graph of f and which is the graph of f'?

Problem 16 n^{th} derivative of a degree n monomial

(a) Compute the 5^{th} derivative

$$\frac{d^5}{dx^5} f(x)$$

of the degree 5 monomial $f(x) = x^5$.

(b) Generalizing the previous part to an arbitrary positive integer n, find a formula for

$$\frac{d^n}{dx^n} f(x),$$

where $f(x) = x^n$.

Problem 17 Exercise 3.7.18

Compute the derivative df/dx using derivative rules that have been introduced so far.

$$f(x) = (x^3 + 3x + 9)^{-4/3}.$$

Problem 18 Practice with Product, Quotient, and Chain Rule

Compute the derivative df/dx where

$$f(x) = \frac{(x^2 + x + 3)(x^5 + 4x^4 + x^3 + 1 + e^x)}{e^{x^3}(x^2 + 1)}.$$

Problem 19 Exercise 3.7.76

Compute the higher derivative

$$\frac{d^2}{dx^2}\Big[(x^2 + 9)^5\Big].$$