Lecture 10 - Extrema, MVT, Concavity

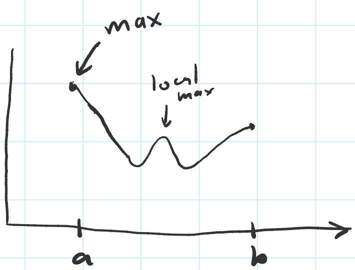
Monday, August 23, 2021 9:23 AM

minima - value of $f(c)$
minimizer - input c

· Read sections 4.2, 4.3, 4.4

Thm: If $f:[a,b]\to\mathbb{R}$ continuous, then a minimizer (or maximizer) c is either an endpoint $c=a$ or $c=b$ or a critical point.

Proof: If c is an endpoint, nothing to prove.
Then, if c isn't an endpoint, c is a minimizer for $f:(a,b)\to\mathbb{R}$, i.e. c is a local minimizer. $\Rightarrow$ c is a critical point.



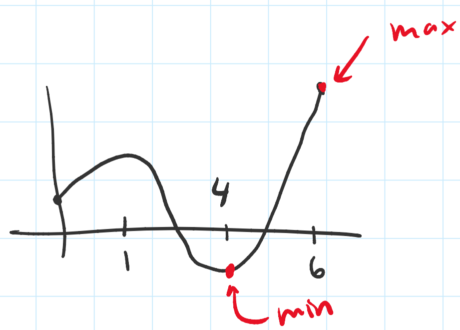
Have to check critical points & endpoints for $f:[a,b]\to\mathbb{R}$

ex/ Find the absolute minima & maxima of
$$f(x) = 2x^3 - 15x^2 + 24x + 7 \text{ on } [0,6].$$
Check $c=0$ & $c=6$ (endpoints of $[0,6]$).
Check critical points $f'(x)=0$ or ~~$f'(x) = DNE$~~. f is differentiable on $(0,6)$.

$$\begin{aligned} 0 = f'(x) &= 6x^2 - 30x + 24 = 6(x^2-5x+4) \\ &= 6(x-4)(x-1) \iff x=1 \text{ or } x=4. \end{aligned}$$

$\boxed{0,1,4,6}$

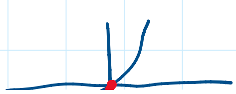
$f(0) = 7$
$f(1) = 18$
$f(4) = -9$ ← min.
$f(6) = 43$ ← max.



4.3

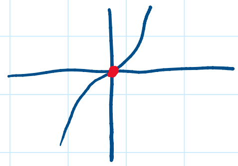
Motivation: When is a critical point a local min/max?

e.g., $f(x) = x^3$



$0 = f'(x) = 3x^2 \iff x = 0$

e.g., $f(x) = x^3$

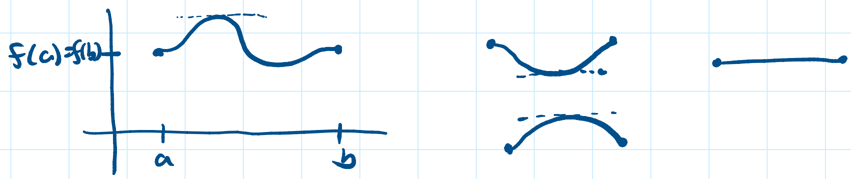


$0 = f'(x) = 3x^2 \Leftrightarrow x = 0$

$\Rightarrow$ 0 is a critical point of f but 0 is not a local min/max.

Rolle's Theorem:

- Let f be continuous on $[a,b]$ and differentiable on (a,b), and assume $f(a) = f(b)$, then there exists $c \in (a,b)$ s.t. $f'(c) = 0$



Proof: f achieves a min. & max by previous theorem, they are endpoints or critical points ($f'(c) = 0$).

- If both the min & max are endpoints, then f is constant $\Rightarrow f'(c) = 0$ for all $c \in (a,b)$.
- Otherwise, atleast one is a critical point $\Rightarrow$ there exists $c \in (a,b)$ s.t. $f'(c) = 0$.

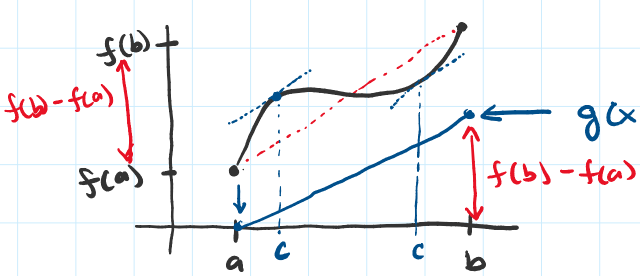
$\square$

The Mean Value Theorem (MVT)

- Assume f is continuous on $[a,b]$ and diff. on (a,b), then there exists $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

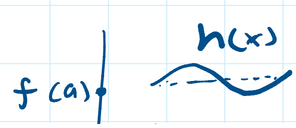
$\leftarrow$ slope of line connecting $(a, f(a))$ with $(b, f(b))$



$g(x) = \frac{f(b) - f(a)}{b-a}(x-a)$ Linear in x.

$g(a) = 0$

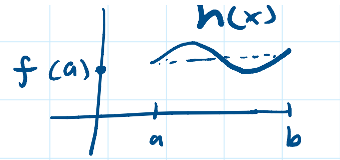
$g(b) = f(b) - f(a)$.



proof: Use Rolle's theorem

a c c b

$g(b) = f(b) - f(a).$



<u>proof</u>: Use Rolle's theorem

Consider

$$h(x) = \underline{f(x)} - \underline{g(x)} = f(x) - \frac{f(b)-f(a)}{b-a}(x-a).$$

$$h(a) = f(a) - \cancel{g(a)}^{0} = f(a)$$

$$\text{"} h(b) = f(b) - g(b) = f(b) - \big(f(b) - f(a)\big) = f(a)$$

Apply Rolle's theorem (h is cont. on $[a,b]$ & diff. on (a,b))

$\Rightarrow$ there exists $c \in (a,b)$ s.t. $h'(c) = 0$.

$$\Rightarrow \quad 0 = h'(c) = f'(c) - \frac{d}{dx}\left[\frac{f(b)-f(a)}{b-a}(x-a)\right]_{x=c}$$

$$= f'(c) - \frac{f(b)-f(a)}{b-a}.$$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}.$$

$\square$

<u>Corollary:</u>

If f is diff. on (a,b) and $\overline{f'(x) = 0}$, for all $x \in (a,b)$, then $f(x) = d$ for all $x \in (a,b)$, for some constant $d \in \mathbb{R}$.

$\llcorner$ constant.

<u>proof:</u> let $u, v \in (a,b)$, $u \neq v$, so assume $u > v$.

Consider the interval $[v, u]$, MVT applies

$\Rightarrow$ there exists $c \in (v,u)$ s.t.

$$\underset{\uparrow \text{ by assumption}}{0 =} f'(c) = \frac{f(u) - f(v)}{u - v} \iff f(u) = f(v).$$

$\Rightarrow f$ constant

$\square$

Monotonicity:

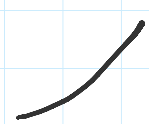
Definition:

Say f is <u>strictly</u> monotone

- increasing on (a,b) if

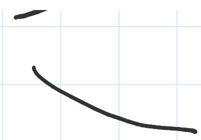
$$x > y \Rightarrow f(x) > f(y) \quad \text{for any } x, y \in (a,b)$$

~~strict. $x > y \Rightarrow f(x) \geq f(y)$~~



- decreasing on (a,b) if

$x > y \Rightarrow f(x) > f(y)$ for any $x, y \in (a,b)$

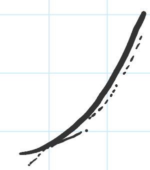


- decreasing on (a,b) if

$x > y \Rightarrow f(x) < f(y)$ for any $x, y \in (a,b)$

$\uparrow$ strict inequalities

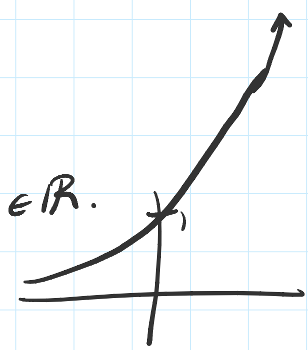
Thm: Let f be differentiable on (a,b),



- If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly monotone increasing
- If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly monotone decreasing.

proof: Similar to above corollary $\square$

ex/ Show that $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$, is strictly monotone increasing (on $\mathbb{R}$).



$\rightarrow$ $f'(x) = \dfrac{d}{dx} e^x = e^x > 0$ for any $x \in \mathbb{R}$.

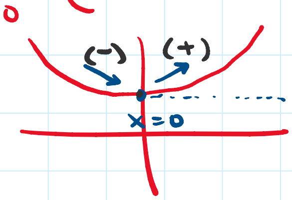
$\rightarrow$ Thus, $x > y \Rightarrow e^x > e^y$.

ex/ Find the regions in $\mathbb{R}$ where $f(x) = e^{x^2}$ is increasing & decreasing.

$$f'(x) = \frac{d}{dx}\left(e^{x^2}\right) = \underbrace{e^{x^2}}_{>0} \cdot \underbrace{2x}_{\substack{>0,\ x>0 \\ <0,\ x<0}} \begin{cases} >0, & x > 0 \\ <0, & x < 0 \end{cases}$$

$\Rightarrow$ f increasing on $x > 0$

f decreasing on $x < 0$



$x = 0$ is a local minimizer

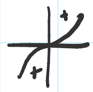
Thm (1st Derivative Test for Critical Points)

- Let c be a critical point of f (where f is differentiable in some open interval containing c), then

$\rightarrow$ If $f'(x)$ changes from $(+)$ to $(-)$ at c, then c is a local max.

$\rightarrow$ If $f'(x)$ changes from $(-)$ to $(+)$ at c, then c is a local min

→ If $f'(x)$ changes from $(-)$ to $(+)$ at c, then c is a local min.

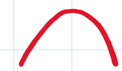
(e.g. $f(x) = x^3$,  $(+)$ to $(+)$, neither local min/max).

ex/ Determine the local minima and maxima of $f(x) = x^3 - 27x - 20$.

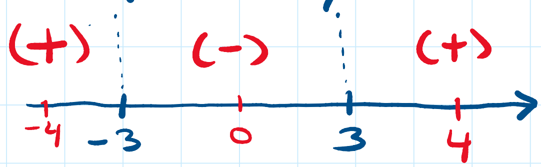
1) Find critical points

$$0 = f'(x) = 3x^2 - 27 = 3(x^2 - 9) \iff x = \pm 3.$$

2) Test the signs away from the critical pts



sign f'



⇒ -3 is a local maximizer
$f(-3)$ is a local maxima

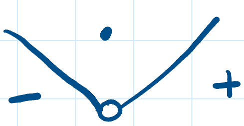
$$f'(-4) = 3(-4)^2 - 27 = 48 - 27 > 0$$

$$f'(0) = -27 < 0$$

$$f'(4) = 3(4)^2 - 27 > 0$$

⇒ $+3$ is a local minimizer.

remark: 1st deriv. needs either continuity or differentiability at the critical point c.

e.g. 

$$f(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$x = 0$ is a critical point $f'(0) = $ DNE.

1st deriv test does not work: $x = 0$ is a local minimizer, which is wrong
$f(0) = 1$

ex/ Determine local min./max. of $e^{-(x-1)^2} = f(x)$, $f : \mathbb{R} \to \mathbb{R}$, if any. f is a "Gaussian centered at $x=1$"

$= f(x)$,

$f: \mathbb{R} \to \mathbb{R}$, if any. f is a "Gaussian centered at $x=1$"

$$f'(x) = \frac{d}{dx} e^{-(x-1)^2} = e^{-(x-1)^2} \cdot \frac{d}{dx}\left(-(x-1)^2\right)$$

$$= \underbrace{e^{-(x-1)^2}}_{\neq 0} \underbrace{(-2)}_{\neq 0}(x-1).$$

$f'(x) = 0 \Leftrightarrow x = 1.$

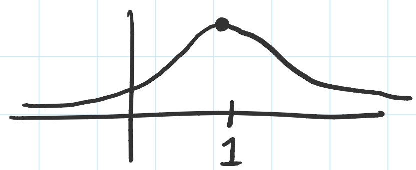
$(+)$ | $(-)$

$x=1.$

local maximizer $x=1$

$x>1$: $f'(x) = \underbrace{e^{-(x-1)^2}}_{>0} \underbrace{(-2)}_{<0} \underbrace{(x-1)}_{>0} < 0$

$x<1$: $\underbrace{}_{>0} \underbrace{}_{<0} \underbrace{}_{<0} > 0$



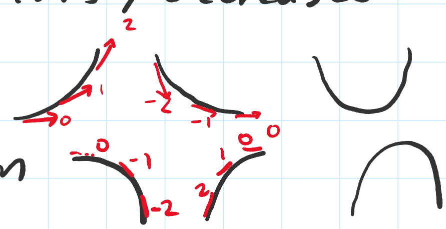
4.4 | Second Derivative & Concavity.

- We saw f' determines if f increases / decreases

$\rightarrow$ f'' determines if f' increases / decreases; determines if the slope of f increases / decreases

concave up: bends up

concave down: bends down



Def: Let f be diff. on (a,b). Say

- f is concave up on (a,b) if f' is increasing on (a,b).
- f is concave down on (a,b) if f' is decreasing on (a,b).

increasing $\Uparrow$ $x>y \Rightarrow f'(x) > f'(y)$

decreasing $\Updownarrow$ $x>y \Rightarrow f'(x) < f'(y)$.

Assume f is twice differentiable on (a,b), then

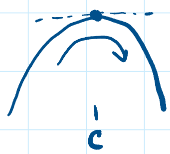
- $f''(x) > 0$ for all $x \in (a,b) \Rightarrow f$ is concave up on (a,b)

Assume f is twice differentiable on (a,b), then

- $f''(x) > 0$ for all $x \in (a,b) \Rightarrow f$ is concave up on (a,b)
- $f''(x) < 0$ for all $x \in (a,b) \Rightarrow f$ is concave down on (a,b).

Second Derivative Test

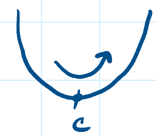
local max



concave down

$f''(c) < 0$

local min



concave up

$f''(c) > 0$

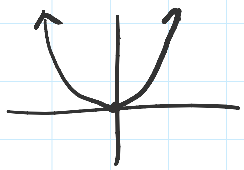
- Let c be a critical of f; if $f''(c)$ exists, then

$f''(c) < 0 \Rightarrow c$ is a local maximizer

$f''(c) > 0 \Rightarrow c$ is a local minimizer.

($f''(c) = 0$ (or DNE) $\Rightarrow$ inconclusive, use 1^{st} derivative test.)

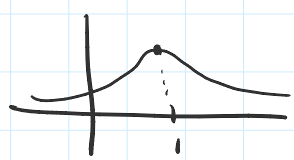
ex/ $f(x) = x^2$



- Critical pts: $0 = f'(x) = 2x \Leftrightarrow x = 0$.
- Check the sign of f'' at critical point

$f''(x) = 2$, $f''(0) = 2 > 0$. $\Rightarrow$ local minima.

ex/ $f(x) = e^{-(x-1)^2}$



- Critical pt, $0 = f'(x) = -2(x-1)e^{-(x-1)^2}$ $\quad x = 1$.
- Second Deriv.

$$f''(x) = \frac{d}{dx}\left[-2(x-1)e^{-(x-1)^2}\right]$$

$$= -2e^{-(x-1)^2} + 4(x-1)^2 e^{-(x-1)^2}$$

$$f''(1) = -2e^{-0^2} = -2 < 0. \Rightarrow \text{local max.}$$

ex/ An example where the 2^{nd} deriv. test does not work

$$f(x) = |x|^3$$

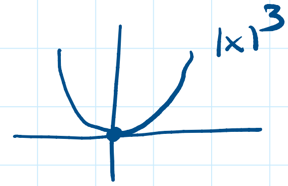
$$\left.\begin{aligned} f'(x) &= 3x|x| \\ f''(x) &= 6|x| \end{aligned}\right\} \text{see midterm 1.}$$

- Critical point $0 = f'(x) = 3x|x| \iff x=0$.
- $f''(0) = 6|0| = 0 \Rightarrow$ inconclusive.

$\Rightarrow$ use 1^{st} deriv. test

sign of $f'(x)$	$(-)$		$(+)$
	$x<0$	0	$x>0$

$f'(x) = 3x|x|$



$\Rightarrow$ local min.

- Next time: L'Hôpital's rule, Graphs, Optimization
- After: integration.
- OH tomorrow 11 – 12.