Lecture 11 - LH, Analyzing Graphs, Optimization] last of material covered.

Monday, August 23, 2021 11:16 AM

- Read sections 4.5, 4.6, 4.7
- Midterm 2 on Monday 8/30/21. (format similar to MT1)
  - → Covering all topics up to 4.7 and all homeworks (1,2,3,4) [focus: 3, 4] excluding problems on homework 4 regarding integration (chapter 5)
  - → Suggestion: try to finish most of hw 4 before Monday, except problems on integration (which you can do after the exam since integration isn't tested; e.g. you can do these problems on Tuesday). Problems 16-19 on hw4 WILL NOT be tested on MT2.
- To give more time to finish homework 4 after the exam, it will be due Wednesday 9/1/21 at 11:59 pm. ← Week 5
  - Similarly, homework 5 will be due Wednesday 9/8/21 at 11:59 pm. ← Week 6
- OH today at 11 am - 12 pm.
- Next Monday lecture - review lecture
  M 8/30/21 extra OH 11 am - 12 pm (go over hw 3 & 4).

4.5 L'Hôpital's Rule

- Indeterminate forms $0/0$, $\frac{\infty}{\infty}$ ← + or -

$$\left[\lim_{x \to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0. \right]$$

Thm: L'Hôpitals Rule (LH)

1) Assume f, g are diff. on an open interval containing a and that $f(a) = 0 = g(a)$ and $g'(x) \neq 0$ (except possibly at a)

Then, $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \underbrace{\frac{f'(x)}{g'(x)}}_{\text{not } \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]}$$ ← if the right hand side exists or is infinite.

$$\lim_{x\to 0} \frac{x^2}{x} \overset{(LH)}{=} \lim_{x\to 0} \frac{2x}{1} = 0.$$

2) If f and g are diff. for x near (but not at) a, and $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$,

Then, $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \leftarrow$ If the right hand side exists or is infinite.

proof: $(x \neq a)$

1) $$\frac{f(x)}{g(x)} = \frac{f(x) - \overset{0}{\overset{\shortparallel}{f(a)}}}{g(x) - \underset{0}{\underset{\shortparallel}{g(a)}}} = \frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{\left(\frac{f(x)-f(a)}{x-a}\right)}{\left(\frac{g(x)-g(a)}{x-a}\right)} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} \quad \square$$

ex/ $$\lim_{x\to 2}\frac{x^3-8}{x^4+2x-20}$$ $2^3-8=0$ ✓ $2^4+2\cdot 2-20=0$ ✓

2 is a root of polynomial $P_n(x) \iff x-2$ divides $p_n(x)$.

$$\overset{(LH)}{=} \lim_{x\to 2}\frac{\frac{d}{dx}(x^3-8)}{\frac{d}{dx}(x^4+2x-20)} = \lim_{x\to 2}\frac{3x^2}{4x^3+2}$$

$$= \frac{\lim_{x\to 2}3x^2}{\lim_{x\to 2}(4x^3+2)} = \frac{3(2)^2}{4(2)^3+2} = \frac{3\cdot 4}{4\cdot 8+2}$$

ex/ $$\lim_{x\to 0}\frac{e^x-x-1}{\cos(x)-1}$$

$$\overset{(LH)}{=} \lim_{x\to 0}\frac{e^x-1}{-\sin(x)} \overset{(LH)}{=} \lim_{x\to 0}\frac{e^x}{-\cos(x)} = \frac{e^0}{-\cos(1)} = -1.$$

ex/ $$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$$ $\frac{1}{0} - \frac{1}{0} \sim \pm\infty \mp \infty$

$$= \lim_{x\to 0}\frac{(x-\sin x)^{=0 \text{ at } x=0}}{(x\sin x)} \overset{(LH)}{=} \lim_{x\to 0}\frac{(1-\cos x)^{=0 \text{ at } x=0}}{(\sin x + x\cos x)^{=0 \text{ at } x=0}}$$

$$= \lim_{x \to 0} \frac{\overbrace{(x - \sin x)}^{=0}}{\underbrace{(x \sin x)}_{=0}} \underset{\text{at } x=0}{\overset{(LH)}{=}} \lim_{x \to 0} \frac{(1-\cos x)''}{(\sin x + x\cos x)} \underset{}{=} 0 \text{ at } x=0$$

$$\overset{(LH)}{=} \lim_{x \to 0} \frac{\sin x}{\cos x + \cos x - x \sin x} = 0.$$

Rmk:

L'Hôpital's also applies for limit point $\pm\infty$; $+\infty$

- Assume f and g are diff. on (b, ∞) and $g'(x) \neq 0$ on (b,∞), $b \in \mathbb{R}$.
- If $\lim_{x\to\infty} f(x)$, $\lim_{x\to\infty} g(x)$ are both zero

  OR are both infinite,

  then $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$.

(Similar for $x \to -\infty$: $(-\infty, a)$, $a \in \mathbb{R}$).

Two algorithms

$f(x)$ runtime of algorithm 1, where x is data size

$g(x)$ " " 2, "

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0 \Rightarrow$ g runs slower, f runs faster.

$\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty \Rightarrow$ g runs faster, f runs slower.
(as data size gets large)

"Complexity"

ex/ $f(x) = e^x$ exponential time

$g(x) = x^n$ polynomial time, n positive integer.

$n = 100$, e^x vs x^{100}

$x = 2$ $\quad e^2 \approx (2.7)^2$ vs 2^{100}

$x = 10000$ $\quad e^{10000}$ vs 10000^{100}

$$\lim_{x\to\infty} \frac{e^x}{x^n} \overset{(LH)}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\cdots n)}$$

$$\lim_{x\to\infty} \frac{e^x}{x^n} \overset{(LH)}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^n)}$$

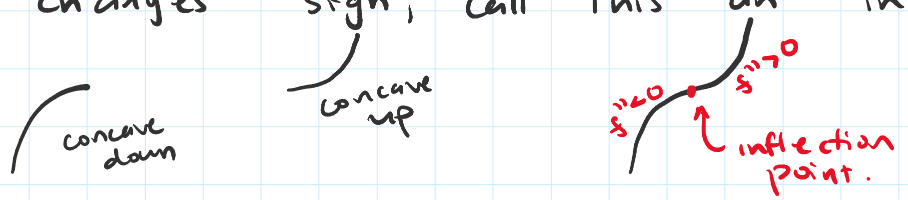
$$\overset{(\text{LH } n\text{-}1 \text{ more times})}{=} \lim_{x\to\infty} \frac{\frac{d^n}{dx^n}(e^x)}{\frac{d^n}{dx^n}(x^n)} = \lim_{x\to\infty} \frac{e^x}{n!}$$

$$= \infty.$$

Exponentials (eventually) grow faster than any polynomial.

$\boxed{4.6}$ Analyzing & Sketching Graphs

- Remember when f' changes sign, have local extrema
  - $(+)$ to $(-)$ local max
  - $(-)$ to $(+)$ local min
- When f'' changes sign, call this an inflection point



These are both called transition points.

To sketch the graph:

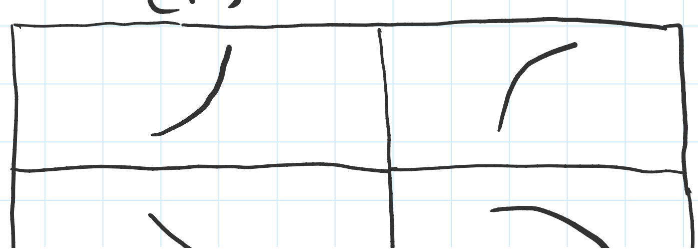
- find all transition points (where f' & f'' change sign)
- Determine signs of f' and f'' between them
- Connect points with appropriate arcs

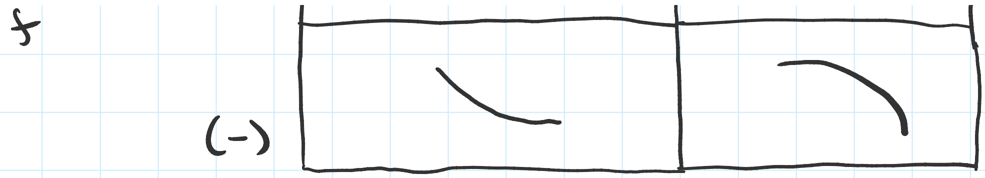
sign of f''

$(+)$ $(-)$

sign of f'

$(+)$





- For domains $\pm\infty$, compute $\lim_{x\to\pm\infty} f(x)$ for asymptotic behavior.

ex/ Sketch $f(x) = xe^x$ on $\mathbb{R}$.

Find zeroes of f' and f'':

$$0 = f'(x) = e^x + xe^x = e^x(x+1) \iff x = -1$$

$$0 = f''(x) = e^x + e^x + xe^x = e^x(x+2) \iff x = -2$$

sign of f': $(-)$ on $(-\infty, -1)$, $(+)$ on $(-1, \infty)$ $\qquad f'(x) = e^x(x+1)$

sign of f'': $(-)$ on $(-\infty, -2)$, $(+)$ on $(-2, \infty)$ $\qquad f''(x) = e^x(x+2)$

$(-\infty, -2)$ $\quad f' < 0,\ f'' < 0$

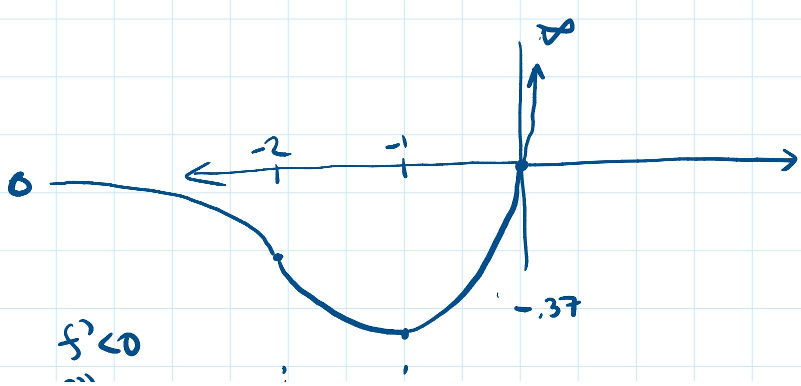
$(-2, -1)$ $\quad f' < 0,\ f'' > 0$

$(-1, \infty)$ $\quad f' > 0,\ f'' > 0$

$f(x) = xe^x$ $\qquad \lim_{x\to\infty} xe^x = \infty.$

$$\lim_{x\to-\infty} xe^x \underset{(s=-x)}{=} \lim_{s\to\infty} -se^{-s} = \lim_{s\to\infty} \frac{-s}{e^s}$$

$$\overset{(LH)}{=} \lim_{s\to\infty} \frac{-1}{e^s} = 0.$$



$f(x) = xe^{-x}$

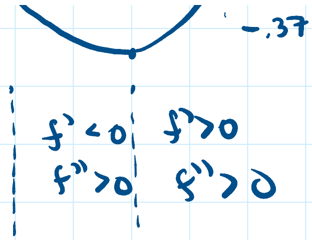
$f(-1) = (-1)e^{-1}$

$= \frac{-1}{e} \approx \frac{-1}{2.7} \approx -0.37$

$f(-2) \approx \frac{-2}{(2.7)^2} \approx -0.27$

$-.37$ $f(e) \approx \frac{\;}{(2.7)^2} \approx -0.27$

$f'<0$
$f''<0$



$\boxed{4.7}$ Applied Optimization (last section on exam).

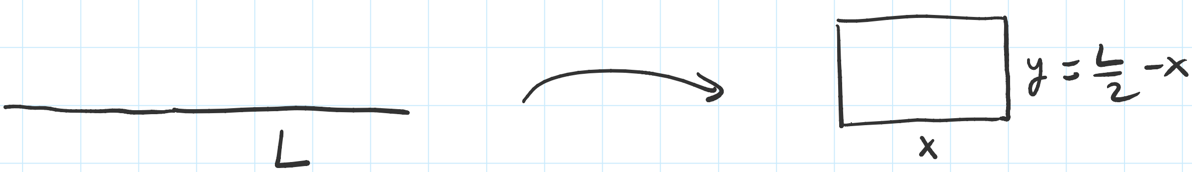
- We know how to find minima/maxima of functions, using calculus. Now, let's apply it.

Optimization on closed intervals:

$f:[a,b]\to\mathbb{R}$, min & max exist (either at critical pts OR endpoints)

- Identify variables for problem
- Identify objective function (min./max.)

ex/ Piece of wire length L bent into a rectangle. What is the configuration which maximizes the area?



$$2x+2y=L \Rightarrow y=\frac{L}{2}-x$$

$$A=xy \Rightarrow A(x)=x\left(\frac{L}{2}-x\right)$$ objective function.

$$A:[0,\tfrac{L}{2}]\to\mathbb{R}.$$

$$A(0)=0,\quad A\left(\tfrac{L}{2}\right)=0$$

$$0=A'(x)=\frac{d}{dx}\left(\frac{L}{2}x-x^2\right)=\frac{L}{2}-2x \Leftrightarrow x=\frac{L}{4}$$

$\Rightarrow$ square

(of all rectangles with fixed perimeter)

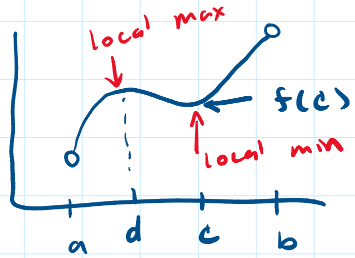
$\Rightarrow$ square

Of all rectangles with fixed perimeter L, the square maximizes the area. $\square$

Optimization over Open Intervals

Min/max aren't guaranteed to exist

$f:(a,b) \to \mathbb{R}$ cont. given by

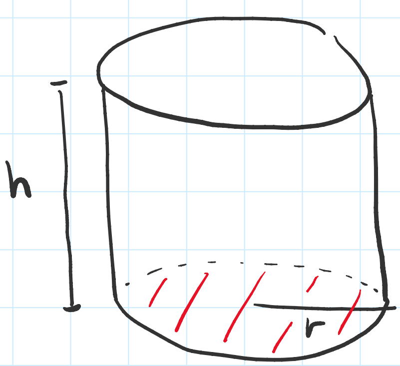


this fcn has no min/max.

$f(c)$ is not less than $\lim_{x \to a^+} f(x)$

$f(d)$ is not greater than $\lim_{x \to b^-} f(x)$.

ex/ Make a cylinder of volume 100 (cm^3) s.t. it has minimal surface area.



volume $\quad$ area of base $\times$ height

$$V = (\pi r^2) \times h = \pi r^2 h.$$

surface area

$$A = \underbrace{\pi r^2}_{\text{(top)}} + \underbrace{\pi r^2}_{\text{(bottom)}} + \underbrace{2\pi r h}_{\text{(side)}}$$

$$= 2\pi r^2 + 2\pi r h.$$

$$100 = V = \pi r^2 h \Rightarrow h = \frac{100}{\pi r^2}$$

$$A(r) = 2\pi r^2 + \frac{200}{r} \qquad A: \overbrace{(0,\infty)}^{\text{open domain}} \longrightarrow \mathbb{R}$$

$$0 = A'(r) = 4\pi r - \frac{200}{r^2} \iff r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$

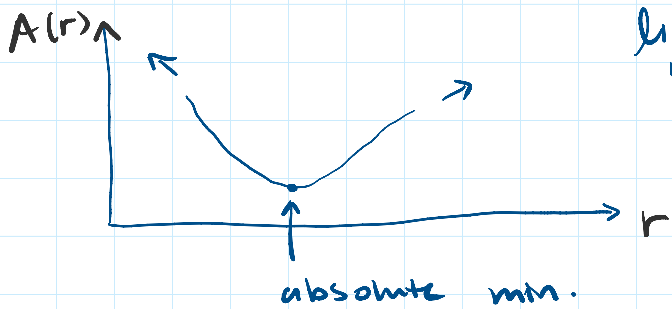
$$r = \left(\frac{50}{\pi}\right)^{1/3} \leftarrow$$

local min / max?

First / Second Derivative Test.

$$A''(r) = 4\pi + \frac{400}{r^3} > 0 \quad \text{for any } r \in (0, \infty).$$

$\Rightarrow r = \left(\frac{50}{\pi}\right)^{1/3}$ is a local minimizer.

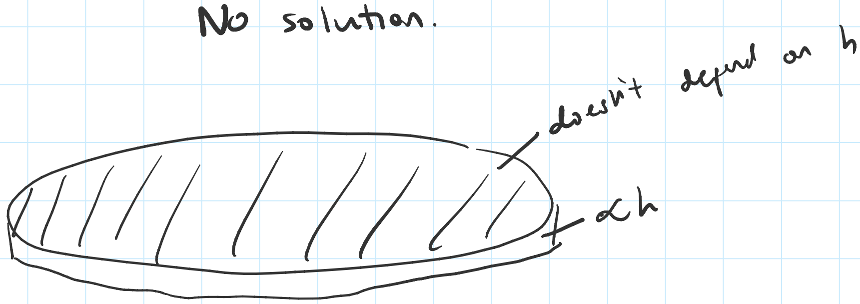


$$\lim_{r \to 0^+} A(r) = \infty$$

$$\lim_{r \to \infty} A(r) = \infty.$$

· Make a cylinder of volume 100 s.t. it has maximal surface area?

No solution.



□

OH in ~ 10 mins.

(end of material on exam).