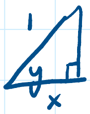
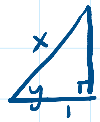
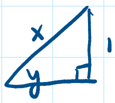
Math 20A MT2 Review

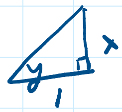
Wednesday, August 25, 2021 2:04 PM

1) Suppose we want to differentiate the inverse secant function, which of the following would be used to determine the derivative:

A) 

$\boxed{\text{C)}}$ 

B) 

D) 

$\frac{d}{dx}\sec^{-1}(x)$?

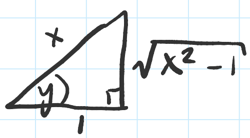
Implicit diff.

$y = \sec^{-1}(x)$

$\Leftrightarrow \quad \sec(y) = x$

$\Leftrightarrow \quad \frac{1}{\cos(y)} = x$

$\Leftrightarrow \quad \cos(y) = \frac{1}{x} = \frac{adj}{hyp}$



$y = \sec^{-1}(x)$

$\sec(y) = x$

$\frac{d}{dx}\sec(y) = \frac{d}{dx}x = 1$

$\frac{d}{du}\sec(u)$

$= \frac{d}{du}\frac{1}{\cos(u)}$ $\begin{matrix} = f \\ = g \end{matrix}$

$= \frac{fg' - gf'^{\,0}}{g^2} = \frac{-\sin(u)}{\cos^2(u)}$

quotient

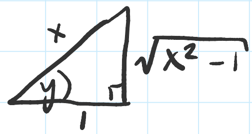
$= -\frac{\sin(u)}{\cos(u)}\cdot\frac{1}{\cos(u)}$

$= -\tan(u)\sec(u)$

$-\tan(y)\sec(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{-\tan(y)\sec(y)}$

$$\frac{dy}{dx} = \frac{1}{-\tan(y)\sec(y)}$$



e.g. $\tan(y) = \sqrt{x^2-1}$

$\sec(y) = x.$

2) LH

$\text{limit}_{x \to \infty} \frac{f(x)}{g(x)}$ $\qquad g'(x) \neq 0$ $\;(b, \infty)$

e.g. $\lim_{x \to \infty} x e^{-x}$

$$\left[\begin{array}{l} \lim_{x \to a} \frac{f(x)}{g(x)} \qquad a \neq \infty,\ a \neq -\infty \\ \text{LH} \quad g'(x) \neq 0 \text{ near (but not including) } x = a \\ \lim_{x \to a} \frac{f'(x)}{g'(x)} \end{array}\right]$$

$g'(x) \neq 0$ in a neighborhood of a (not including a)

$\lim_{x \to \infty} x \underbrace{e^{-x}}$

Indeterminate Form $\infty \cdot 0$

$= \lim_{x \to \infty} \frac{x = f}{e^x = g}$ $\qquad$ I.F. $\frac{\infty}{\infty}$

$e^x = g'(x) \neq 0$ for $x \in \mathbb{R}$.

$= \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1}{e^x} = 0.$

3) T or F: To find all critical points of a function $f(x)$, we need to find all points x s.t. $f'(x) = 0$.

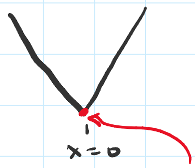
if it said differentiable (true)

False.

False.

Critical points : either
$f'(x) = 0$ OR $f'(x)$ DNE.

e.g. $f(x) = |x|$



$f'(0)$ DNE

critical point, local min.

(first deriv test)

$$f'(x) = \begin{cases} -1, & x<0 \\ \text{DNE}, & x=0 \\ +1, & x>1 \end{cases}$$

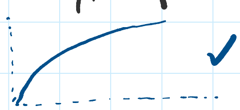
e.g. $f(x) = x^2$. Differentiable

Critical pts $f'(x) = 0 \Leftrightarrow x = 0$

4) extra question:
What is a function (excluding a polynomial) that is increasing & concave down?

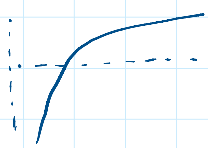
concave down

$f(x) = \sqrt{x}$



✓

$f(x) = \ln(x)$



e.g. $\rightarrow f(x) = \ln(x) \quad x \in (0, \infty)$

$$f'(x) = \frac{d}{dx}\ln(x) = \frac{1}{x} > 0$$

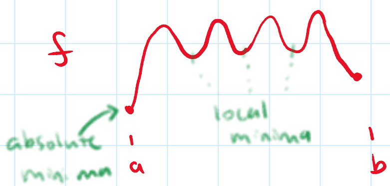
$$f''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1}$$

$$\underset{\text{power rule}}{=} -x^{-2} < 0$$

$$\underbrace{f' > 0}_{\text{increasing}}, \quad \underbrace{f'' < 0}_{\text{concave down}}$$

5) Which is true about optimization of a continuous function on a closed interval?

- ~~To determine the absolute minima and maxima, search for all of the critical points and determine the function's values at those points.~~ endpoints too
- ~~If the function has multiple local minima, one of them is guaranteed to be the absolute minimum.~~ see drawing below
- ~~It is not guaranteed that the function has an absolute maxima or minima.~~
- None of the above



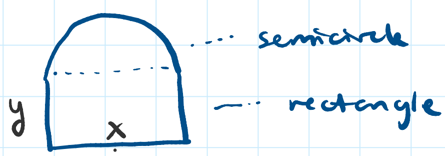
f cont. on $[a, b]$

Thm: A continuous function on a closed interval has an absolute minimum (and maximum), which occurs at either critical points or at an endpoint of the interval.

minimum y

minimizer $x = a, b,$ or c

6) **Suppose we want to build a fence with perimeter 100, which consists of a semicircle on top of a rectangle (see figure), in order to maximize the area enclosed. Which is the correct equation which gives the length y in terms of the width x?**

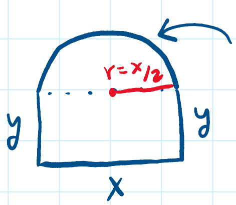


A) $y = 50 - x/2 - \pi x/2$

B) $y = 50 - x/2 - \pi x/4$

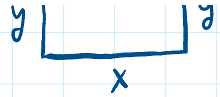
C) $y = 100 - x - \pi x/2$

D) $y = 100 - x - \pi x$



arclength semicircle

$\pi r = \pi x/2$



Perimeter

$$100 = 2y + x + \pi x/2$$

$$\Rightarrow \quad y = 50 - \frac{x}{2} - \pi x/4 \,. \quad !$$

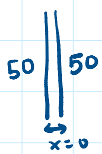
Area

$A = $  $+$ 

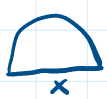
$$= x \cdot y + \frac{\pi r^2}{2}$$

$$A(x) = \underbrace{x \cdot \left(50 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{\pi}{2}\left(\frac{x}{2}\right)^2}_{\text{quadratic}} \,.$$

$x = 0$



$y = 0$



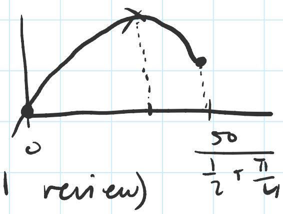
$$0 = 50 - \left(\frac{1}{2} + \frac{\pi}{4}\right)x$$

$$x = \frac{50}{\frac{1}{2} + \frac{\pi}{4}}$$

A(x)

$$A : \left[0, \frac{50}{\frac{1}{2} + \frac{\pi}{4}}\right] \longrightarrow \mathbb{R}$$

$$A'(x) = 0$$



(Start of additional review)

7) $\frac{d}{dx}\left[g(x)^{h(x)}\right]$

True or false: Let's say we know how to differentiate two functions g and h. To differentiate the function f(x) = g(x)^h(x), that is g(x) raised to the h(x), we should apply the power rule with the chain rule.

$f(x) = x^x$

. power rule $\leftarrow$ n is constant (does not depend on x)

$u(x) = x^n$

. chain rule : function $v(x)$

$$u(v(x)) = v(x)^n$$

. exponential $u(x) = b^x$ (b constant) $(b>0, b \neq 1)$

$$u(v(x)) = \underline{\underline{b}}^{v(x)}$$

h(x)

$u(v(x)) = \underline{\underline{b}}^{v(x)}$

$\frac{d}{dx} f(x)$, where $\underline{f(x) = g(x)^{h(x)}}$

Logarithmic $\ln(a^b) = b \ln(a)$ $(a>0)$

$$\frac{d}{dx} \ln(f(x)) \underset{\substack{\uparrow \\ \text{chain} \\ \text{rule}}}{=} \frac{1}{f(x)} \cdot f'(x)$$

$$f'(x) = f(x) \frac{d}{dx} \ln(f(x))$$

$$f'(x) = f(x) \frac{d}{dx} \ln(g(x)^{h(x)})$$

$$= f(x) \underbrace{\frac{d}{dx}\left[h(x) \ln(g(x))\right]}_{\text{prod rule} + \text{chain rule}}$$

$$\rightarrow = g(x)^{h(x)} \left(h'(x) \ln(g(x)) + \frac{g'(x)}{g(x)} \right)$$

e.g. $f(x) = x^x$ $g(x) = x$, $h(x) = x$

$$f'(x) = x^x \left(\ln(x) + \frac{1}{x} \right),$$

e.g.

$f(x) = x^{3^x}$ $g(x) = x$

$= g(x)^{h(x)}$ $h(x) = 3^x$

$g'(x) = 1$

$h'(x) = \frac{d}{dx}(3^x) = 3^x \cdot \ln(3)$

8) $f(x) = \ln(x^2)$

$f: \underbrace{(-\infty, 0) \cup (0, \infty)}_{(x \neq 0)} \rightarrow \mathbb{R}.$

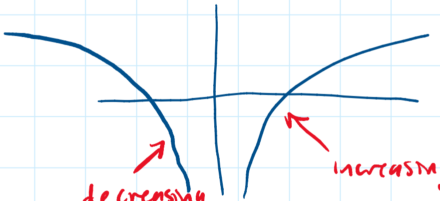
Consider the function f(x) = ln(x^2), the natural logarithm of x squared, defined for all numbers except x = 0. Which of the following is correct? 0 0 0

- f is increasing and concave up on x > 0, f is decreasing and concave down on x < 0
- f is decreasing and concave up on x > 0, f is increasing and concave down on x < 0
- f is increasing and concave down on x > 0, f is decreasing and concave down on x < 0
- f is not differentiable for x < 0

$f(x) = \ln(x^2)$ $f(-x) = \ln((-x)^2)$

$f(x) = \ln(x^2)$ $\qquad$ $f(-x) = \ln((-x)^2)$

$= \ln(x^2)$



decreasing, concave down $\qquad$ increasing, concave down

$f'(x)$?

$\cancel{\ln(x^2) = 2\ln(x)}$ $\qquad$ $\ln(a^b) = b\ln(a)$ $(a>0)$

$\cancel{\frac{d}{dx}(2\ln(x)) = \frac{2}{x}}$.

does not work since x can be negative. Instead, use chain rule.

• $\frac{d}{dx}\ln(x^2) = \frac{2x}{x^2} = \frac{2}{x}$

differentiable for positive inputs $\qquad$ differentiable and outputs positive values for $x \neq 0$

• $f''(x) = \frac{d}{dx}\left(\frac{2}{x}\right) = \frac{-2}{x^2} < 0$

for $x \neq 0$.

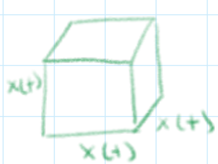
9) **Imagine a cube which is expanding. Suppose the length of the cube as a function of time is given by x(t) and the volume as a function of time is V(t). Which of the following gives the rate of expansion of the volume? (Also, see additional question)**

A) $V'(t) = x(t)^3$

B) $V'(t) = (x'(t))^3$

$\boxed{C)}$ $V'(t) = 3x(t)^2\, x'(t)$

D) $V'(t) = 3x'(t)^2$



$V(t) = x(t)^3$

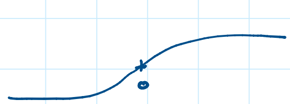
$V'(t) = \frac{d}{dt}\left(x(t)^3\right)$

$= 3x(t)^2 \cdot x'(t)$

chain rule

chain rule

Homework 4)



Problem 10)

$$\lim_{x\to 0} \frac{\tan^{-1}(x) = f(x)}{\sin^{-1}(x) = g(x)}$$

$\lim_{x\to 0} \tan^{-1}(x) = \tan^{-1}(0) = 0$,

$y = \tan^{-1}(x)$

$\tan(y) = x = 0$

$\frac{\sin(y)}{\cos(y)} = 0 \Leftrightarrow y = 0$

$\lim_{x\to 0} \sin^{-1}(x) = \sin^{-1}(0) = 0.$

$\Rightarrow$ indeterminate form $0/0$.

$f(0) = 0, \quad g(0) = 0$ ✓

$$\frac{d}{dx} g(x) = \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \neq 0 \text{ near } x=0$$

Apply LH $\quad \lim_{x\to 0} \frac{\tan^{-1}(x)}{\sin^{-1}(x)} \overset{LH}{=} \lim_{x\to 0} \frac{\frac{d}{dx}\tan^{-1}(x)}{\frac{d}{dx}\sin^{-1}(x)} = \lim_{x\to 0} \frac{\left(\frac{1}{1+x^2}\right)}{\left(\frac{1}{\sqrt{1-x^2}}\right)}$

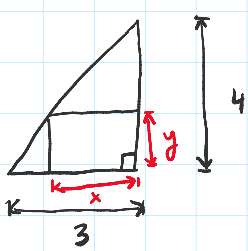
$$= \lim_{x\to 0} \frac{\sqrt{1-x^2}}{1+x^2}$$

$$= \frac{\lim_{x\to 0}\sqrt{1-x^2}}{\lim_{x\to 0} 1+x^2} = \frac{\sqrt{1-0^2}}{1+0^2} = 1.$$

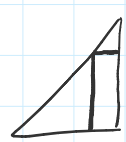
quotient law for limits

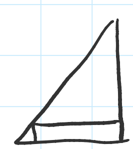
continuity

Problem 13)

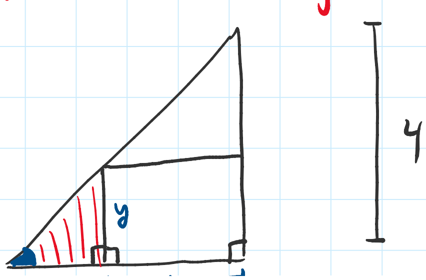


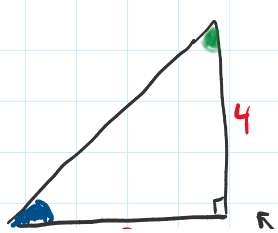
Maximize area of inscribed rectangle.

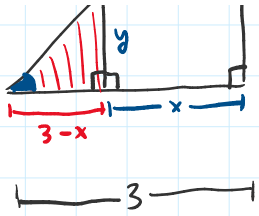


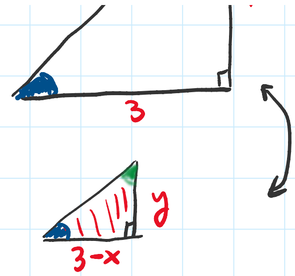


Area of rectangle $A = xy$

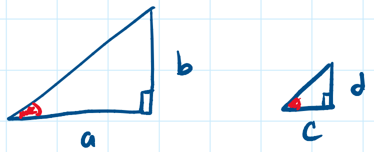








similar (right) triangles



$\Rightarrow \quad \frac{a}{b} = \frac{c}{d} \quad \Rightarrow \quad \frac{3-x}{y} = \frac{3}{4}$

$y = \frac{4}{3}(3-x) \qquad 4 = \frac{4}{3}(3-x) \iff x = 0$

$A = xy$

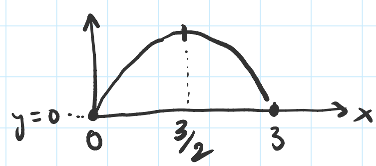
$\Rightarrow A(x) = x\left(\frac{4}{3}\right)(3-x)$

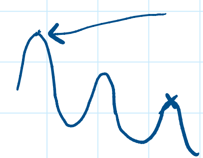
$x=3$ widest

$x=0$ tallest

A is defined on $[0,3]$.

$A(x) = \frac{4}{3}x(3-x)$





Critical pts: $A'(x) = 0$

$A(x) = 4x - \frac{4}{3}x^2$

$0 = A'(x) = 4 - \frac{8}{3}x \iff x = \frac{3 \cdot 4}{8} = \frac{3}{2} \qquad x = 3/2.$

$\left(\text{2nd deriv. test} \quad A''(x) = -\frac{8}{3} \qquad A''\left(\frac{3}{2}\right) = -\frac{8}{3} \quad \text{local max} \right)$

endpoints $\quad A(0) = \frac{4}{3}(0)(3-0) = 0$

$A(3) = \frac{4}{3} \cdot 3 \cdot (3-3) = 0$

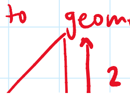
critical pt. $\quad A\left(\frac{3}{4}\right) = \frac{4}{3}\left(\frac{3}{2}\right)\underbrace{\left(3 - \frac{3}{2}\right)}_{>0} > 0$

$A\left(\frac{3}{4}\right) > A(0) \qquad A\left(\frac{3}{4}\right) > A(3)$

$\Rightarrow$ absolute max of area occurs at $x = 3/2$.

often, solution to geometric optimization problems is the most "symmetric"

e.g.



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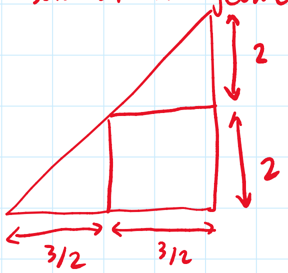

e.g.
lecture: