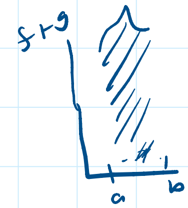
Lecture 14 - The Indefinite Integral and the FTC I

Monday, August 30, 2021 11:14 PM

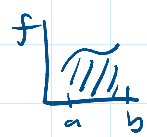
Read sections 5.3 and 5.4

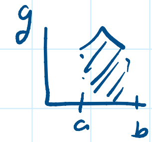
Properties of the Definite Integral:



Theorem:

Let f and g be integrable on $[a,b]$. Then





Linearity

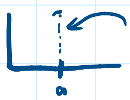
(i) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(ii) $\int_a^b (c\,f(x))\,dx = c\int_a^b f(x)\,dx$

(iii) Order of Integration

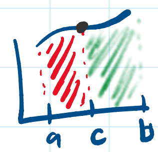
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(iv) $\int_a^a f(x)\,dx = 0$

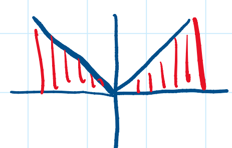


(v) Let $c \in (a,b)$. Then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

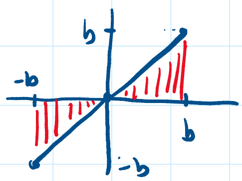


ex/ Compute $\int_{-b}^{b} x\,dx$, $\int_{-b}^{b} |x|\,dx$
(b > 0)



$\int_{-b}^{b} x\,dx$

$\overset{(v)}{=}$

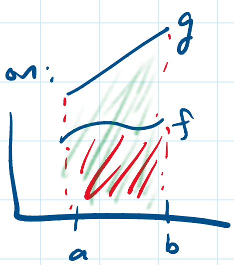


$\underbrace{\int_{-b}^{0} x\,dx}_{-\frac{1}{2}b\cdot b} + \underbrace{\int_{0}^{b} x\,dx}_{\frac{1}{2}b\cdot b} = 0$

$= \frac{1}{2}b\cdot b + \frac{1}{2}b\cdot b = b^2$

(vi) If $f(x) \le g(x)$ on $[a,b]$, comparison:

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



If f is bounded,

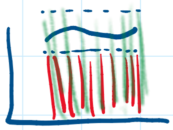
$m \le f(x) \le M$

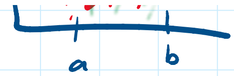
$\int^b C\,dx$

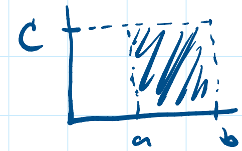
if f is bounded, $m \le f(x) \le M$,

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\int_a^b m\,dx = m(b-a) \qquad \int_a^b M\,dx = M(b-a)$$







$$\int_a^b C\,dx = c(b-a)$$

5.3 Indefinite Integrals

Def: Antiderivative

A function F is an antiderivative of a function f on an open interval (a,b) if: $F'(x) = f(x)$ for all $x \in (a,b)$

ex/

$f(x) = \sin(x)$

$F(x) = -\cos(x) + C = \int \sin(x)\,dx$, $C \in \mathbb{R}$

$F'(x) = -\frac{d}{dx}\cos(x) = -(-)\sin(x) = \sin(x) = f(x)$

$\Rightarrow$ $-\cos(x)$ is an antiderivative for $\sin(x)$.

ex/

$f(x) = x^3$

$F(x) = \frac{1}{4}x^4 + C = \int x^3\,dx$

$F'(x) = \frac{1}{4}\frac{d}{dx}x^4 = \frac{1}{4}\cdot 4x^3 = x^3 = f(x)$.

Thm: Let F be an antiderivative of f on (a,b). Then, every other antiderivative is of the form $F + C$ where C is any constant function on (a,b).

proof: Let's say F and G are antiderivatives of f

$F + C$ where C is any constant function on (a,b).

proof: Let's say F and G are antiderivatives of f on (a,b). $F'(x) = f(x)$ & $G'(x) = f(x)$ on (a,b).

· Let $D = F - G$ be their difference.

$$D'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \text{for all } x \in (a,b).$$

$\Rightarrow D$ is constant,

or in other words, $G(x) = F(x) + C$. $\square$

The indefinite integral of a function f denotes the collection of all antiderivatives of f. Say F is an antiderivative of f

$$\int f(x)\,dx = F(x) + C \iff F'(x) = f(x)$$

(for x in the domain of f)

ex/ (Power rule for antiderivatives)

For $n \neq -1$, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \iff \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$

For $n = -1$, $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$ $\quad$ $\frac{1}{x}$ is defined on $x < 0$ & $x > 0$

$x > 0$, $\displaystyle\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$.

$x < 0$, $\displaystyle\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$.

$\Rightarrow$ For $x \neq 0$, $\displaystyle\frac{d}{dx}\ln|x| = \frac{1}{x} \iff \int \frac{1}{x}\,dx = \ln|x| + C$.

Indefinite integral is linear:

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx \quad (\text{for any } \alpha \in \mathbb{R})$$

$$\int \alpha f(x)\,dx = \alpha \int f(x)\,dx \qquad (\text{for any } \alpha \in \mathbb{R})$$

ex/ $$\int (x^5 + \cos(x) + e^x)\,dx$$

$$= \int x^5\,dx + \int \cos(x)\,dx + \int e^x\,dx$$

$$= \left(\frac{x^6}{6} + C_1\right) + \left(\sin(x) + C_2\right) + \left(e^x + C_3\right)$$

$$= \frac{x^6}{6} + \sin(x) + e^x + C \qquad (C = C_1 + C_2 + C_3)$$

ex/ $$\int \frac{x^2 - x^4}{x^5}\,dx$$

$$= \int (x^{-3} - x^{-1})\,dx$$

$$= \int x^{-3}\,dx - \int x^{-1}\,dx$$

$$= \frac{x^{-2}}{-2} + C_1 - \ln|x| + C_2$$

$$= -\frac{1}{2}x^{-2} - \ln|x| + C.$$

ex/ Let's say we know $F(x) + C = \int f(x)\,dx$.

Then, $\int f(kx)\,dx = \dfrac{F(kx)}{k} + C.$ $(k \neq 0)$

proof: $F'(x) = f(x)$

$$\frac{d}{dx}\left(\frac{F(kx)}{k}\right) \underset{\uparrow}{=} \frac{F'(kx)}{k} \cdot k = F'(kx) = f(kx).$$

chain rule.

e.g. $\int e^{2x}\,dx = \dfrac{e^{2x}}{\ }+ C$

e.g., $\int e^{2x}\,dx = \frac{e^{2x}}{2} + C$

e.g., $\int \sin(kx)\,dx = -\frac{\cos(kx)}{k} + C.$

spatial freq. space

$\sqrt{1+x^2+\tan(x)}$

$\boxed{5.4}$ Fundamental Theorem of Calculus I (FTC I)

<u>Theorem</u> (FTC I)

Let f be continuous on $[a,b]$. Let F be an antiderivative of f on (a,b), then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\left(\int_a^b F'(x)\,dx = F(b) - F(a)\right)$$

<u>proof</u>: Partition

$a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$

$F(b) - F(a)$

$= F(x_N) - F(x_0)$

$= \big(F(x_N) - F(x_{N-1})\big) + \big(F(x_{N-1}) - F(x_{N-2})\big)$ telescoping sum

$+ \dots + \big(F(x_2) - F(x_1)\big) + \big(F(x_1) - F(x_0)\big)$

$= \sum_{i=1}^{N} \big(F(x_i) - F(x_{i-1})\big)$

MVT:

$F'(x) = f(x)$

F is differentiable; there exists $c_i \in (x_{i-1}, x_i)$

F is differentiable; there exists $c_i \in (x_{i-1}, x_i)$ such that by MVT

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) = f(c_i)$$

$$\Leftrightarrow \quad F(x_i) - F(x_{i-1}) = f(c_i)\cdot(x_i - x_{i-1})$$

$$= \sum_{i=1}^{N} f(c_i)\cdot(x_i - x_{i-1}) \quad \leftarrow \text{Riemann sum of } f$$

$$\Rightarrow \quad F(b) - F(a) = \sum_{i=1}^{N} f(c_i)\cdot(x_i - x_{i-1})$$

taking limit as $N \to \infty$ $(\|P\| \to 0)$

$$F(b) - F(a) = \sum_{i=1}^{N} f(c_i)\cdot(x_i - x_{i-1})$$

$$\Big\downarrow \qquad\qquad \Big\downarrow\ N \to \infty\ (\|P\| \to 0)$$

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

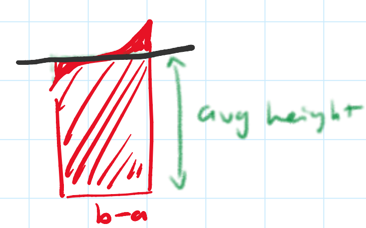
$\square$

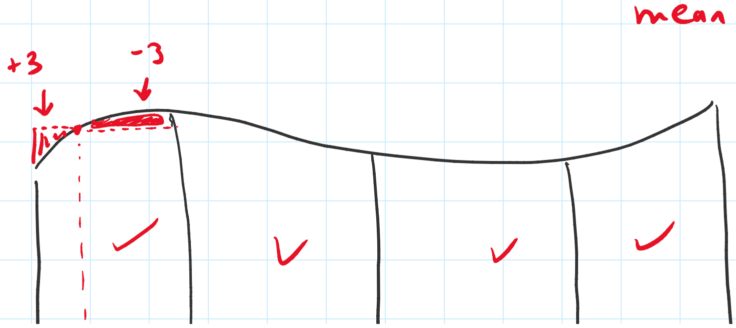
intuition behind proof:

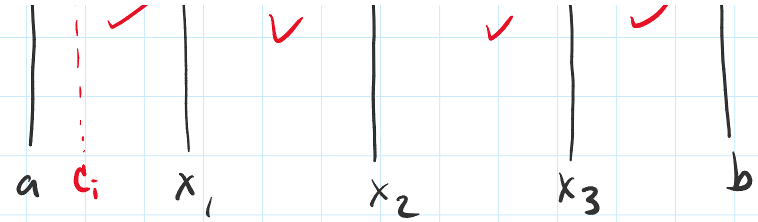
MVT said
$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) = f(c_i)$$

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

$$\Rightarrow \quad \frac{F(b) - F(a)}{b-a} = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{mean value of } f}$$



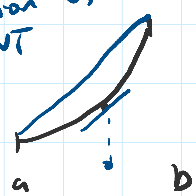




ex/ Show that quadratic polynomial satisfies the MVT at midpoint $c = \frac{a+b}{2}$ for any [a,b].

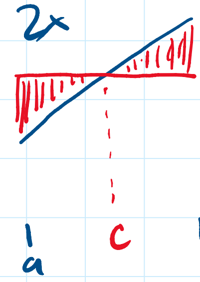
derivative version of MVT

$x^2 = F$



Integral version

$F' = 2x$



Integral version of MVT:

Let f be continuous on $[a,b]$. Then, there exists $c \in (a,b)$ s.t.

$$f(c) = \underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{mean value of } f}$$

$$\Longleftrightarrow \int_a^b f(x)\,dx = f(c)\cdot(b-a)$$