Lecture 15 - FTC I and FTC II

Tuesday, August 31, 2021 12:03 PM

- Read sections 5.4 and 5.5
- **Homework 4**: Due tonight 9/1 at 11:59 pm. I will post partial solutions onto canvas after the due deadline.
- **Final Exam**: You will be given 3 hours 30 minutes (or 210 minutes) to complete and upload the exam to gradescope. You should spend about 3 hours 15 minutes working on the exam, and leave about 15 minutes to upload the exam.
  - The exam will be available on Thursday 9/9 starting at 12 pm in the afternoon and will be available to access on Gradescope for 36 hours (closing on Friday 9/10 at 11:59 pm midnight). Once you access it, the 210 minute timer begins. Thus, you can start the exam any time on Thursday or Friday, but make sure to start the exam before 8:30 pm on Friday for the full time.
  - You can use 2 sheets of notes on the exam; no outside sources as usual.
- **Final Exam Content**: The final will consist of 9 equally weighted problem (20 points each), which breaks down as follows: 3 questions on topics tested on the first midterm, 3 questions on topics tested on the second midterm, and 3 questions on integration. There will also be 1 extra credit problem (10 points extra credit).
  - I've posted a practice final on canvas. Although I will go over the solutions to the practice final next week (see below), I recommend that you do the practice final yourself before I go over the solutions. To make most use of the practice test, try to treat it as the real final: when you are doing it, give yourself 3 hours 15 minutes.
- **Next week** (week 6)
  - <u>Monday 9/6</u>: No class (Labor day holiday). Enjoy the 3 day weekend ^_^.
  - <u>Tuesday 9/7</u>: Introductory lecture on differential equations; NOT tested. However, some of the material covered during this lecture gives good examples and perspective on the FTC, so I recommend paying attention anyway. Also, office hours at usual time 11 - 12.
  - <u>Wednesday 9/8</u>: Review lecture (homework 5 and practice final).
    - Furthermore, instead of office hours on Thursday, I will have an additional review office hours on Wednesday from 11 am - 12, similar to what we did for Midterm 2. I will discuss the rest of homework 5 and the practice final problems that we do not finish in lecture. I will record it for those that are unable to attend (if you can't attend, make sure to watch this additional review session before taking the final).
    - Homework 5 due at 11:59 pm. Homework 5 will be slightly shorter (14 questions), since you also have to study for the final; it is already available to view on the course webpage. Homework 5 and the 3 integration problems on the final will focus on: the definite integral, antiderivatives, FTC I and II.
  - <u>Thursday 9/9</u>: The Final will be released at 12 pm in the afternoon. No office hours.
  - <u>Friday 9/10</u>: The Final closes at 11:59 pm. View before 8:30 pm on Friday for the full 210 minutes.
    - No discussion section on Friday 9/10.
- **Post-Diagnostic Readiness Test**: As part of the Summer Bridge assessment, you have to take the post-diagnostic Calculus 2 readiness test at the end of the course. The readiness test will be accessible starting on Saturday 9/4 and will close on Thursday 9/9 at 10 pm.
- **Course and Professor Evaluation (CAPE)** - Please fill out your CAPEs for this course. I would really appreciate the feedback from all of you. Just like you all, I am learning as well, so I really appreciate any input that you have. It is anonymous, so please be honest. You should have received an email regarding filling out your CAPEs.
  - If more than 75% of the class fills out their CAPEs for this course, I will add another 1% extra credit to everyone's grades (again, all extra credit is applied after any curves for the course).

$g(u(x))$

chain rule

$$\frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx}$$

Recall FTC I:

notation

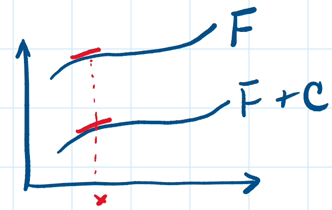
$$\int_a^b f'(x)\,dx = f(b) - f(a) \doteq f\Big|_a^b$$

$$\int_a^b \frac{df}{dx}\,dx = \int_a^b df = \text{change in } f \text{ from } a \text{ to } b$$

$$= f(b) - f(a)$$

$$f(b) + \cancel{C} - (f(a) + \cancel{C})$$

$$\int^1 (2x^2 + e^{2x} + 4)\,d \quad \int^1$$



ex/ $\int_0^1 \underbrace{(3x^2 + e^{2x} + 4)}_{\text{integrand } f(x)}\,dx = \int_0^1 f(x)\,dx$

Antiderivative: $x^3 + \frac{e^{2x}}{2} + 4x = F(x)$

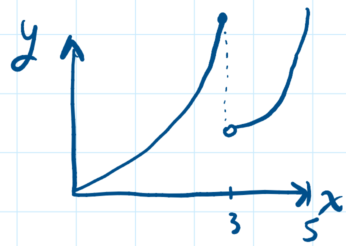
$F'(x) = \frac{d}{dx}\left(x^3 + \frac{e^{2x}}{2} + 4x\right) = f(x)$ ✓

$$= \int_0^1 F'(x)\,dx = F\Big|_0^1$$

$$= F(1) - F(0) = \left(1^3 + \frac{e^{2\cdot 1}}{2} + 4\cdot 1\right) - \left(0^3 + \frac{e^0}{2} + 4\cdot 0\right)$$

$$= 1 + \frac{e^2}{2} + 4 - \frac{1}{2}.$$

ex/ $\int_0^5 f(x)\,dx$, where $f(x) = \begin{cases} e^x, & 0 \le x \le 3 \\ x^2, & 3 < x \le 5 \end{cases}$



$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= \int_0^3 e^x\,dx + \int_3^5 x^2\,dx$$

$$= \int_0^3 \frac{d}{dx}(e^x)\,dx + \int_3^5 \frac{d}{dx}\left(\frac{x^3}{3}\right)dx$$

$$\xrightarrow{\text{FTC I}} = e^x\Big|_0^3 + \frac{x^3}{3}\Big|_3^5$$

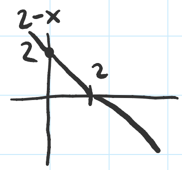
$$= \left(e^3 - e^0\right) + \left(\frac{5^3}{3} - \frac{3^3}{3}\right).$$

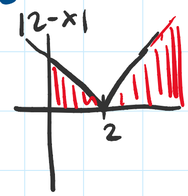
2-x

|2-x|

signed area

ex/ $\int_0^6 |2-x|\,dx$





signed area +

$$|t| = \begin{cases} t, & t \geq 0 \\ -t, & t < 0 \end{cases} \quad \Rightarrow \quad |2-x| = \begin{cases} 2-x, & 2-x \geq 0 \\ x-2, & 2-x < 0 \end{cases}$$

$$= \begin{cases} 2-x, & x \leq 2 \\ x-2, & x > 2 \end{cases}$$

$$\Rightarrow \int_0^6 |2-x|\,dx = \int_0^2 (2-x)\,dx + \int_2^6 (x-2)\,dx$$

$$= \int_0^2 \frac{d}{dx}\left(2x - \frac{x^2}{2}\right)dx + \int_2^6 \frac{d}{dx}\left(\frac{x^2}{2} - 2x\right)dx$$

$$= \left(2x - \frac{x^2}{2}\right)\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^6$$

$$= \left[\left(2\cdot 2 - \frac{2^2}{2}\right) - \left(2\cdot 0 - \frac{0^2}{2}\right)\right] + \left[\left(\frac{6^2}{2} - 2\cdot 6\right) - \left(\frac{2^2}{2} - 2\cdot 2\right)\right]$$

> 0

integrate with respect to 't'

ex/ $\int_{-3}^{-1} \frac{1}{t}\,dt$

$$= \int_{-3}^{-1} \frac{d}{dt}\left(\ln|t|\right)dt$$

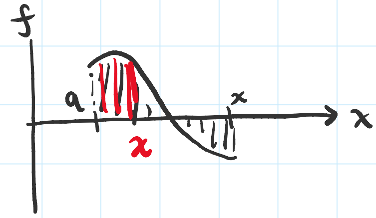
$$\overset{\text{FTC I}}{=} \ln|t|\,\Big|_{-3}^{-1} = \ln|-1| - \ln|-3|$$

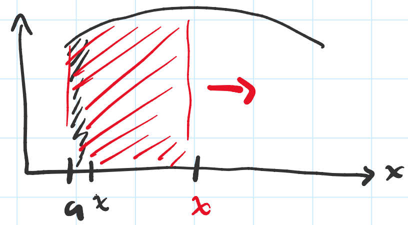
$$= \cancel{\ln(1)}^{0} - \ln(3) = \ln(1/3).$$

$\boxed{5.5}$ FTC II

Consider the area function associated to f, based at a,

$$A(x) = \int_a^x f(t)\,dt$$

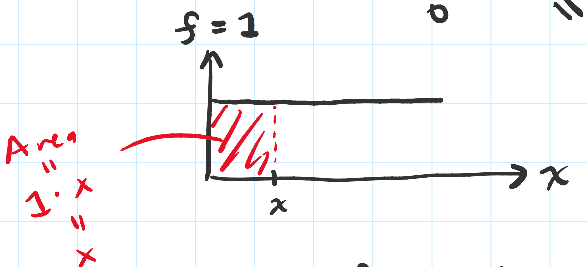




ex/ $A(x) = \int_0^x 1\,dt$ area function of a constant

$$= \int_0^x \frac{d}{dt}(t)\,dt$$

$$= t\Big|_0^x = x - 0 = x$$



$f(x) = 1$

$A(x) = x$ $\Rightarrow$ A is an antiderivative of f.

Theorem:

Let f be continuous on an open interval I and let $a \in I$. Then, the area function

$A(x) = \int_a^x f(t)\,dt$ (this is defined for any $x \in I$)

is the antiderivative of f in I:

$$A'(x) = f(x) \quad \text{for any } x \in I,$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad (\text{for any } x \in I).$$

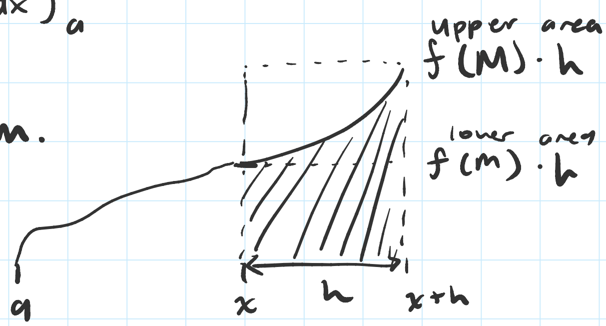
(rmk: Any continuous function has an antiderivative.)

(rmk: Any continuous function has an antiderivative.)

rmk: $\int_b^x f(t)\,dt = \int_a^x f(t)\,dt + \underbrace{\int_b^a f(t)\,dt}_{\text{constant (no dependence on } x)}$

$$\Rightarrow \frac{d}{dx}\int_b^x f(t)\,dt = \frac{d}{dx}\int_a^x f(t)\,dt.$$

proof: squeeze theorem.



$$f(m)\cdot h \le A(x+h) - A(x) \le f(M)\cdot h$$

$$\Rightarrow f(m) \le \frac{A(x+h)-A(x)}{h} \le f(M)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$f(x) \qquad \frac{d}{dx}A(x) \qquad f(x)$$

$\square$

ex/ Consider $f(t) = t - \cos t$, $f: \mathbb{R} \to \mathbb{R}$.
Find the area function based at $a = 0$.
Verify FTC II, $A'(x) = f(x)$, $x \in \mathbb{R}$.

$$A(x) = \int_0^x f(t)\,dt = \int_0^x (t - \cos t)\,dt$$

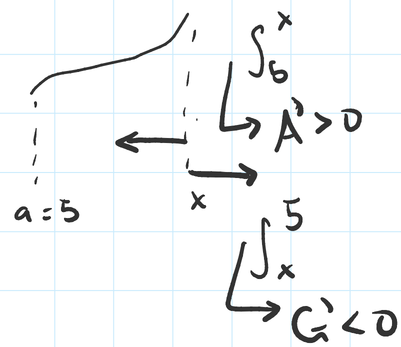
$$= \int_0^x \frac{d}{dt}\left(\frac{t^2}{2} - \sin t\right) dt$$

$A'(x) = f(x)$

$$= \frac{t^2}{2} - \sin t \Big|_0^x = \frac{x^2}{2} - \sin(x).$$

ex/ Compute $\displaystyle \frac{d}{dx}\int_3^x \underbrace{\sin\left(e^{t^2+1}\right)}_{f(t)}\,dt$

$$\overset{\text{FTC II}}{=} f(x) = \sin\left(e^{x^2+1}\right)$$



ex/ Compute $\displaystyle \frac{d}{dx}\overbrace{\int_x^5 \sin\left(e^{t^2+1}\right)dt}^{G}$

$$= \frac{d}{dx}\left(-\int_5^x \sin\left(e^{t^2+1}\right)dt\right)$$

$$= -\sin\left(e^{x^2+1}\right)$$

ex/ Find the derivative of

$$H(x) = \int_1^{x^2} \sin(t)\,dt$$

$$= \int_1^{x^2} \frac{d}{dt}(-\cos(t))\,dt$$

$$= -\cos(t)\Big|_1^{x^2} = -\cos(x^2) - (-\cos(1))$$

$$= \cos(1) - \cos(x^2)$$

$H'(x) = 2x\sin(x^2)$.

ex/ Compute $H'(x)$ where

$$H(x) = \int_1^{x^2} \sin(t^2 + e^t)\,dt$$

$$\frac{d}{\cdot}\int^{x^2} \sin(t^2 + e^t)\,dt$$

$$\frac{d}{dx}\int_1^{x^2} \sin(t^2+e^t)\,dt$$

$$H(x) = f(g(x)), \quad g(x) = x^2$$

$$f(u) = \int_1^u \sin(t^2+e^t)\,dt$$

$$H'(x) = f'(g(x))\cdot g'(x)$$

$$\frac{df}{du} = \frac{d}{du}\int_1^u \sin(t^2+e^t)\,dt \overset{\text{FTC II}}{=} \sin(u^2+e^u)$$

$$\Rightarrow f'(g(x)) = \sin(x^4+e^{x^2})$$

$$g'(x) = \frac{d}{dx}(x^2) = 2x$$

$$\Rightarrow \underline{\frac{d}{dx}\int_1^{x^2} \sin(t^2+e^t)\,dt = \sin(x^4+e^{x^2})\cdot 2x}$$

Chain Rule Version of the FTC II:

Let

$$H(x) = \int_a^{g(x)} f(t)\,dt.$$

Observe $H(x) = A(g(x))$ where $A(u) = \int_a^u f(t)\,dt$ area fcn

$$\Rightarrow H'(x) = A'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x).$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$$

$$\int^b f\,dx = \int^c f\,dx + \int^b f\,dx$$

$\frac{d}{dx}\int_a f(t)\,dt \quad f(g(x))\cdot g'(x)$

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$

ex/ Compute

$$\frac{d}{dx}\int_{x^2}^{e^{2x}} \frac{t}{t+1}\,dt$$

$$= \frac{d}{dx}\left(\int_{x^2}^{0} \frac{t}{t+1}\,dt + \int_0^{e^{2x}} \frac{t}{t+1}\,dt \right)$$

$$= \frac{d}{dx}\left(-\int_0^{x^2} \frac{t}{t+1}\,dt + \int_0^{e^{2x}} \frac{t}{t+1}\,dt \right)$$

$$= -\left(\frac{x^2}{x^2+1}\right)\cdot 2x + \left(\frac{e^{2x}}{e^{2x}+1}\right)\cdot 2e^{2x}.$$

FTC I & II:

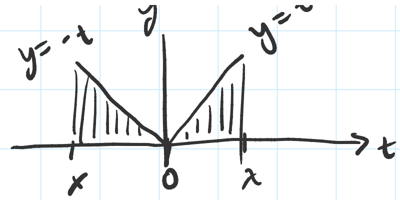
Differentiation & integration are inverses.

$$\left[\; f(x) \xrightarrow{\text{integrate}} \int_a^x f(t)\,dt \xrightarrow{\text{diff.}} \frac{d}{dx}\int_a^x f(t)\,dt = f(x) \;\right]$$

$$\left[\; f(x) \xrightarrow{\text{differentiate}} f'(x) \xrightarrow{\text{integrate}} \int_a^x f'(t)\,dt = f(x) - \underbrace{f(a)}_{\text{up to a constant}}. \;\right]$$

ex/ $A(x) = \int_0^x |t|\,dt.$

Show $A(x) = \frac{1}{2}x|x|.$ check $A'(x) = |x|.$



$x > 0$: $$A(x) = \int_0^x |t|\,dt = \int_0^x t\,dt = \int_0^x \frac{d}{dt}\left(\frac{t^2}{2}\right)dt$$
$$= \frac{t^2}{2}\Big|_0^x = \frac{x^2}{2}.$$

$x < 0$: $$A(x) = \int_0^x |t|\,dt = \int_0^x (-t)\,dt = \int_0^x \frac{d}{dt}\left(-\frac{t^2}{2}\right)dt$$
$$= \frac{-t^2}{2}\Big|_0^x = -\frac{x^2}{2}.$$

$$A(x) = \begin{cases} x^2/2, & x \geq 0 \\ -x^2/2, & x < 0 \end{cases} = \frac{1}{2}x \cdot \underbrace{\begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}}_{|x|}$$
$$= \frac{1}{2}x|x|.$$

Check that $A'(x) = |x|$. Practice final problem.
($\frac{d}{dx}(x|x|)$ is defined & continuous everywhere, $2|x|$.

- End of tested material
- OH tomorrow, 11 - 12