Lecture 16 - An Intro. To Differential Equations

Monday, September 6, 2021 4:51 PM

- **Reminder**: Final is available to view on Gradescope starting this Thursday at 12 pm noon until Friday at midnight. Once you start viewing the final, the timer begins; you have 3 hours 30 minutes to complete and upload your exam, so start before 8:30 pm on Friday for the full time. You can prepare 2 sheets of notes to use, but no other outside resources.
- **Homework 5** is due tomorrow, Wednesday, at 11:59 pm.
- Your lowest percentage homework will be dropped. Don't use this as a reason to skip Homework 5. Instead, try to do well on Homework 5 so one of your lower homework scores can be ignored.
- Both midterms and homework 1-4 are graded. Please review these on Gradescope. Also, I sent everyone individually an email regarding their current grade in the class. Email me if you have any questions.
- I will have office hours today after lecture, from 11 to 12 as usual.
- **Tomorrow** (Wednesday), we will review the practice final in its entirety and some of homework 5. Also, if you have any other questions, feel free to ask them during the review session. Make sure to take the practice final on your own before lecture tomorrow. This is the best way to make full use of the practice final.
  - There is an additional review session from 11 to 12 on Wednesday, to cover the rest of the review material that we did not finish in lecture. This will be recorded and posted to the media gallery in Canvas, for those of you who cannot attend the additional review session.
  - Due to the Labor Day holiday, Nicholas did not have office hours yesterday. Instead, he will be having an office hours on Wednesday at 2 pm. This is the last office hours in the course, so if you have any remaining questions, please attend Nicholas' office hours.
- I will not have office hours on Thursday, and there will be no discussion section on Friday.
- Remember to do to the MDTP Post-Diagnostic test before Thursday at 10 pm; this is part of the requirements of the Summer Bridge program. The link can be found under the module "MDTP Diagnostic Testing" on Canvas.

Differential Equations (not tested)

[
- Math 20D: Differential Equations
- Math 110: Partial Differential Equations
]

What are differential equations (DE)?

- A DE is a relation between a independent variable (input of function), a dependent variable (function), and the derivatives of the dep. variable with respect to the independent variable.
- Used throughout science & engineering

ex/ Newton's Equation position $y(t)$

force = mass × acceleration

$$\left[F(\underline{t}, \underline{y(t)}, \underline{y'(t)}) = m \cdot \underline{y''(t)} \right]$$ 2nd-order DE

ex/ Maxwell's Equations - electromagnetism

ex/ Navier-Stokes Equation - fluid dynamics

ex/ Black-Scholes Equation

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- economics, price evolution of calls/puts

ex/ Lotka-Volterra Equation
- predator/prey

ex/ Diffusion & reaction equation - chemistry

ex/ Equations of solid mechanics & elasticity
- structural engineering.

Think about an algebraic equation

1 var.
$F(x) = 0$
e.g., $0 = F(x) = x^2 - 1 \iff x = +1$ or $x = -1$
solutions to this system are points.

2 var.
$F(x, y) = 0$ indep. variable
dep. variable
e.g. $F(x,y) = y - x^2$. $F(x,y) = 0 \iff y = x^2$
solution is a function $y(x) = x^2$.

In contrast, a DE is an equation relating indep. variable, dep. variable, and some number of derivatives of dep. variable w.r.t indep. variable

$$F(t, y(t), y'(t), y''(t), \ldots, y^{(n)}(t)) = 0$$

order of the differential equation

1st-Order DEs:

$$\underline{F(t, y(t), y'(t))} = 0.$$

ex/ $\underline{y'(t) = 1}$, $\underline{y(0) = y_0}$

ex/ $\underbrace{y'(t) = 1}_{\text{differential equation}}$, $\underbrace{y(0) = y_0}_{\text{initial condition}}$

$y'(t) = 1 \iff$ $y(t)$ is an antiderivative of 1

$$y(t) = t + \underline{C} \xrightarrow{\text{plug in } t=0} y_0 = y(0) = 0 + C \Rightarrow C = y_0.$$

(FTC

integrate both sides

$$y'(s) = 1$$

$$\underset{y(t)-y(a)}{\overset{\|}{\int_a^t y'(s)\,ds}} = \underset{t-a}{\overset{\|}{\int_a^t 1\,ds}}$$

$$\iff y(t) = t + \underbrace{(y(a) - a)}_{C}$$
)

(For n^{th} order differential equation, you have to specify n points of data for unique solution)

ex/ $\underbrace{y'(t) = \overset{\text{integrable}}{f}(t)}_{y \text{ is an antiderivative of } f}$, $\underbrace{y(\overset{\text{time}}{0}) = y_0.}_{\text{initial condition}}$

$$\frac{d}{dt}[y(t)] = f(t)$$

FTC II: $\frac{d}{dt}\left[\int_{\underset{\text{free to choose}}{a}}^t f(s)\,ds\right] = f(t)$

$a=0$

$$y(t) - y(0) \overset{\text{FTC I}}{=} \int_0^t y'(s)\,ds = \int_0^t f(s)\,ds$$

$$\Rightarrow y(t) = y(0) + \int_0^t f(s)\,ds$$

$$\Rightarrow \boxed{y(t) = y_0 + \int_0^t f(s)\,ds.}$$

differential eqn:

$$\frac{d}{dt} y(t) = \frac{d}{dt}\left(y_0 + \int^t f(s)\,ds \right)$$

$$\frac{d}{dt}y(t) = \frac{d}{dt}\left(y_0 + \int_0^t f(s)\,ds\right)$$

$$= \cancel{\frac{d}{dt}y_0} + \frac{d}{dt}\int_0^t f(s)\,ds \underset{\substack{\uparrow \\ \text{FTC II}}}{=} f(t). \quad \checkmark$$

Initial condition

$$y(0) = y_0 + \cancelto{0}{\int_0^0 f(s)\,ds} = y_0 \quad \checkmark$$

ex/ $\underbrace{y'(t) = t^2}_{\text{y is an antiderivative of } t^2}$, $\quad y(0) = 2.$

$$\Rightarrow y(t) = \frac{1}{3}t^3 + C$$

$$2 = y(0) = \frac{1}{3}(0)^3 + C = C$$

$$\Rightarrow y(t) = \frac{1}{3}t^3 + 2.$$

$$y(t) = y_0 + \int_0^t f(s)\,ds$$

$$y(t) = 2 + \int_0^t s^2\,ds = 2 + \int_0^t \frac{d}{ds}\left(\frac{1}{3}s^3\right)ds$$

$$= 2 + \left.\frac{s^3}{3}\right|_0^t = \frac{t^3}{3} + 2.$$

ex/ $\left[\begin{matrix} y'(t) = \overset{\substack{\text{unknown} \\ |}}{y(t)} \\ y(0) = 1 \end{matrix}\right]$ DIFFERENT than $y'(t) = \overset{\substack{\text{known} \\ |}}{\underline{\underline{f}}}(t).$

$$\rightarrow y(t) = C \cdot e^t \qquad y'(t) = \frac{d}{dt}(Ce^t) = C\frac{d}{dt}e^t$$

$$= Ce^t = y(t).$$

$$1 = y(0) = Ce^0$$

$$= C$$

$$\Rightarrow y(t) = e^t.$$

$$\left(\begin{matrix} y'(t) = y(t) \\ y(0) = 1 \end{matrix}\right) \leftarrow$$ define the exponential function as the

$$\Rightarrow y(t) = e^t.$$

$$\left(\begin{array}{c} y'(t) = y(t) \\ y(0) = 1 \end{array} \right) \leftarrow$$

exponential function as the solution to this differential equation.

$$y' = y$$

$$\frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt$$

separation of variables.

$$\int_{y(0)}^{y(t)} \frac{1}{y}\, dy = \int_0^t dt$$

$$\ln y(t) - \ln(y(0)) = t$$

$$\Rightarrow \ln(y(t)) = \ln(y(0)) + t$$

$$\Rightarrow y(t) = e^{\ln(y(t))} = e^{\ln(y(0))} \cdot e^t$$

$$= y(0) \cdot e^t.$$

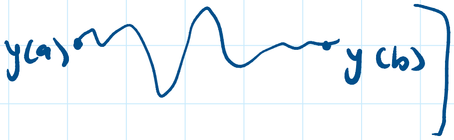
2nd order differential equations

$$F(t, y(t), y'(t), y''(t)) = 0.$$

- specify two points of data

initial value problem $\left\{ \begin{array}{l} \text{function at a time } y(0) \\ \text{derivative at } \uparrow \text{ time } y'(0) \end{array} \right.$

boundary value problem $\left\{ \text{function at two different times, } y(a) \text{ \& } y(b). \right.$



ex/ object under constant acceleration

y position of object

y position of object

$$\text{DE}\begin{cases} y''(t) = a \\ y(0) = y_0 \\ y'(0) = v_0 \end{cases}$$

($a \in \mathbb{R}$ is constant in time)

$$my''(t) = \underbrace{ma}_{\text{constant force.}}$$

$$y''(s) = a$$

Integrate

$$\int_0^t y''(s)\,ds = \int_0^t a\,ds$$

$$\int_0^t y''(s)\,ds = \int_0^t \frac{d}{ds}\big(y'(s)\big)\,ds \overset{\text{FTC I}}{=} y'(s)\Big|_0^t = y'(t) - y'(0)$$

$$\int_0^t a\,ds = a \cdot t$$

$$\Leftrightarrow \quad y'(t) = y'(0) + at$$

$$= V_0 + at.$$

Integrate again

$$y(t) - \underset{\substack{\uparrow \\ y_0}}{y(0)} \underset{\substack{\uparrow \\ \text{FTC I}}}{=} \int_0^t y'(s)\,ds = \int_0^t (V_0 + as)\,ds$$

$$= V_0 t + \frac{1}{2}at^2$$

$$\Leftrightarrow \quad y(t) = y_0 + V_0 t + \frac{1}{2}\underset{\substack{\uparrow \\ \text{ex gravity} \\ a = -g}}{a}t^2 \longleftarrow \text{Kinematic Equations (object under constant acceleration)}$$

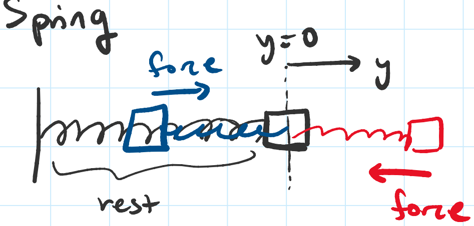
Newton's 2nd Law

- mass × acceleration = force
- position of object $y(t)$

$$my''(t) = F(t, y(t), y'(t))$$

$$m\,y''(t) = F(t, \underline{\underline{y(t)}}, y'(t))$$

ex/ Spring



spring constant $k>0$.

Hooke's law $F = -k\,y(t)$

Newton's 2^{nd} $\Longleftrightarrow$ $m\,y''(t) = -k\,y(t).$

$m=1, \; k=1$

$\Rightarrow \; y''(t) = -y(t).$ 2^{nd} deriv. of function = -function

$$(*)\begin{cases} \frac{d}{dt}\frac{d}{dt}\sin(t) = \frac{d}{dt}\cos(t) = -\sin(t) \\ \frac{d}{dt}\frac{d}{dt}\cos(t) = \frac{d}{dt}(-\sin(t)) = -\cos(t) \end{cases}$$

$$\begin{cases} y''(t) = -y(t) \\ \left.\begin{array}{l} y'(0) = V_0 \\ y(0) = y_0 \end{array}\right\} \text{fix } A, B \end{cases}$$

$$\frac{d^2}{dt^2}\left[\underline{A}\sin(t) + \underline{B}\cos(t)\right]$$

linear combination of both particular solutions

$$= A\frac{d^2}{dt^2}\sin t + B\frac{d^2}{dt^2}\cos t$$

$$= -A\sin t - B\cos t = -\left[A\sin t + B\cos t\right]$$

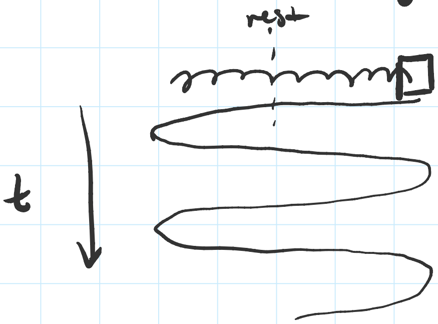
$$y(t) = A\sin(t) + \overset{y_0}{\overset{\shortparallel}{B}}\cos(t)$$

$$y_0 = y(0) = A\cancel{\sin(0)} + B\cos(0) = B$$

$$y_0 = y(0) = A\cancel{\sin(0)} + B\cos(0) = B$$

$$v_0 = y'(0) = A\cos(0) - \cancel{B\sin(0)} = A$$

$$\Rightarrow \quad y(t) = v_0\sin(t) + y_0\cos(t).$$



$v_0 = 0$

$y(t) = y_0\cos(t)$

$$m y''(t) = F(t, y(t), y'(t))$$

We say a system is conservative if the force is given by

$$F(y(t)) = -V'(y(t))$$

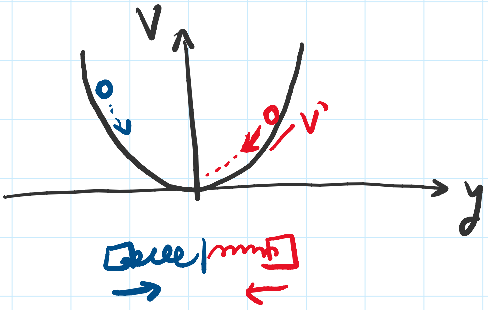
V potential energy.

force wants to decrease potential energy

ex

$$F(y) = -ky$$

$$V(y) = \frac{1}{2}ky^2 \Rightarrow -V'(y) = -ky$$



Theorem:

For a solution of conservative force

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conservative force

$$my''(t) = -V'(y(t)),$$

the energy

$$E = \underbrace{\frac{1}{2}m(y'(t))^2}_{K=\frac{1}{2}mv^2 \text{ kinetic energy.}} + \underbrace{V(y(t))}_{\text{potential energy}}$$

is conserved.

proof: Chain rule

$$\frac{d}{dt}E = \frac{d}{dt}\left(\frac{1}{2}m(y'(t))^2 + V(y(t))\right)$$

$$= \frac{1}{2}m\,\frac{d}{dt}\left[(y'(t))^2\right] + \frac{d}{dt}\left[V(y(t))\right]$$

↓ chain rule

$$= \frac{1}{2}m \cdot 2\,y'(t) \cdot y''(t) + V'(y(t)) \cdot y'(t)$$

$$= \left(my''(t) + V'(y(t))\right)y'(t)$$

$$\overset{\text{Newton's law}}{=} \left(\underbrace{-V'(y(t)) + V'(y(t))}_{0}\right)y'(t)$$

$$= 0$$

$$\Rightarrow \frac{d}{dt}E = 0. \Rightarrow E \text{ is constant in time.} \quad \square$$

OH in 10 mins.