Lecture 4 - Squeeze Thm (& trig limits), IVT.

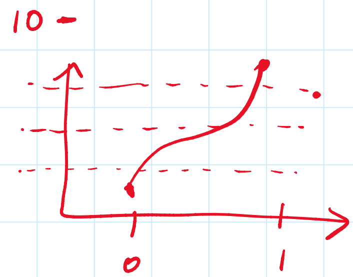
Thursday, August 5, 2021 2:53 PM

· Read Sections 2.6 and 2.8

$f: A \to B$

Injectivity: $x \neq y \Rightarrow f(x) \neq f(y)$

$\boxed{f(x) = y}$ $x \in A$, $y \in B$



$f(x) = 10$

$f: [0,1] \to \mathbb{R}$

UNIQUENESS

at most one solution

Surjectivity: $y \in B$, there is some $x \in A$ s.t. $f(x) = y$

at least one solution

EXISTENCE

$\Downarrow$

Bijectivity: there exists <u>exactly</u> one solution x to $f(x) = y$ for each $y \in B$.

$x = f^{-1}(y)$.

eg. $f: \mathbb{R} \to [0, \infty)$ given $f(x) = x^2$.

$x^2 = f(x) = 4$ $\quad x = 2$, $x = -2$ $\quad x \neq f^{-1}(4)$ not invertible

$f: [0,\infty) \to [0,\infty)$ injective & surjective.

$x = 2$

$f^{-1} = \sqrt{\ }$

Limits at infinity:

$\lim_{x \to \pm\infty} f(x) = L$ if $f(x)$ approaches L as x goes to $\pm\infty$.

<u>Polynomials</u>: $a_n x^n + a_n x^{n-1} + \dots + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$

x^n ← monomial

<u>Prop</u> $(n > 0)$

$\lim_{x \to +\infty} x^n = \infty$, $\quad \lim_{x \to +\infty} x^{-n} = 0$.

$$\lim_{x\to+\infty} x^{\cdot\cdot} = \infty, \qquad \lim_{x\to+\infty} x^{\cdot\cdot} = 0.$$

(e.g. x^2, $\sqrt{x}$ $\qquad$ $\frac{1}{x^2}$, $\frac{1}{\sqrt{x}}$)

for n integer, $\displaystyle\lim_{x\to-\infty} x^n = \begin{cases} +\infty, & n \text{ even} \\ -\infty, & n \text{ odd} \end{cases}$

$$\lim_{x\to-\infty} x^{-n} = 0.$$

Polynomials

$$\lim_{x\to\infty}\left(\underset{\neq 0}{a_n} x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0\right) \qquad \lim_{x\to\infty} (a_n x^n)$$

e.g. $x^4 + x^2$ $\qquad x = 10$, $x^4 = 10^4$, $x^2 = 10^2$ (factor of 100) $\qquad x = 1000$, $x^4 = 10^{12}$, $x^2 = 10^6$ (factor of 1,000,000)

Rational Function

$$(*)\quad \lim_{x\to\pm\infty}\left[\frac{\underset{\neq 0}{a_n} x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0}{\underset{\neq 0}{b_m} x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0}\right]$$

$$= \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m}.$$

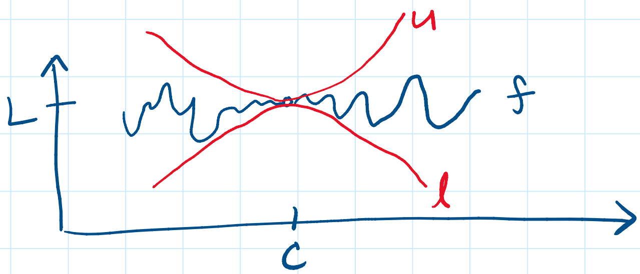
e.g., $$\lim_{x\to\infty} \frac{x^5 + 7}{2x^5 + 4x + 1} \overset{(*)}{=} \frac{1}{2} \lim_{x\to\infty} \frac{x^5}{x^5} = \frac{1}{2} \lim_{x\to\infty} 1 = 1/2.$$

[Squeeze Theorem]

Assume for $x \neq c$ (limit point) in some open interval containing c,
one has: $\quad l(x) \leq f(x) \leq u(x)$
and $\displaystyle\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$,

and $\lim_{x \to c} l(x) = L = \lim_{x \to c} u(x)$,

then $\lim_{x \to c} f(x)$ exists and equals L.

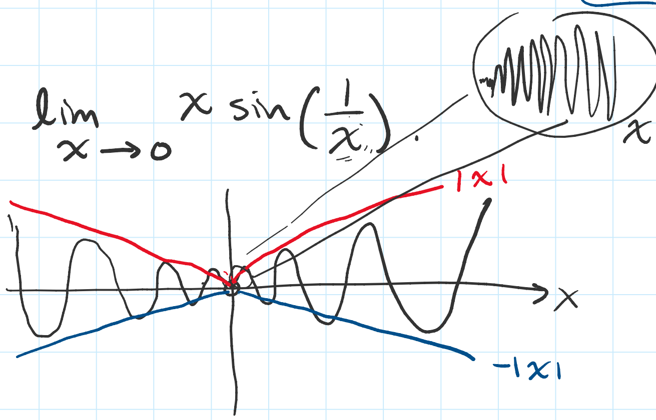


proof:

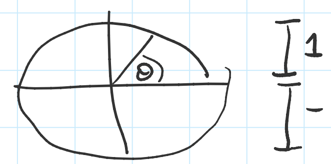
$$\underset{\downarrow (x\to c) \atop 0^-}{l(x) - L} \le \underset{\downarrow (x\to c) \atop 0}{\underline{f(x) - L}} \le \underset{\downarrow (x\to c) \atop 0^+}{u(x) - L}$$

$\square$

ex1 $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.



$x=0$ $\cancel{0 \cdot \sin\left(\frac{1}{0}\right)}$



$\underline{|\sin(\theta)| \le 1}$ for any θ

$\left|\sin\left(\frac{1}{x}\right)\right| \le 1$ for any $x \neq 0$.

$$\left|x \sin\left(\frac{1}{x}\right)\right| = |x| \underbrace{\left|\sin\left(\frac{1}{x}\right)\right|}_{\le 1} \le |x|.$$

$$|a| \le b \overset{0}{\wedge} \iff -b \le a \le b$$

$$\underline{-|x|} \le \underline{x \sin\left(\frac{1}{x}\right)} \le \underline{|x|}$$

$\left.\begin{array}{l} \lim_{x \to 0} |x| = 0 \\ \lim_{x \to 0} -|x| = 0 \end{array}\right\}$ squeeze thm. $\Longrightarrow \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

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ex2 $\lim_{x \to 0} x^2 e^{\cos\left(\frac{1}{x}\right)}$

$\left|\cos\left(\frac{1}{x}\right)\right| \le 1$, $x \neq 0$.

$$\left|x^2 e^{\cos\left(\frac{1}{x}\right)}\right| \le x^2 e^1$$

$$\| \quad |x^2| \cdot \left|e^{\cos\left(\frac{1}{x}\right)}\right|$$

$|\cos(\frac{1}{x})| \le 1, \quad x \neq 0.$

e^x is monotone increasing $x \geq y \Rightarrow e^x \geq e^y$.

so $e^1 \geq e^{\cos(\frac{1}{x})}$

$|x^2 e^{\cos(1/x)}| \le x^2 e$

$$\Rightarrow \underset{\downarrow \atop 0}{-x^2 \cdot e} \le \underbrace{x^2 e^{\cos(1/x)}}_{\lim_{x\to 0}(\cdots) = 0.} \le \underset{\downarrow \atop 0}{x^2 \cdot e}$$

$|\sin\theta| \le 1$

$\left|\frac{\sin\theta}{\theta}\right| \le \frac{1}{|\theta|}$

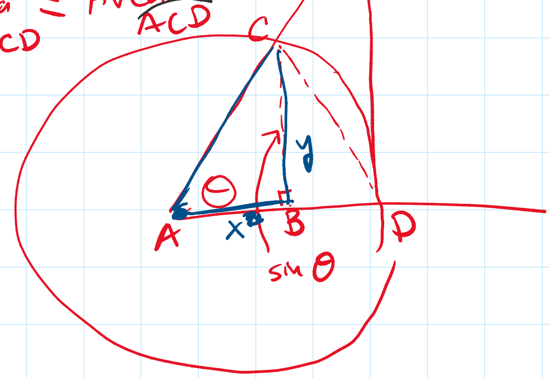
$\lim_{\theta \to 0}$

examples in 2.6:

a) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ ✓

b) $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$ ✓

Area $\triangle ACD$ $\le$ Area ~~ACD~~ $\le$ Area $\triangle APE$.



a) $\boxed{\cos\theta \le \frac{\sin\theta}{\theta} \le 1}$ ✓

$\cos\theta \xrightarrow{\theta\to 0} = \cos(0) = 1$

$\frac{\sin\theta}{\theta} \xrightarrow{\text{squeeze}} 1$

$1 \xrightarrow{\theta\to 0} 1$

$\boxed{\cos^2\theta + \sin^2\theta = 1}$

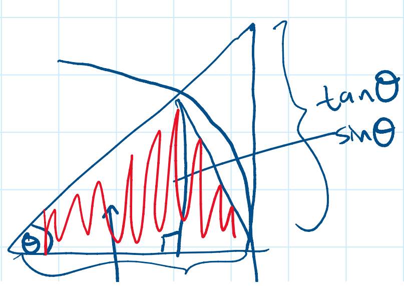
b) $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta \to 0} \left(\frac{1-\cos\theta}{\theta}\right) \frac{1+\cos\theta}{1+\cos\theta}$

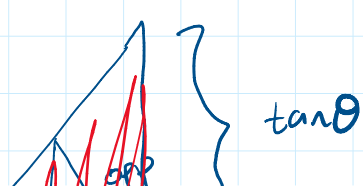
$= \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)}$

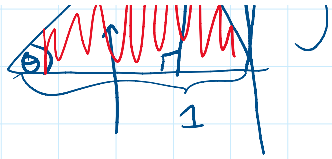
$= \lim_{\theta \to 0} \left[\sin\theta \left(\frac{\sin\theta}{\theta}\right) \frac{1}{1+\cos\theta} \right]$

$\quad\;\; \downarrow \qquad\quad \downarrow (a) \qquad \downarrow$ ✓

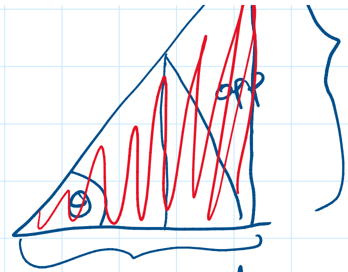
$\quad\;\; 0 \quad \cdot \quad 1 \quad \cdot \quad 1/2 \quad = 0$







Area $= \frac{1}{2}\sin\theta$

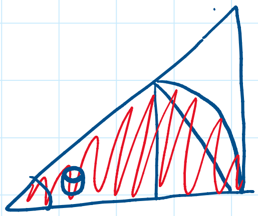


$\tan\theta$

$\tan\theta = \frac{opp}{adj} = \frac{opp}{1}$.

$1 = adj$

$\hookrightarrow$ Area $= \frac{1}{2}\tan\theta$



Area $= \pi = \frac{2\pi}{2} = \theta$

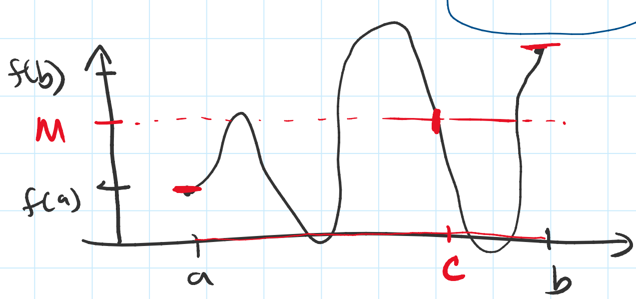
Area $= \boxed{\frac{1}{2}\theta} \leftarrow$

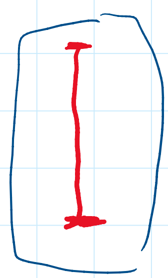
$$\frac{1}{2}\sin\theta \le \frac{1}{2}\theta \le \frac{1}{2}\tan\theta \quad (\text{Area: } \triangle \le ⌔ \le \triangle)$$

$$\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta} \quad \left(\Rightarrow \cos\theta \le \frac{\sin\theta}{\theta} \le 1\right)$$

Intermediate Value Theorem (IVT)

Thm: Let f be continuous on a closed interval $[a,b]$. Then, for any M strictly between $f(a)$ and $f(b)$ ($f(a) < M < f(b)$ OR $f(b) < M < f(a)$), then there exists $c \in (a,b)$ such that $f(c) = M$.

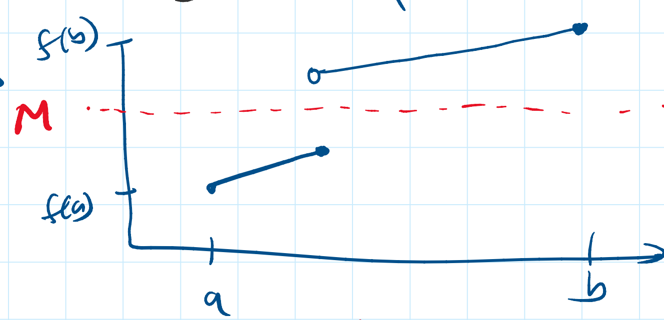


pf: 

$[f(a), f(b)] \subseteq f([a,b])$

subset

e.g., discontinuous

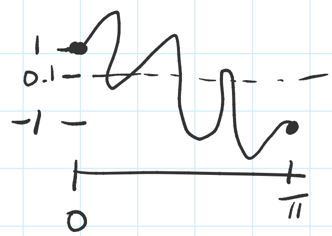


there is no c such that $f(c) = M$

need continuity of f!

need continuity of f!

ex1: Prove $\cos x = 0.1$ has at least one solution for $x \in (0, \pi)$.



cos is continuous

$\cos(0) = 1$, $\cos(\pi) = -1$ IVT.

ex2: Prove $\cos : [0, \pi] \to [-1, 1]$ is a surjection.

$\cos(x) = y$, $y \in [-1, 1]$.

$y = -1$, $x = \pi$

$y = 1$, $x = 0$

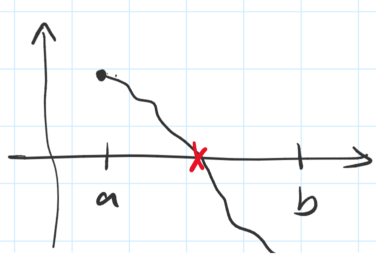
$\cos(x) = y$, $y \in (-1, 1)$ $\Rightarrow$ IVT, there is a solution x, $x \in (0, \pi)$, for any $y \in (-1, 1)$.

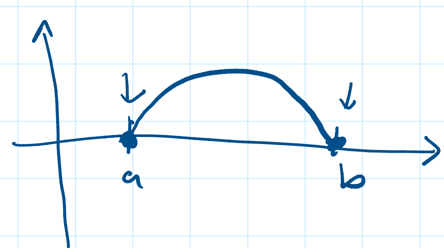
Solving equation $\boxed{f(x) = 0}$

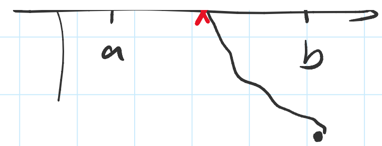
Corollary: Immediate consequence of a theorem. (not much to prove)

Corollary (Existence of zeroes)

If f is continuous on $[a, b]$ and one of $f(a)$ or $f(b)$ is nonpositive and the other is nonnegative, then there is atleast one solution $x \in [a, b]$ to $f(x) = 0$.







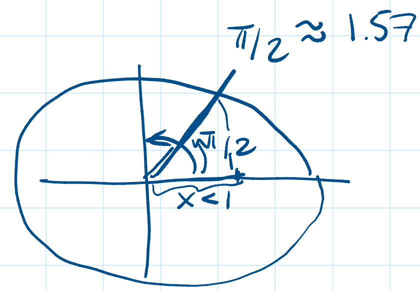
ex/ Show that $\cos x = x$ has a solution for x in $[0,1]$.

$\to$ Consider $f : [0,1] \to \mathbb{R}$ given by $f(x) = x - \cos x$.

Is f continuous? Yes, by sum law.

$\to f(0) = 0 - \cos(0) = -1$.

$\underline{\underline{f(1)}} = 1 - \underbrace{\cos(1)}_{<1} > 0$



By IVT corollary, there exists a zero of f at x in $[0,1]$.

ex/ Show $\tan^{-1}(x) = \cos^{-1}(x)$ has a solution.

Office Hours: Tu Th 11 am - 12 pm (or by appointment).