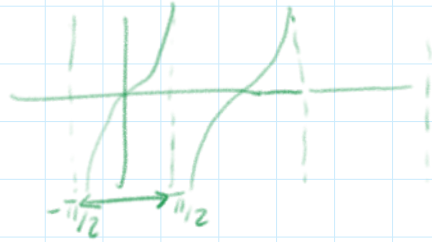
Lecture 5 - The Derivative, The Derivative as a Function, The Product and Quotient Rule

Monday, August 9, 2021 10:26 PM

· Read sections 3.1, 3.2, 3.3.

ex Show $\boxed{\tan^{-1}(x) = \cos^{-1}(x)}$ has a solution.

- Trig functions & inverses
- Continuity of inverses
- Intermediate value theorem



trig fcns & prop.

$\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$

$\cos : [0, \pi] \to [-1, 1]$

are both continuous bijections.

inverses

$\Rightarrow$ $\tan^{-1} : (-\infty, \infty) \to (-\pi/2, \pi/2)$

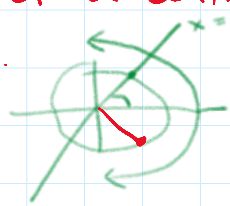
$\cos^{-1} : [-1, 1] \to [0, \pi]$

are both continuous. (by Thm in Lecture 3).

$\underbrace{\tan^{-1}(x) - \cos^{-1}(x)}_{f(x)} = 0$

$f(x) = \underbrace{\tan^{-1}(x)}_{\text{defined on } \mathbb{R}} - \underbrace{\cos^{-1}(x)}_{\text{defined on } [-1,1]}$ $\quad f : [-1,1] \to \mathbb{R}$, f is continuous

(• note: we are restricting the domain of $\tan^{-1}$ from $\mathbb{R}$ to instead be $[-1,1]$. The restriction of a continuous to a smaller domain is still continuous.)



$f(1) = \tan^{-1}(1) - \cos^{-1}(1)$

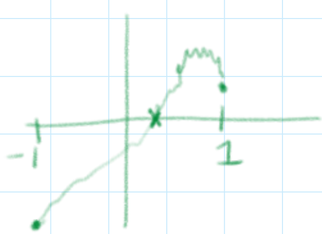
$\theta = \cos^{-1}(1) \Leftrightarrow$ unique θ in $[0, \pi]$ such that $\cos\theta = 1 \Rightarrow \theta = 0$.

$\alpha = \tan^{-1}(1) \Leftrightarrow$ unique α in $(-\pi/2, \pi/2)$ such that $\tan\alpha = 1$

$1 = \tan\alpha = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \alpha = \pi/4$

$f(1) = \tan^{-1}(1) - \cos^{-1}(1) = \frac{\pi}{4} - 0 > 0$.

$f(-1) = \underbrace{\tan^{-1}(-1)}_{-\pi/4} - \underbrace{\cos^{-1}(-1)}_{=\pi} = -\frac{\pi}{4} - \pi = -\frac{5\pi}{4} < 0$.



$\Rightarrow$ [By corollary to IVT, there is a zero to f, i.e. $f(x) = 0$ for some x.

some x.

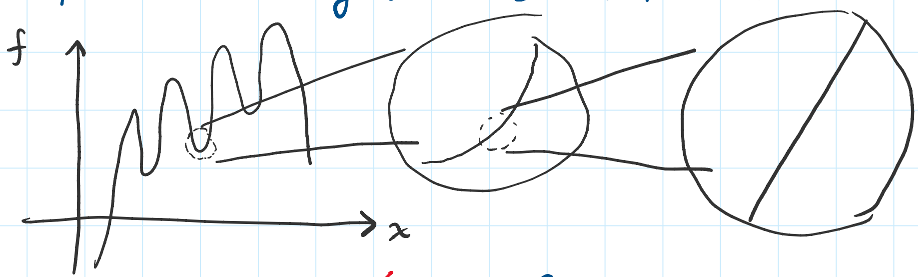
$\boxed{f(x)=0}$ "Bisection method"

Derivative

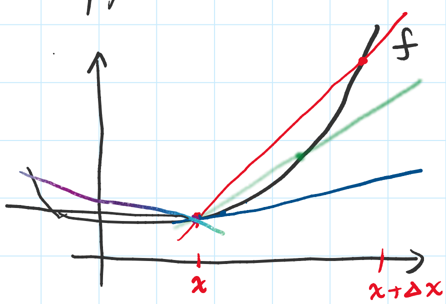
- Newton $F = ma$ (force, mass, acceleration)

$a \approx \frac{\Delta v}{\Delta t \to 0}$, $v \approx \frac{\Delta x}{\Delta t \to 0}$.

- Also, simultaneously Leibniz invented calculus.



"Linear Approximation"



$$\left[m_1 = \frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$ Difference Quotient.

$\Delta x \to 0$ secant line goes to a tangent line.

Definition:

The derivative of a function f at $x=a$ is the slope of the tangent line to f at $x=a$ (if it exists), denote
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

- If this limits exists, we say f is differentiable at $x=a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$x = a + \Delta x$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ Indeterminate form $(0/0)$

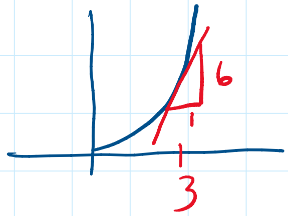
ex/ Compute slope of the tangent line to $f(x) = x^2$ at $x=3$.

$$\Big[f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 3^2}{h} \Big]$$

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$$

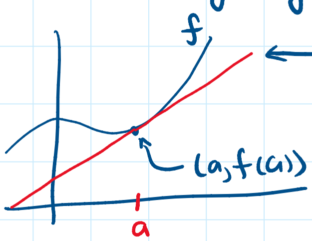
$$= \lim_{h\to 0} \frac{\cancel{9} + 6h + h^2 - \cancel{9}}{h} = \lim_{h\to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= 6.$$



· Find the equation of the tangent line (at a point). to a graph

$y - y_0 = m(x - x_0)$ point-slope form for a line.



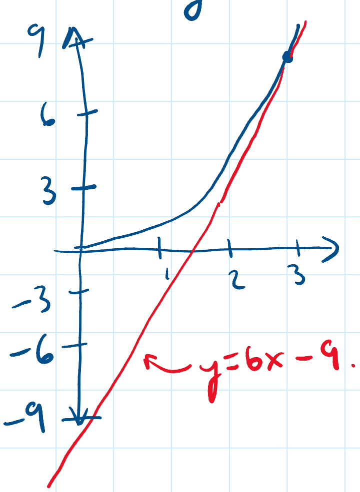
← slope is $f'(a)$, located $(a, f(a))$

$$y - f(a) = f'(a)\cdot(x-a).$$

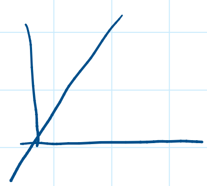
ex/ Find the equation of tangent line to $f(x) = x^2$ at $x=3$.

$$y - f(3) = f'(3)\cdot(x-3)$$

$$y - 9 = 6(x-3) \longrightarrow y = 6x - 9.$$



Linear Functions



$f(x) = mx + b$ ←

claim: $f'(a) = m$.

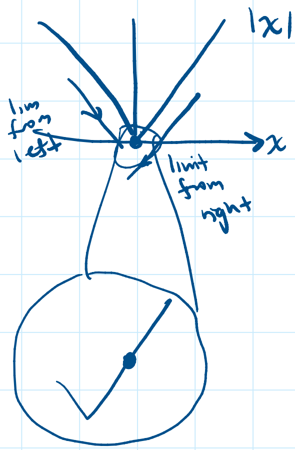
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{m(a+h) + b - (ma + b)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{m(a+h) + b - (ma+b)}{h}$$

$$= \lim_{h \to 0} \frac{m\cancel{h}}{\cancel{h}} = \lim_{h \to 0} m = m.$$

Corollary: $f(x) = c$, a constant function.
$f'(a) = 0$.

e.g. of non-differentiable function $f(x) = |x|$ at $x=0$

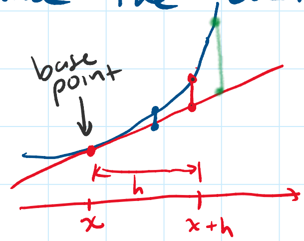


$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} +1 & \text{from right} \\ -1 & \text{from left} \end{cases} = \text{DNE}.$$

$\Rightarrow$ f is not differentiable at $x=0$, but it is differentiable away from $x=0$. Because the derivative is local.

Linear Approximation:



$$\overline{f'(a)} = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

By defn. of limit,

$\Rightarrow$ $\left[f'(a) \approx \dfrac{f(a+h) - f(a)}{h} \right]$ for h small.

base point

$$\Rightarrow \boxed{f(a+h) \approx f(a) + h \cdot f'(a)}$$ Linear approximation

e.g. compute approximately $\overset{f}{\overset{\|}{\sin}}(0.1)$. $h = 0.1$, $\underline{\underline{a=0}}$

we choose $a=0$ since we know $\sin(0) = 0$.

$$f(0.1) \approx \underset{\underset{0}{\|}}{f(0)} + 0.1 \cdot f'(0)$$

$$\approx 0.1 \cdot f'(0)$$

$$f'(0) = \lim \frac{\sin(0+h) - \sin(0)}{h} = \lim \frac{\sin(h)}{h}$$

$\approx 0.1 \cdot f'(0)$

$$f'(0) = \lim_{h\to 0} \frac{\sin(0+h) - \cancel{\sin(0)}}{h} = \lim_{h\to 0} \frac{\sin(h)}{h} = 1$$

$\Rightarrow \left[\sin(0.1) = f(0.1) \approx 0.1\right]$ $\quad \sin(0.1) = 0.0998334\ldots$

$\sin\theta \approx \theta$ for θ small. $\quad \approx 0.1$

3.2 | The derivative as a function.

- f to be differentiable at a point $\rightsquigarrow$ $f'(a) = \lim(\ldots)$ exists.
- Let $f: I \to \mathbb{R}$, where I some interval.

  Define the derivative of f as a function

  ($f': I_1 \to \mathbb{R}$ where $I_1 \subseteq I$ such that the derivative of f is defined at each point in I_1.

$f(x)$

$f'(x)$

ex/ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$

$f': (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$

$I = \mathbb{R}$, $I_1 = \mathbb{R} \setminus \{0\}$.

- If f' exists on all of I, then we call differentiable (on the domain I).

e.g. $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = mx + b$

$f'(x) = m$ for any $x \in \mathbb{R}$ $\Rightarrow f': \mathbb{R} \to \mathbb{R}$

$\Rightarrow$ a linear function is differentiable (on all of $\mathbb{R}$)

<u>notation</u>:

$f'(x)$, $\quad$ Leibniz $\dfrac{df}{dx} \approx \dfrac{\Delta f}{\Delta x \to 0}$ $\quad dx$ represents $\Delta x \to 0$.

$$\frac{df}{dx} = \underbrace{\frac{d}{dx}}(f). \qquad \text{e.g. } \frac{d}{dx}(mx+b) = m.$$

ind. variable

$f'(c) = \dfrac{df}{dx}\Big|$

(Physicists: independent variable being time t, $\dfrac{df}{dt}$, $\dot{f}$)

ind. variable $f'(c) = \left.\dfrac{df}{dx}\right|_{x=c}$. (Physicists: independent variable being time t, $\dfrac{df}{dt}$, $\dot{f}$)

Power rule: For any exponent n,

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

$$\frac{d}{dx}(x^n) = n\, x^{n-1}$$

lower exp.

subtract one

$\left(\text{e.g. } \left.\frac{d}{dx}(x^2)\right|_{x=3} = \left.2x\right|_{x=3} = 6\right)$

proof: n positive integer.

$$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Binomial expansion: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

$\binom{n}{k} = n \text{ choose } k = \dfrac{n!}{k!\,(n-k)!}$

e.g. $(x+y)^2 = x^2 + 2xy + y^2$

$n=2$ $= \underset{\substack{\shortparallel\\1}}{\binom{2}{0}} x^2 y^0 + \underset{\substack{\shortparallel\\2}}{\binom{2}{1}} x^1 y^1 + \underset{\substack{\shortparallel\\1}}{\binom{2}{2}} x^0 y^2$

$n=2 \to$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$\binom{n}{0}$

$$= \lim_{h\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} x^{n-k} h^k - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\overset{(k=0)}{\cancel{x^n}} + \overset{\substack{n\\ \shortparallel}}{\binom{n}{1}} \overset{(k=1)}{x^{n-1} \cdot h} + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \left(n x^{n-1} + \underbrace{\sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^{k-1}}_{\to 0}\right)$$

$$= n x^{n-1}$$

$\square$

Rule: If f and g are differentiable, then

Rule: If f and g are differentiable, then so is af and $f+g$ (where a is some constant).

(follows from sum & constant multiple laws for limits).

$\Rightarrow$ polynomials are differentiable.

$$\frac{d}{dx}(x^4 + x^3 + 2x^2 + 1)$$

$$= \frac{d}{dx}(x^4) + \frac{d}{dx}(x^3) + \frac{d}{dx}(2x^2) + \cancel{\frac{d}{dx}(1)}$$

$$= 4x^3 + 3x^2 + 4x + 0.$$