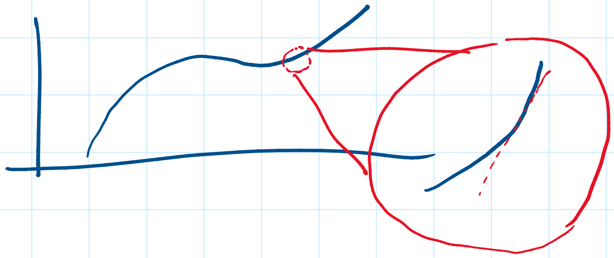
Read sections 3.3, 3.4, 3.5, 3.7

- Derivative gives slope of tangent line to f (at a pt.)
  - If derivative > 0 at a point, function locally increasing.
  - If derivative < 0 " , function locally decreasing.



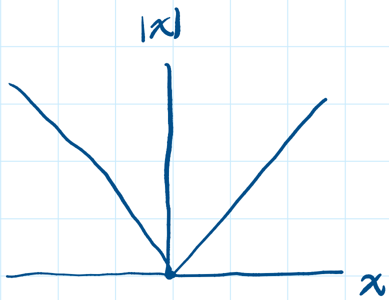
Linear Approx:

$$f(x+h) \approx f(x) + \underbrace{h}_{>0} \cdot \underbrace{f'(x)}_{>0}$$

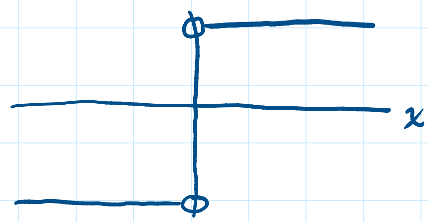
h small

$$f(x+h) \gtrsim f(x)$$

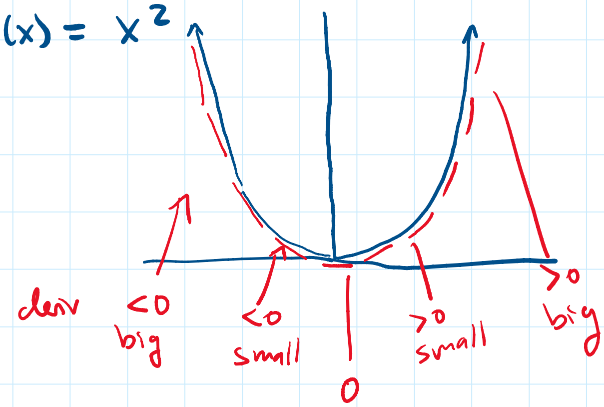
e.g., $|x|$



$\frac{d}{dx} \Longrightarrow$

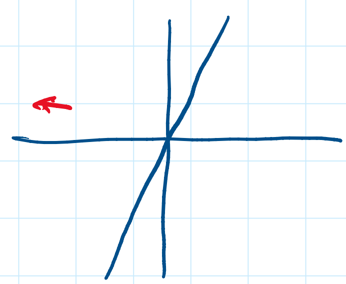


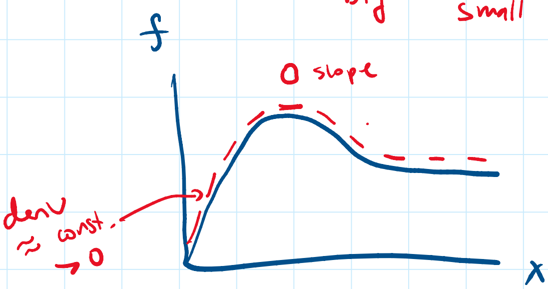
e.g., $f(x) = x^2$

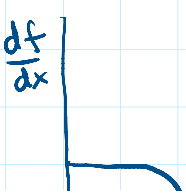


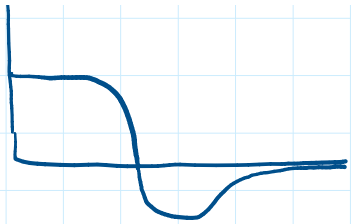
$\frac{d}{dy} \Longrightarrow$

$f'(x) = 2x.$









Polynomials, Exponentials, Trig, and Logarithm

Exponential

$$\frac{d}{dx}[e^x] = \overset{f(x)}{\overset{\shortparallel}{e^x}}. \qquad \frac{d}{dx}(2^x) \neq 2^x. \qquad \boxed{e^h = \sum_{k=0}^{\infty} \frac{h^k}{k!}}$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = e^x \underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{=1.}$$

$$= e^x.$$

Differentiability implies continuity:

Thm If f is differentiable at $x=c$, then it is continuous at c.

pf:

$$\lim_{x\to c} f(x) = f(c)$$

$$0 \overset{?}{=} \lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\left(\frac{[f(x)-f(c)]}{x-c}\cdot(x-c)\right)$$

$$= \left[\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right] \lim_{x\to c}(x-c) = f'(c)\cdot 0 = 0$$

Converse: not true

(Weierstrass: continuous, but nowhere differentiable.)

(Section 3.3)

Product Rule

Thm:

Let f and g be differentiable, then the product fg is and

(Newton) $(fg)' = f'g + fg'$ ←

product fg is and

(Newton) $(fg)' = f'g + fg'$ ←

(Leibniz) $\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$.

pf: $\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$

$= \lim_{h\to 0}\left(\underbrace{\frac{[f(x+h) - f(x)]}{h}}_{\to f'(x)} \underbrace{g(x+h)}_{\to g(x)} + \underbrace{f(x)}_{\to f(x)} \underbrace{\frac{[g(x+h) - h]}{h}}_{\to g'(x)} \right)$ □

ex/ $\frac{d}{dx}\left[\overbrace{(x^5+4x^4+3x^3+x^2-x+1)}^{f}\overbrace{(x^4-x^3+2x^2+x+1)}^{g}\right]$

$= \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}$

$= (5x^4+16x^3+9x^2+2x-1+0)\cdot(x^4-x^3+2x^2+x+1)$
$+ (x^5+4x^4+3x^3+x^2-x+1)\cdot(4x^3-3x^2+4x+1+0)$.

ex/ $\frac{d}{dx}[\underbrace{e^x}_{= f(x)} \underbrace{(x^2+1)}_{= g(x)}]$

$= \frac{df}{dx}g + f\frac{dg}{dx} = \left(\frac{d}{dx}e^x\right)(x^2+1) + e^x\frac{d}{dx}(x^2+1)$

$= e^x(x^2+1) + e^x(2x+0)$

$= e^x(x^2+2x+1) = e^x(x+1)^2$.

Quotient Rule:

Thm: If f and g are differentiable, then f/g is differentiable

Quotient Rule:

<u>Thm</u>: If f and g are differentiable, then f/g is differentiable at x whenever $g(x) \neq 0$, and

$$\frac{d}{dx}\left[\frac{f}{g}\right]_x = \left(\frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}\right)_x$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

<u>pf</u>: Textbook $\quad Q \overset{\text{def}}{:=} \frac{f}{g} \qquad f = Qg$

$$f' = Q'g + Qg' \qquad Q' = \left(\frac{f}{g}\right)'$$

(This proof is wrong because it assumes Q is diff.) $\square$

<u>e.g.</u> $\displaystyle \frac{d}{dx}\left(\frac{x^3}{x^2+1}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$

$$= \frac{(x^2+1)\,3x^2 - x^3\cdot(2x+0)}{(x^2+1)^2}.$$

<u>e.g.</u> $\displaystyle \frac{d}{dx}[x^n] = nx^{n-1}.$

$$\frac{d}{dx}[x^{-n}] = \frac{d}{dx}\left[\frac{1}{x^n}\right]$$

$$= \frac{x^n\cancelto{0}{\frac{d}{dx}(1)} - 1\cdot\frac{d}{dx}(x^n)}{(x^n)^2} = \frac{-nx^{n-1}}{x^{2n}} = \frac{-nx^{-n-1}}{} \qquad \surd\, x^{-n-1}$$

$\square$

Section 3.4: Rates of Change

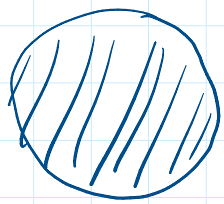
Derivative of f at x_0 represents instantaneous rate of change

Derivative at x_0 represents instantaneous rate of change of f with respect to x (at x_0)

$$\frac{\text{units } y}{\text{units } x} \rightarrow f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x)}{\Delta x}$$

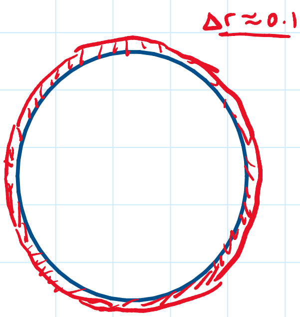
e.g. car dist ~ mi time ~ hr velocity ~ mi/hr.

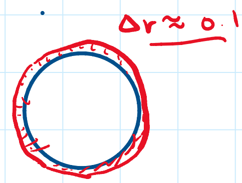
ex/ $A = \pi r^2$



Compare $\frac{dA}{dr}$ at $r=1$ and $r=3$.

$$\frac{dA}{dr} = 2\pi r \qquad \left.\frac{dA}{dr}\right|_{r=1} = 2\pi \qquad \left.\frac{dA}{dr}\right|_{r=3} = 6\pi$$





e.g.) velocity $V(t) = \frac{d}{dt} x(t)$ ← position

x m
v m/s
a m/s²

Object under constant acceleration, a,

$$x(t) = x_0 + V_0 t + \frac{1}{2} a t^2$$

x_0 initial pos.
V_0 initial vel.

Calculate $\frac{d}{dt} x$, calculate $\frac{d}{dt}\left(\frac{d}{dt} x\right)$

$$V(t) = \frac{d}{dt} x = V_0 + at$$

$$a(t) = \frac{d}{dt}\frac{d}{dt} x = a$$

□

Section 3.5

[Higher Derivatives]

Newton $\quad$ Leibniz

$$f' = \frac{df}{dx} \quad \text{1st derivative}$$

$$f'' = \frac{d^2}{dx^2} f \quad \text{2nd derivative}$$

$$\frac{d^n}{dx^n}(x^n).$$

$$\vdots \qquad \qquad \vdots$$

$$f^{(n)} = \frac{d^n}{dx^n} f \quad n\text{th derivative.}$$

The more derivatives a function has, the <u>smoother</u> it is (or the graph behaves nicer).

eg. Polynomial

$$f(x) = \sum_{k=0}^{n} a_k x^k. \quad \text{degree}$$

$\frac{df}{dx}$ is a degree $n-1$ polynomial.

$\frac{d^2f}{dx^2}$ is a degree $n-2$ "

$$\downarrow$$

$$\frac{d^{n+1}f}{dx^{n+1}} = 0$$

[Polynomials are infinitely differentiable.
Polynomials are smooth functions.
eg. $f(x) = e^x$

Chain Rule:

- Consider the composition

$$(f \circ g)(x) = f(g(x)).$$

$(f \circ g)(x) = f(g(x))$.

Thm (Chain Rule)

- If f is differentiable at $g(x)$ and g is differentiable at x, then $f \circ g$ is differentiable at x:

Newton:

$$\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$$

Leibniz: $y = f(u) = \boxed{f(g(x))}$ $\quad u \sim g(x)$

$$\frac{d}{dx}(f \circ g)\Big|_x = \frac{df}{du}\Big|_{u=g(x)} \cdot \frac{dg}{dx}\Big|_x$$

ex

$$\frac{d}{dx}\, \underline{e^{x^2+1}} \qquad f'(u) = \frac{d}{du} e^u = e^u$$

$e^{x^2+1} = f(g(x))$ where $f(u) = e^u$, $g(x) = x^2+1$

$$\frac{d}{dx}\left(e^{x^2+1}\right) = \underbrace{f'(g(x))}_{e^{g(x)}} \cdot \underbrace{g'(x)}_{2x}$$

$$= e^{x^2+1} \cdot 2x$$

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x).$$

ex $\quad \frac{d}{dx}\left[(x^2+1)^{100}\right]$

$(x^2+1)^{100} = f(g(x))$ where $f(u) = u^{100}$, $g(x) = x^2+1$

$$\frac{d}{dx}\left[(x^2+1)^{100}\right] = \frac{d}{dx} f(g(x))$$

$$\frac{d}{dx}\left[(x^2+1)^{100}\right] = \frac{d}{dx} f(g(x))$$

$$= \left.\frac{df}{du}\right|_{u=g(x)} \cdot \left.\frac{dg}{dx}\right|_{x}$$

$$= \left.\frac{d}{du}\left[u^{100}\right]\right|_{u=g(x)} \cdot \left.\frac{d}{dx}\left[x^2+1\right]\right|_{x}$$

$$= \left.\left(100u^{99}\right)\right|_{u=g(x)} \cdot \left.2x\right|_{x}$$

$$= 100(x^2+1)^{99} \cdot 2x$$

$$\frac{d}{dx}\left(\underline{x^2+1}\right)^{100} \qquad 100\left(x^2+1\right)^{99} \cdot x^2+1$$

chain rule: derivative of composition

≈ derivative of outside

times derivative of inside.

ex/

$$\frac{d}{dx}\left[\frac{e^{x^2+5x}}{xe^{x}}\right] \begin{matrix} \leftarrow f \\ \leftarrow g \end{matrix}$$

quotient rule

$$= \frac{\frac{df}{dx}\cdot g - f\frac{dg}{dx}}{g^2} = \frac{\overbrace{\frac{d}{dx}\left(e^{x^2+5x}\right)}^{\substack{(A)\\ \text{chain rule}}}\cdot xe^{x} - e^{x^2+5x}\overbrace{\frac{d}{dx}\left(xe^{x}\right)}^{\substack{(B)\\ \text{product rule}}}}{x^2e^{2x}}$$

$$(A) = \frac{d}{dx}\left(e^{x^2+5x}\right) \overset{\text{chain}}{=} e^{x^2+5x}\cdot(2x+5)$$

$$(B) = \frac{d}{dx}\left(xe^{x}\right) \overset{\text{product}}{=} \frac{d}{dx}(x)\cdot e^{x} + x\frac{d}{dx}\left(e^{x}\right) = e^{x} + xe^{x}$$

$$= \quad e^{x^2+5x}\cdot(2x+5)\cdot xe^{x} - e^{x^2+5x}\cdot\left(e^{x}+xe^{x}\right)$$

$$= \frac{e^{x^2+5x}\cdot(2x+5)\cdot xe^x - e^{x^2+5x}\cdot(e^x+xe^x)}{x^2e^{2x}}.$$

ex/ Consider inflating a spherical balloon, whose radius increases at 2 cm/s (constant rate).

- At $t = 1$ s, $r = 2$ cm, what rate is the volume increasing?

$$\boxed{\begin{aligned} \frac{dV}{dr} &= 4\pi r^2 \\ \frac{dA_0}{dr} &= \frac{d}{dr}(\pi r^2) \\ &= 2\pi r \end{aligned}}$$

→ $V(r) = \frac{4}{3}\pi r^3$ $\quad r(t)$

→ $V(r(t)) = \frac{4}{3}\pi r(t)^3.$

rate of change of volume:

$$\frac{d}{dt}V(r(t)) = \left.\frac{dV}{dr}\right|_{r=r(t)} \cdot \left.\frac{dr}{dt}\right|_t$$

$$= 4\pi r(t)^2 \cdot \left.\frac{dr}{dt}\right|_t$$

$\frac{d}{dt}V(r(1))$ $\quad$ at $t=1$, $r(1) = 2$ cm

$\left.\frac{dr}{dt}\right|_{t=1} = 2$ cm/s

$$\parallel$$

$$4\pi\,(2\text{cm})^2\cdot\left(2\,\frac{\text{cm}}{\text{s}}\right) = 32\pi\ \frac{\text{cm}^3}{\text{s}}.$$

end of material covered for Exam 1.

(OH tmrw)