Lecture 7

Monday, August 16, 2021 9:36 AM

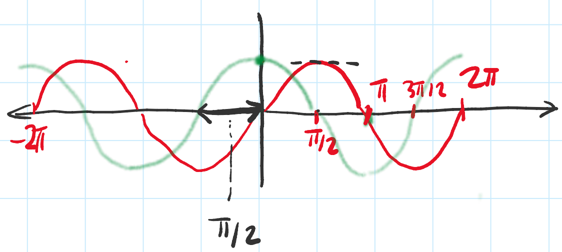
Read sections 3.6, 3.8, & 3.9.

3.6 → trig, 3.8 → implicit, 3.9 → logarithms

$\boxed{3.6}$ Derivatives of Trig. Functions

(*) $\lim_{h\to 0} \frac{\sin h}{h} = 1$, $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.

Thm: $\frac{d}{dx}[\sin x] = \cos x$; $\frac{d}{dx}[\cos x] = -\sin x$



= $\sin x$

= $\cos x$

Proof:

$\frac{d}{dx}[\sin x] = \cos x$

monomial x^n $(x+h)^n - x^n$

$\sqrt{-1}$ $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$\sin(x+h) = \sin x \cos h + \cos x \sin h$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\underbrace{\sin x}_{\to \sin x} \underbrace{\left(\frac{\cos h - 1}{h}\right)}_{\to 0} + \underbrace{\cos x}_{\to \cos x} \underbrace{\frac{\sin h}{h}}_{\to 1} \right] = \cos x.$$

$\lim_{h\to 0} f(h) = L$

$f(h) \xrightarrow{(h\to 0)} L$

(similarly for $\frac{d}{dx} \cos x$). $\square$

ex/ Compute $\frac{d}{dx}\left[\cos(e^x)\right]\Big|_{x=\ln(\pi/2)}$.

Chain Rule

· $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

$\cdot\ x = \ln(\pi/2).$

Chain Rule

- $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$
- $f(u) = \cos(u), \quad g(x) = e^x \qquad f(g(x)) = \cos(e^x).$

$$\frac{d}{dx}[\cos(e^x)]\Big|_{x=\ln(\pi/2)} = -\sin(e^x)\cdot e^x \Big|_{x=\ln(\pi/2)}$$

$$= -\sin(e^{\ln(\pi/2)})\cdot e^{\ln(\pi/2)}$$

$$= -\sin\left(\frac{\pi}{2}\right)\cdot \frac{\pi}{2} = -\pi/2$$

$$\tan x = \frac{\sin x}{\cos x} \ (*), \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \csc x = \frac{1}{\sin x}$$

quotients ✓

Recall quotient rule: $h(u) = u^{-1}$, $g(x)$ $\quad h(g(x))$ $\quad h'(u) = -u^{-2}$

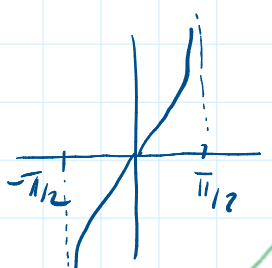
$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left[f(x)\cdot(g(x))^{-1}\right]$$

$$\overset{\text{prod.}}{=} f'(x)\cdot(g(x))^{-1} + f(x)\cdot\frac{d}{dx}(g(x))^{-1}$$

$$= f'(x)\cdot(g(x))^{-1} + f(x)(-1)(g(x))^{-2}\cdot g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \quad \square$$

$$\tan x = \frac{\sin x}{\cos x}, \quad 0 \neq \cos x$$



$\sec^2(x) \equiv [\sec(x)]^2.$

Thm: $\frac{d}{dx}\tan x = \sec^2 x$, $\quad \frac{d}{dx}\sec x = \sec x \tan x$,

$\frac{d}{dx}\cot x = -\csc^2 x$, $\quad \frac{d}{dx}\csc x = -\csc x \cot x$

(wherever the denominators in (*) are nonzero)

ex/ ... $\frac{d}{dx}$ ... $\frac{d}{dx}$ ...

ex

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\frac{d}{dx}(\sin x)\cdot\cos x - \sin x\cdot\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{\cos x\cdot\cos x + \sin x\cdot\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \quad \square$$

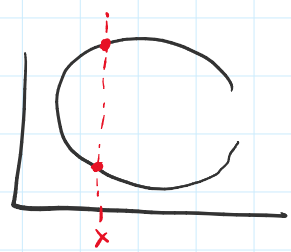
(other 3 are similar) $\square$

$\boxed{3.8}$ Implicit Differentiation

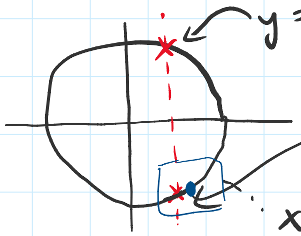
Take derivatives of a curve that's a graph:

(graph) $\underline{y = f(x)}$

vertical line test



• Not all curves are graphs

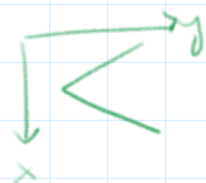


$y = +\sqrt{1-x^2}$

$y = -\sqrt{1-x^2}$

$$\underbrace{x^2+y^2=1}$$

Implicitly defines the unit circle



Graphs $y = f(x)$

Implicit differentiation: $\underline{g(x,y) = 0}$ e.g., $g(x,y) = y - f(x)$

Implicit Diff: think of y <u>locally as a function x</u>,

$$\frac{d}{dx}[g(x,y)] = \frac{d}{dx}[0] = 0$$

e.g., $g(x,y) = x^2 + y^2 - 1$

$$0 = \frac{d}{dx}g(x,y) = \frac{d}{dx}(x^2+y^2-1)$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}\underline{(y(x))^2} + \cancel{\frac{d}{dx}(-1)}^{\,0}$$

$f(u) = u^2$

$a(x) = u(x)$

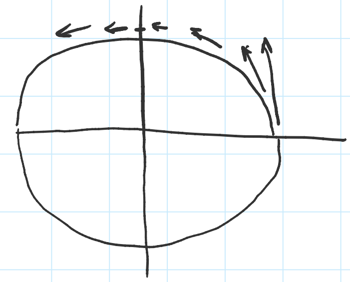
$$= \frac{d}{dx}(x^2) + \frac{d}{dx}\underbrace{(y(x))^2}_{f(g(x))} + \cancel{\frac{d}{dx}(-1)}^{\,0}$$

$f(u) = u^2$
$g(x) = y(x)$

$$= 2x + 2(y(x)) \cdot y'(x)$$

$$= 2x + 2y \cdot \frac{dy}{dx}$$

$$\Longleftrightarrow \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$



General steps for implicit diff.

1) Define a curve implicitly by the equation $g(x,y) = 0$

2) Differentiate this equation w.r.t. (with respect to) x and viewing y as a local function x (or vice-versa)

Inverse Trig Derivatives:

$\sin : [-\pi/2, \pi/2] \to [-1, 1]$

Inverse: $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$

$\frac{d}{dx} \sin^{-1}(x)$?

I know how to differentiate sin.

$y = \sin^{-1}(x) \Longleftrightarrow x = \sin(y)$

$$\frac{d}{dx}[x = \sin y]$$

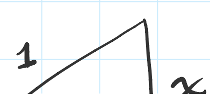
$$\frac{d}{dx}[x] = \frac{d}{dx} \sin(y(x))$$

$$1 = \cos(y(x)) \cdot y'(x)$$

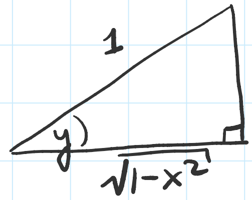
$y = \sin^{-1}(x)$
$y' = \frac{d}{dx} \sin^{-1}(x)$

$$\Rightarrow y'(x) = \frac{1}{\cos(\underbrace{y(x)}_{\text{angle.}})}$$

$y = \sin^{-1}(x)$ $\qquad$ $x = \sin y$



angle.



$$\Rightarrow \quad y'(x) = \frac{1}{\sqrt{1-x^2}} \qquad \Big\} \cos(y(x)) = \sqrt{1-x^2}$$

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad (\text{for } x \in (-1,1))$$

(read textbook for other inverse trig fcns)

ex Find eqn. of the tangent line to $\underset{(x,y)}{\overset{}{(1,1)}}$ of curve: $\boxed{y^4 + xy = x^3 - x + 2.}$

$$\frac{d}{dx}\left[y^4(x) + x\,y(x)\right] = \frac{d}{dx}\left[x^3 - x + 2\right]$$

$$4y^3(x)\cdot y'(x) + 1\cdot y(x) + x\cdot y'(x) = 3x^2 - 1$$

$$\left[4y^3 + x\right]\cdot y' = 3x^2 - y - 1$$

$$(x,y) = (1,1) \Rightarrow \quad 5\cdot y' = 3(1)^2 - 1 - 1 \quad \Rightarrow y'\Big|_{(1,1)} = 1/5.$$

$$y - y_0 = m(x - x_0)$$

$$y - 1 = \frac{1}{5}(x-1).$$

Thm: For $x > 0$, $\boxed{\frac{d}{dx}[\ln x] = \frac{1}{x}}$ base e.

$\ln = f^{-1}$

$f(x) = e^x$

$f : \mathbb{R} \to (0,\infty)$ bijection

$\ln : (0,\infty) \to \mathbb{R}.$

$$y = \ln x$$

$$e^y = x$$

$$\frac{d}{dx}\left[e^{y(x)}\right] = \frac{d}{dx}[x]$$

$$\underbrace{e^{y(x)}}_{\neq 0}\cdot y'(x) = 1$$

$$\Rightarrow y'(x) = \frac{1}{e^{y(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

$$\Rightarrow y'(x) = \frac{1}{e^{y(x)}} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

More generally, $\frac{d}{dx}(f^{-1})\Big|_x = \frac{1}{f'(f^{-1}(x))}$. ← Check: $f(x) = \sin(x)$, $f(x) = e^x$

$$\frac{d}{dx}\left[f(f^{-1}(x))\right] = [x]$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1})\Big|_x = 1$$

Inverse Function Thm

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Logarithmic Differentiation;

$f(x) > 0$ for all x in its domain, and f is differentiable.

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x)$$

$$\Leftrightarrow \boxed{f'(x) = f(x)\frac{d}{dx}\left[\ln(f(x))\right]} \quad (*)$$

Logarithmic Diff.

ex $f(x) = x^x$ $(x>0)$ monomial x^n ← const. exponential b^x ← const.

$$\cancel{\frac{d}{dx}(x^x) = x \cdot (x)^{x-1}}$$

$$\frac{d}{dx}(x^x)\ ?$$

↳ $f(x) = x^x$ $\quad \ln(f(x)) = \ln(x^x) = x \cdot \ln(x)$

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}(x \cdot \ln x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\frac{d}{dx}(x^x) = x^x \frac{d}{dx}\ln(f(x)) = x^x(\ln x + 1).$$

ex/ $f(x) = x^{\sin x}$

$f'(x) = x^{\sin x}\left(\cos x \cdot \ln x + \frac{\sin x}{x}\right).$

Try yourselves.

Different Bases: $b>0,\ b \neq 1$

base e: $\frac{d}{dx} e^x = e^x$, $\frac{d}{dx} \ln x = \frac{1}{x}$.

base b: $\frac{d}{dx} b^x = b^x \cdot \ln b$, $\frac{d}{dx} \log_b x = \frac{1}{x \ln b}$

(If $b=e$, $\ln b = \log_e b = \log_e e = 1$.)

$$\frac{d}{dx} b^x = \frac{d}{dx} e^{\ln(b^x)} = \frac{d}{dx} e^{x \ln b}$$

$$= e^{x\ln b} \cdot \frac{d}{dx}(x \ln b) = e^{x \ln b} \cdot \ln b$$

$$= b^x \cdot \ln b \quad \square$$

$$\frac{d}{dx} \log_b x = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b} \frac{d}{dx} \ln x$$

change of base Logarithms

$$= \frac{1}{x \ln b} \quad \square$$