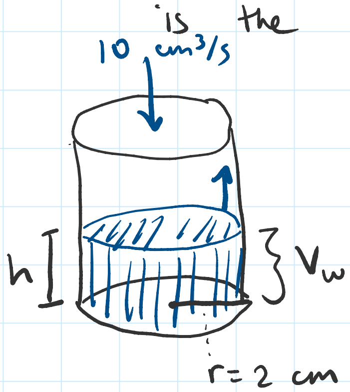
Read sections 3.10, 4.1, 4.2

$\boxed{3.10}$ Related Rates

- Two quantities $f(t)$, $g(t)$
  Relation between them $F(f(t), g(t)) = 0$.
- Steps:
  1) Identify variables & their rates (derivatives w.r.t. time)
  2) Identify the relationship between them
  3) Differentiate relationship to get relation between derivatives/rates
  4) Plug-in given info & solve unknown.

ex/ Filling a cylinder with water:

- Water flows into a cylinder at $\underline{10\ cm^3/s}$; the cylinder has a radius of 2 cm. How fast is the water level rising in the cylinder?



<u>step 1 & 2</u>

$$V_w = \pi r^2 h$$

rates:

$r'(t) = 0$

$h'(t)$ = rate of water level rising

$V_w'(t)$ = rate of change of volume of water $= \underline{10\ cm^3/s}$

units $\dfrac{(\text{length})^3}{\text{time}}$ $\quad \dfrac{dV}{dt}$

<u>step 3</u>: Differentiate

$$\frac{d}{dt} V_w(t) = \frac{d}{dt}\left[\pi r(t)^2 h(t)\right]$$

$$V_w'(t) = \pi\left(\cancel{\frac{d}{dt}[r(t)^2]}^{\,0}\, h(t) + r(t)^2 \frac{d}{dt} h(t)\right)$$

$$V_w'(t) = \pi r(t)^2 h'(t)$$

<u>Step 4</u>: Plug in & solve

$$10\ \underline{cm^3} = \pi (2\ cm)^2 \cdot h'(t)$$

Step 4: Plug in & solve

$$10\ \frac{cm^3}{s} = \pi (2\ cm)^2 \cdot h'(t)$$

$$\Rightarrow h'(t) = \frac{10}{4\pi}\ \frac{cm^3}{s} \cdot \frac{1}{cm^2} = \frac{5}{2\pi}\ \frac{cm}{s} = \frac{dh}{dt}$$

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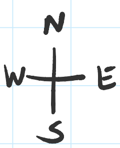
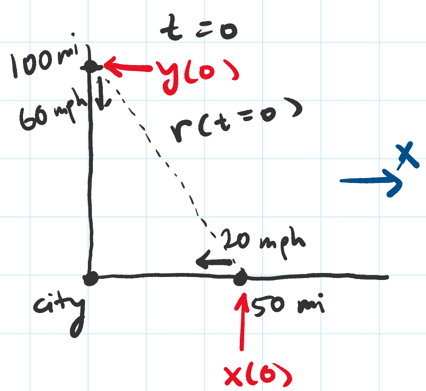
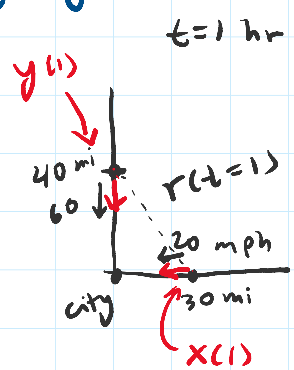
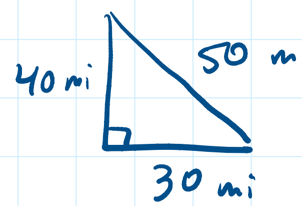
ex 2 A car starts 100 mi north of the city going south 60 mph (= mi/h); another car starts 50 mi east of the city going west 20 mi/h.

- After 1 hour, at what rate is the distance between them changing?









$$r(t) = \sqrt{x(t)^2 + y(t)^2} \qquad \text{(Pythag. thm.)}$$

$$\frac{d}{dt} r(t) = \frac{d}{dt}\sqrt{x(t)^2 + y(t)^2}$$

want to know (t = 1 hr)

$$= \frac{d}{dt}\left(x(t)^2 + y(t)^2\right)^{1/2}$$

$$= \frac{1}{2}\left(x(t)^2 + y(t)^2\right)^{-1/2} \cdot \frac{d}{dt}\left(x(t)^2 + y(t)^2\right)$$

chain rule

$$= \frac{1}{2} \cdot \frac{1}{r(t)} \cdot \left(2x(t)\frac{d}{dt}x(t) + 2y(t)\frac{d}{dt}y(t)\right)$$

✓ ✓ ✓ ✓ ✓

$$= \frac{1}{2}\ \frac{1}{50\ mi}\left(2(30\ mi)\left(-20\ \frac{mi}{h}\right) + 2(40\ mi)\left(-60\ \frac{mi}{h}\right)\right)$$

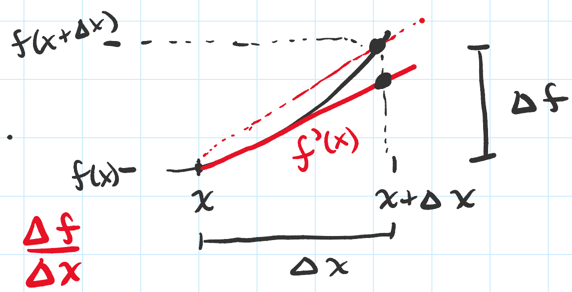
$$\underset{\text{at 1 hr}}{=} \frac{1}{2}\,\frac{1}{50\text{ mi}}\left(2(30\text{ mi})\left(-20\,\frac{\text{mi}}{\text{h}}\right) + 2(40\text{ mi})\left(-60\,\frac{\text{mi}}{\text{h}}\right)\right)$$

$$= \frac{-1200-4800}{100}\,\frac{\text{mi}}{\text{h}} = -60\,\frac{\text{mi}}{\text{h}}$$

cars are getting closer at $t = 1$ hr.

$\boxed{4.1}$ Linear Approximation

- Assume f is differentiable at x.
- $\Delta f = f(x+\Delta x) - f(x)$



$f'(x) \approx \dfrac{\Delta f}{\Delta x}$ for Δx small

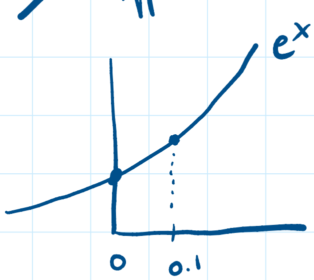
$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \approx \frac{\Delta f}{\Delta x} \text{ for } \Delta x \text{ small.}$$

$$f'(x)\,\Delta x \approx \Delta f = f(x+\Delta x) - f(x)$$

$$\Rightarrow \boxed{f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x} \quad (\text{for } \Delta x \text{ small})$$

Linear Approximation

ex/ Approximate $e^{0.1}$ without a computer. $f(x) = e^x$



Approximate about $x = 0$, $\Delta x = 0.1$.

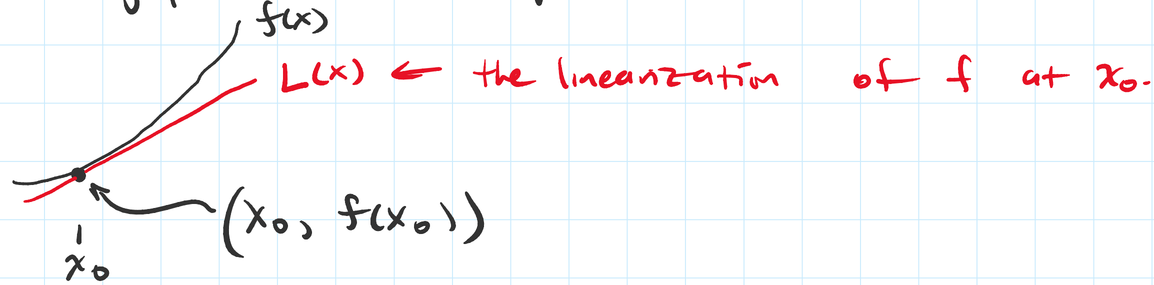
$$f(0.1) \approx f(0) + f'(0)\cdot\Delta x$$
$$= e^0 + e^0\cdot(0.1)$$
$$= 1 + 1(0.1) = 1.1.$$

$$e^{0.1} \approx 1.10517\ldots$$

fairly close.

Definition: (Assume f is differentiable at x_0)

- The linearization of f at x_0 is the function whose graph is the tangent line to f at x_0.



$$\underset{L(x)}{y} - \underset{f(x_0)}{y_0} = \underset{f'(x_0)}{m}\,(\underline{\underline{x}} - \underset{x_0}{x_0}) \Rightarrow L(x) = f(x_0) + f'(x_0)\cdot(x - x_0)$$

For $x \approx x_0$, $f(x) \approx L(x)$.

$\left(L(x_0) = f(x_0) \right)$

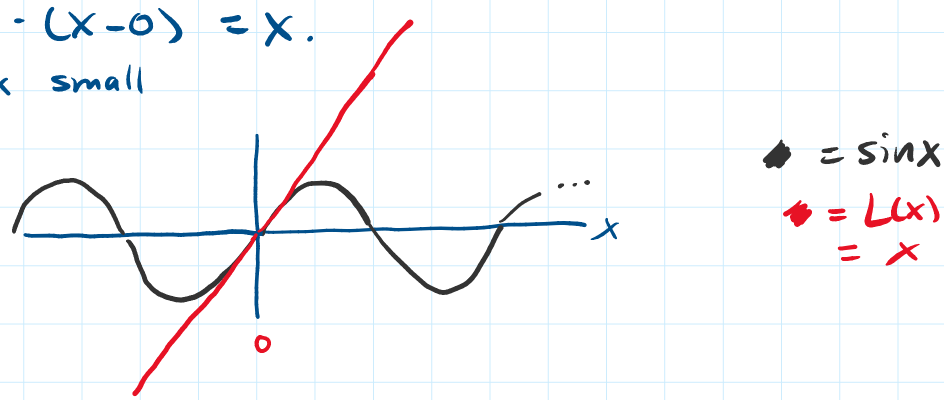
eg Linearize $f(x) = \sin(x)$ about $x = 0$.

$f(0) = \sin(0) = 0.$

$f'(0) = \frac{d}{dx}\sin(x)\Big|_{x=0} = \cos(x)\Big|_{x=0} = \cos(0) = 1.$

$$L(x) = \overset{0}{f(0)} + \overset{1}{f'(0)}\cdot(x - 0) = x.$$

$\sin x \approx x$ for x small



ex Newton's Law of Gravity:

potential $\quad U(r) = -\dfrac{GMm}{r}$

G Newton's constant
M mass body 1
m mass body 2

$F = -\dfrac{d}{dr}U(r)$

acceleration on mass m: $F = ma \qquad a = \dfrac{F}{m} = -\dfrac{1}{m}\dfrac{d}{dr}U(r)$

ex/ Newton's Law of Gravity:

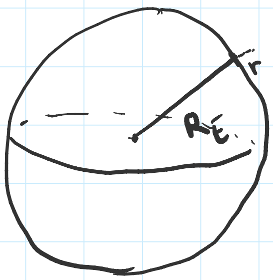
potential $U(r) = -\frac{GMm}{r}$

G Newton's constant
M mass body 1
m mass body 2

$$F = -\frac{d}{dr}U(r)$$

acceleration on mass m: $F = ma$ $\quad a = \frac{F}{m} = -\frac{1}{m}\frac{d}{dr}U(r)$

gravity on surface of Earth:



$$U(R_{Earth} + r) \approx U(R_{Earth}) + U'(R_{Earth})\cdot r$$
$$= U(R_{Earth}) + \underbrace{\frac{GMm}{R_{Earth}^2}}\cdot r$$

$$a = \frac{F}{m} = \frac{-1}{m}\frac{d}{dr}U(r) = -\frac{GM}{R_{Earth}^2} = \underbrace{-9.8\ m/s^2}_{\text{gravity on surface of Earth is approx. constant.}}.$$

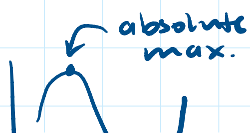
4.2 Extreme Values

- We often care about extreme values of functions

[e.g. optimization ⇝ machine learning.
e.g. space missions, cost function ~ rocket fuel.]

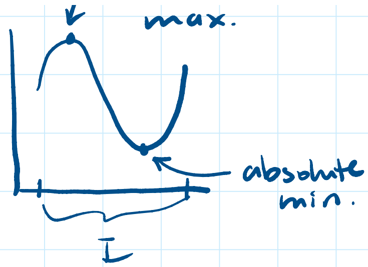
Definition (Extreme Value)
- Let $f: I \to \mathbb{R}$ (I some interval).

- If there is some $b \in I$ such that $f(b) \leq f(x)$ for all $x \in I$, we call b the minimizer of f and $f(b)$ the absolute minimum of f on I.
- If there is some $b \in I$ such that $f(b) \geq f(x)$ for all $x \in I$, we call b the maximizer of f and $f(b)$ the absolute maximum of f on I.

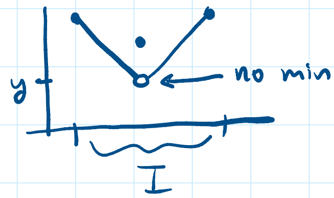
e.g. 

Functions aren't guaranteed to have min/max. $f(x) = 1/x$. $I = (0,1]$

e.g.

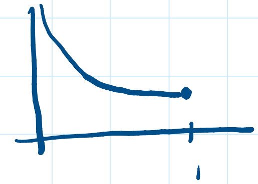


Functions aren't guaranteed to have min/max.



discontinuous

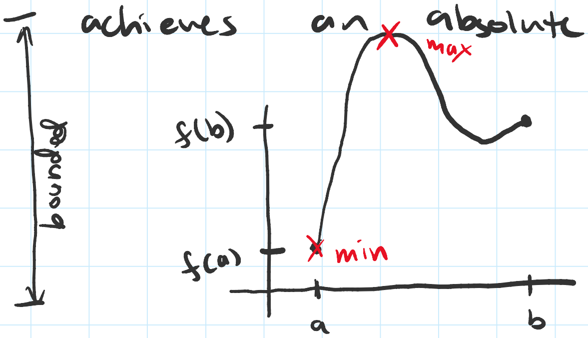
$f(x) = 1/x$, $I = (0,1]$



no max, interval is not closed.

Thm:

A continuous function on a closed interval, $f: [a,b] \to \mathbb{R}$, achieves an absolute min. & max.



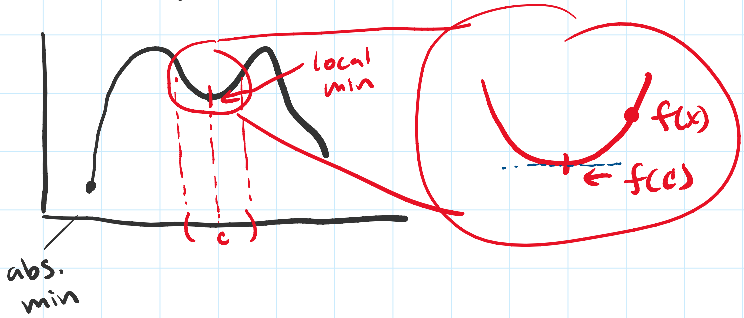
Corollary:

Continuous functions on closed intervals are bounded.

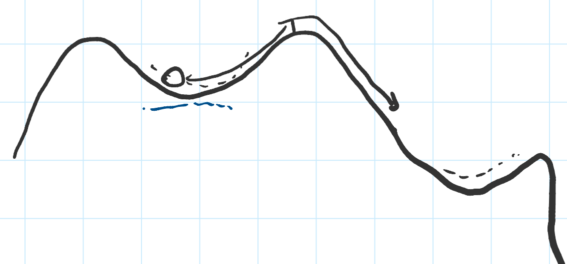
$$\min \le f(x) \le \max$$

minimizer, $f(c)$ local minimum

Def: Say c is a local min of f if there is an open interval I containing c s.t. $f(c) \le f(x)$ for all $x \in I$
(similarly for local max)

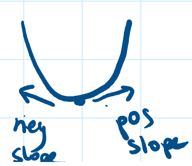


$f(c) \le f(x)$ for all $x \in I$.

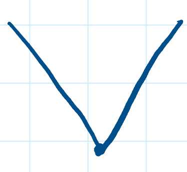


Def:

A critical point c in the domain of f is a point where either $f'(c) = 0$ OR $f'(c) = \text{DNE}$.



$|x|$

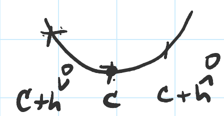


$f'(c) = \text{DNE}$

Thm: If c is a local minimizer/maximizer of f, then c is a critical point of f.

pf: If $f'(c)$ DNE $\Rightarrow$ c is a critical point.

So, assume $f'(c)$ does exist.



Local minimizer:

$h>0$ small, $\dfrac{f(c+h)-f(c)}{h} > 0 \Rightarrow f'(c) \geq 0$

$h<0$ small, $\dfrac{f(c+h)-f(c)}{h} < 0 \Rightarrow f'(c) \leq 0$

$\Rightarrow f'(c) = 0$ $\square$

Today: OH in 10 mins.

Tmrw: review, bring questions.