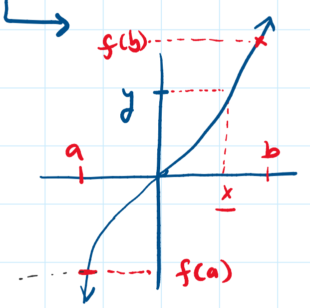
- Midterm released today (W 8/18/21) at 1:30 pm, on gradescope.
- Viewing closes after 12 hours (Th 8/19/21 at 1:30 am)
- Once you view the midterm, you have 90 min. to complete, scan, and upload your work (as a single PDF, like the homework).
  - ≈ 80 mins to work on exam
  - ≈ 10 mins to scan & upload.
- Start the exam before 11:59 pm to make sure you have the full 90 mins.
- There are 4 problems × 20 pts each.
  - → There is one extra credit problem worth 10 pts.
  - → Any extra credit in the class is applied <u>after</u> the grading curve. Not doing the extra credit cannot hurt your grade.

HW1
Problem 1(b) Showing $f(x) = x^3$, $f: \mathbb{R} \to \mathbb{R}$, is surjective.

<u>Show</u>
For every $y \in \mathbb{R}$, there exists an $x \in \mathbb{R}$ s.t. $f(x) = y$.
($x^3 = y$, choose $x = y^{1/3}$, $y = (y^{1/3})^3 = x^3$)



- Note f is continuous on $\mathbb{R}$.
- Fix $y \in \mathbb{R}$.
- Since $\underbrace{\lim_{x \to \infty} f(x) = \infty}$

as x approaches limit pt., f gets arbitrarily large.

$\lim_{x \to \infty} f(x) = L \iff$ $f(x)$ approaches L (gets arbitrarily close to L) as x gets large.

$\Rightarrow$ there exists some $b \in \mathbb{R}$ s.t. $f(b) > y$.

(Similarly since $\lim$ $f(x) = -\infty$, there exist some $a \in \mathbb{R}$

$\Rightarrow$ there exists some $b \in \mathbb{R}$ s.t. $f(b) > y$.

(Similarly, since $\lim_{x \to -\infty} f(x) = -\infty$, there exists some $a \in \mathbb{R}$ s.t. $f(a) < y$)

$$f(a) < y < f(b)$$

$\Rightarrow$ by IVT, there is an x s.t. $f(x) = y$ $\square$

Holds for any $y \in \mathbb{R} \Rightarrow f$ surjective.

HW2 Problem 1

okay at $x=0$ — issue at $x=0$

$$\lim_{x \to 0} \underbrace{(\tan(x) \cos(\sin(1/x)))}_{f(x)}$$

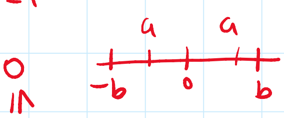
(for $x \neq 0$):

$|ab| = |a| \cdot |b|$

$$|f(x)| = |\tan(x) \cos(\sin(1/x))| = |\tan(x)| \cdot \overbrace{|\cos(\sin(1/x))|}^{\leq 1}$$

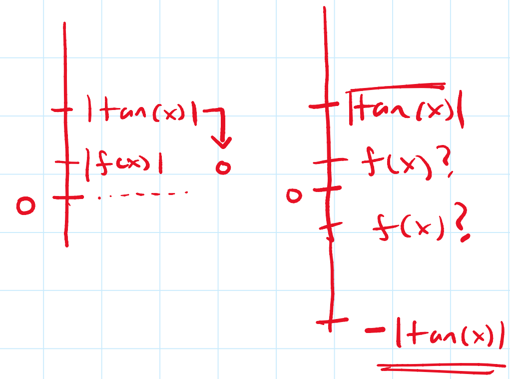
$$\leq |\tan(x)|$$

$|\cos\theta| \leq 1$



$$|\overbrace{f(x)}^{a}| \leq \overbrace{|\tan(x)|}^{b}$$

$|a| \leq b \Leftrightarrow -b \leq a \leq b$



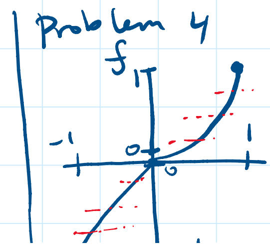
$$\underbrace{-|\tan(x)|}_{\to 0} \leq f(x) \leq \underbrace{|\tan(x)|}_{\to 0}$$

Fact! The composition of continuous functions is continuous. f cont. at x, g cont. at $f(x)$, then $\underbrace{g \circ f}_{g(f(x))}$ is cont. at x

$h(x) = \tan(x)$, $\ell(x) = |x|$ — both cont. (near $x=0$) $\Rightarrow$ composition is continuous

By squeeze thm., $f(x) \xrightarrow[(x \to 0)]{} 0$ $\square$

Problem 4



Ex/

$f: \mathbb{R} \to \mathbb{R}$ Compute the deriv. of piecewise

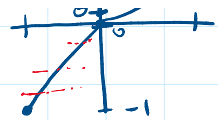
$\{ x^2 \quad x < 0$

Ex/

$f: \mathbb{R} \to \mathbb{R}$ function

Compute the deriv. of piecewise

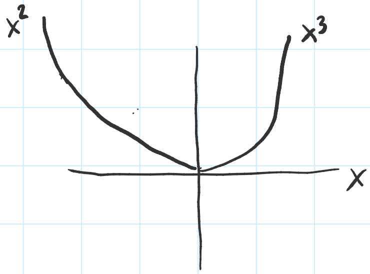
$$f(x) = \begin{cases} x^2, & x < 0 \\ x^3, & x \geq 0 \end{cases}$$



$$f^{-1}(y) = \begin{cases} \sqrt{y}, & y \in [0,1] \\ y, & y \in [-1,0) \end{cases}$$

- Where does f' exist?

  Where is f' continuous or not?



on $x<0$, $f(x) = x^2 \Rightarrow$ $f'(x) = 2x$, $x<0$

on $x>0$, $f(x) = x^3 \Rightarrow$ $f'(x) = 3x^2$, $x>0$

Definition:
$$f'(0) = \lim_{h \to 0} \frac{f(h) - \overset{0}{\overset{\shortparallel}{f(0)}}}{h} = \lim_{h \to 0} \frac{f(h)}{h}$$

Recall a limit exists $\Leftrightarrow$ both one-sided limits exist and are the same.

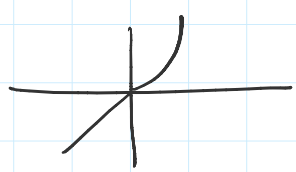
$$\lim_{h \to 0^-} \frac{f(h)}{h} = \lim_{h \to 0^-} \frac{h^2}{h} = 0$$

$\Rightarrow$ $f'(0)$ exists and equals 0.

$$\lim_{h \to 0^+} \frac{f(h)}{h} = \lim_{h \to 0^+} \frac{h^3}{h} = 0$$

$$f'(x) = \begin{cases} 2x, & x < 0 \leftarrow \text{cont.} \\ 0, & x = 0 \\ 3x^2, & x > 0 \leftarrow \text{cont.} \end{cases}$$

$f': \mathbb{R} \to \mathbb{R}$ continuous everywhere.



$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} (2x) = 0$$

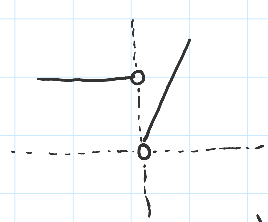
$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} (3x^2) = 0$$

$\Rightarrow$ $\boxed{\lim_{x \to 0} f'(x) = 0 = f'(0)}$

continuity.

f'' ?

$$f''(x) = \begin{cases} 2, & x < 0 \\ 6x, & x > 0 \end{cases}$$

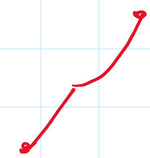


jump discontinuity.

2^{nd} deriv not defined at $x = 0$. (difference quotient).

4)

$f: [-1,1] \to [-1,1]$ $\quad f(x) = \begin{cases} x, & x < 0, \\ x^2, & x \geq 0. \end{cases}$



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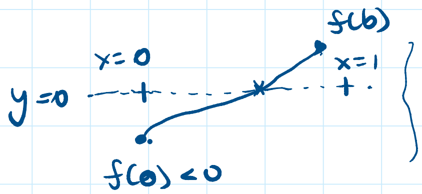
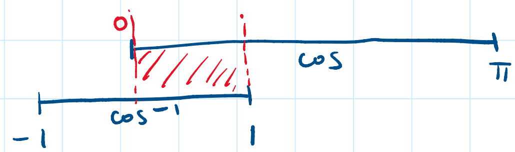
3) Using IVT, show $\boxed{\cos(x) = \cos^{-1}(x)}$ has a solution in $(0,1)$.

$\Leftrightarrow f(x) = 0.$



$f(x) = \cos(x) - \cos^{-1}(x)$

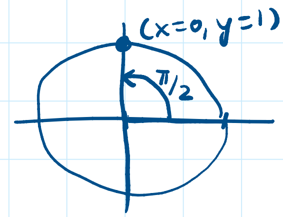


Continuity: $\quad \cos: [0,\pi] \to [-1,1]$

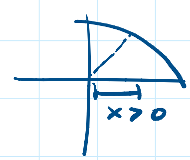
$\cos^{-1}: [-1,1] \to [0,\pi]$

$f: [0,1] \to \mathbb{R}$ is continuous.

$f(0) = \underbrace{\cos(0)}_{=1} - \underbrace{\cos^{-1}(0)}_{\substack{\text{angle in } [0,\pi] \\ \text{such that the} \\ x\text{-value is } 0}} = -\pi/2 < 0$



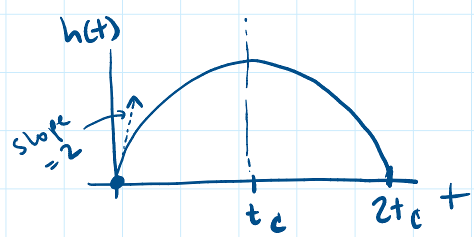
$f(1) = \underbrace{\cos(1)}_{>0} - \underbrace{\cos^{-1}(1)}_{=0} > 0 \qquad 0 < 1 < \pi/2$



Problem 13:

$h(t) = \cancel{h_0} + V_0 t + \frac{1}{2} g t^2$

$h_0 = 0, \quad V_0 = 2, \quad g < 0$



Max height: $\quad h'(t_c) = 0$

$\Rightarrow 0 = \frac{d}{dt}\left(V_0 t + \frac{1}{2} g t^2\right)\Big|_{t=t_c} = V_0 + g t_c \Rightarrow t_c = -\frac{V_0}{g} > 0$

$\frac{\Delta V}{\Delta t} = acc$

$\Delta t = \frac{\Delta V}{acc}$

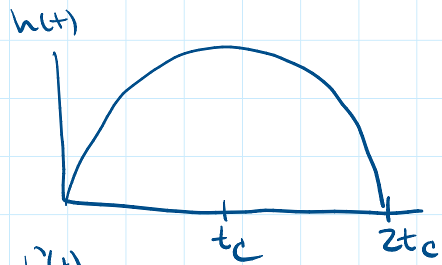
$h(t) = 0$

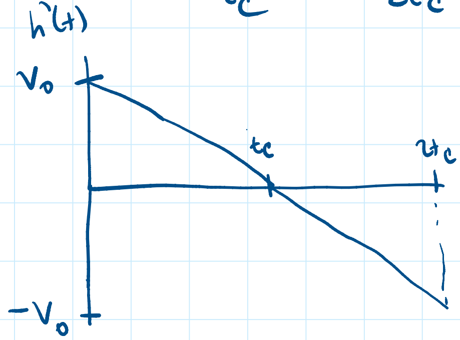
$0 = V_0 t + \frac{1}{2} g t^2 = \underset{=0}{t}\,(\underbrace{V_0 + \tfrac{1}{2} g t}_{=0})$

$t = \frac{-2V_0}{g} = 2t_c$

$$t = \frac{-2v_0}{g} = 2t_c$$

$$\boxed{h'(2t_c) = -v_0}$$





$e^x \in (0,\infty)$

$\cancel{e^{x^3}}\,(\underbrace{x^2+1}_{=0})$

$\Rightarrow x = \pm i$

$$f(x) = \frac{\overset{h(x)}{\left[(x^2+x+3)(x^5+4x^4+x^3+1+e^x)\right]}}{\underset{g(x)}{\left[e^{x^3}(x^2+1)\right]}}$$

Quotient rule: for all x since $g(x) \neq 0$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{h(x)}{g(x)}\right) = \frac{h'(x)\,g(x) - h(x)\,g'(x)}{g(x)^2} = \ldots$$

$$h'(x) = \frac{d}{dx}\Big[\underbrace{(x^2+x+3)}_{u}\underbrace{(x^5+4x^4+x^3+1+e^x)}_{v}\Big]$$

prod. rule

$$= \frac{du}{dx}v + u\frac{dv}{dx}$$

$$= (2x+1)(x^5+4x^4+x^3+1+e^x) + (x^2+x+3)(5x^4+16x^3+3x^2+e^x)$$

$$g'(x) = \frac{d}{dx}\left[e^{x^3}(x^2+1)\right]$$

$$= e^{x^3}\cdot 3x^2\cdot(x^2+1) + e^{x^3}\cdot 2x$$

$\frac{d}{dx}\left(e^{x^3}\right)$

$\frac{d}{dx} e^x$ ✓

$u(x) = x^3$

$\ell'(t) = e^t$

$\frac{d}{dx}\left(\underbrace{e^{x^3}}_{\ell(u(x))}\right)$

$\frac{d}{dx} e^x$ ✓

$\frac{d}{dx} x^3$ ✓

$$\begin{cases} u(x) = x^3 \\ \ell(t) = e^t \\ \ell(u(x)) = e^{x^3}. \end{cases}$$

$\ell'(t) = e^t$

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}(e^{x^3}) = \frac{d}{dx} \ell(u(x))$$

$$= \ell'(t)\Big|_{t=u(x)} \cdot u'(x)\Big|_{x}$$

$$= e^t\Big|_{t=u(x)} \cdot 3x^2 = \boxed{e^{x^3} \cdot 3x^2}$$