• Extending an orthonormal subset to an orthonormal basis.
• Adjoint of a matrix; adjoint of a linear operator.
• Riesz representation theorem for finite-dimensional inner product spaces: every linear functional \( \varphi : V \to F \) is of the form \( \varphi(\cdot) = \langle \cdot, v \rangle \) for some unique \( v \in V \). Morally, why is this true? Dimension theorem!
• Example: trace on \( M_{n \times n}(F) \) with Frobenius inner product \( \langle \cdot_1, \cdot_2 \rangle_{HS} \).
• Example: for a linear operator \( T : V \to V \), what about the functional \( \varphi(v) = \langle T(v), w \rangle \) for a fixed \( w \in V \)? Motivates the definition of the adjoint.
• Adjoint of a linear operator \( T : V \to V \): \( \langle T(x), y \rangle = \langle x, T^*(y) \rangle \). Well-defined by uniqueness.
• Adjoint is unique, linear. The “conjugate transpose” of a linear operator.
• In practice, you can move \( T \) across an inner product at the cost of an adjoint: \( \langle T(x), y \rangle = \langle x, T^*(y) \rangle \) and \( \langle x, T(y) \rangle = \langle T^*(x), y \rangle \).
• We will assume adjoints exist for linear operators on infinite-dimensional inner product spaces.
• If \( T : V \to V \) is a linear operator where \( V \) is a finite-dimensional inner product space, then \( [T]_{\beta}^* = [T^*]_{\beta} \) for any orthonormal basis \( \beta \) of \( V \).
• Corollary: the conjugate transpose of a matrix is the adjoint in the sense that \( (L_A)^* = L_{A^*} \).
• Properties of the adjoint: conjugate linear, anti-homomorphism, involution. Same is true of matrices.