• \( T : V \to V \) for \( V \) a finite-dimensional complex inner product space: orthonormal eigenbasis exists iff \( T \) is normal.
• Not true for infinite-dimensional complex inner product spaces: Fourier example.
• What about real inner product spaces? Already know it’s not enough because of the rotation matrix example.
• Replace normality with the stronger condition of self-adjointness \( T = T^* \).
• Properties: eigenvalues of self-adjoint linear operators are real. Furthermore, if \( V \) is a real inner product space, then the characteristic polynomial of \( T \) still splits.
• \( T : V \to V \) for \( V \) a finite-dimensional real inner product space: orthonormal eigenbasis exists iff \( T \) is self-adjoint.
• Length preserving linear operators: unitary in the case of \( \mathbb{C} \); orthogonal in the case of \( \mathbb{R} \).
• If a linear operator preserves the inner product, then it preserves the norm. Is the converse true? Yes.
• In finite-dimensions, length preserving linear operators are invertible; not true in infinite-dimensions. Example of unilateral shift operator (common counterexample that frequently appears, also an example of a non-normal operator).
• Example of unitary operator on an infinite-dimensional inner product space: point-wise multiplication operator on the continuous functions.
• Characterization of unitary/orthogonal operators on finite-dimensional inner product spaces:
  (1) \( TT^* = T^*T = I \);
  (2) \( \langle T(x), T(y) \rangle = \langle x, y \rangle \) for all \( x, y \in V \);
  (3) \( T \) takes any orthonormal basis \( \beta \) to an orthonormal basis \( T(\beta) \);
  (4) There exists an orthonormal basis \( \beta \) such that \( T(\beta) \) is an orthonormal basis;
  (5) \( \|T(x)\| = \|x\| \) for all \( x \in V \).
• To prove this, we need a lemma: if \( U \) is a self-adjoint linear operator on a finite-dimensional inner product space \( V \) and \( \langle U(x), x \rangle \) for every \( x \in V \), then \( U \) is the zero operator.
• Corollary: every eigenvalue of a unitary or orthogonal linear operator has modulus 1.
• Corollary: In the real case, orthonormal basis of eigenvectors with eigenvalues of modulus 1 exists iff self-adjoint and orthogonal.
• Corollary: In the complex case, orthonormal basis of eigenvectors with eigenvalues of modulus 1 exists iff unitary.
• Example: shift on \( \mathbb{C}^n \), takes \((z_1, \ldots, z_n)\) to \((z_2, \ldots, z_n, z_1)\). Know the eigenvalues have to be of modulus 1 and \( n \)th power equals 1: roots of unity.
• Orthogonal matrices; unitary matrices; characterization in terms of orthonormality of rows/columns; relation to matrix representations of orthogonal/unitary linear operators.
• Unitary equivalence; characterizations of diagonalizability in terms of unitary equivalence.
• Orthogonal equivalence; characterizations of diagonalizability in terms of orthogonal equivalence.
• Characterization of Schur’s theorem in terms of unitary equivalence/orthogonal equivalence.