Goal: decomposition of a normal (in the complex case) or self-adjoint (in the real case) linear operator as a linear combination of orthogonal projections.

If $V = W_1 \oplus W_2$, then we can define the projection $T$ on $W_1$ along $W_2$. Note that $R(T) = W_1$ and $N(T) = W_2$.

$T$ is a projection if and only if $T^2 = T$.

A projection is not uniquely determined by its range. Example: let $W_1 = x$-axis; $W_2 = y$-axis; $W_3 = \text{the line } y = x$. Then $W_1 \oplus W_2 = W_1 \oplus W_3$.

Orthogonal projection: projection such that $N(T)^\perp = R(T)$ and $R(T)^\perp = N(T)$.

$T$ orthogonal projection iff $T = T^2 = T^*$.

Orthogonal projection: projection such that $N(T)^\perp = R(T)$ and $R(T)^\perp = N(T)$.

Matrix representation of a projection; matrix representation of an orthogonal projection.

Complex spectral theorem: Assume $T$ is normal. Then $V$ is a direct sum of eigenspaces $W_i$ with $W_i^\perp = \bigoplus_{j \neq i} W_j$ and $T$ can be written as a linear combination of orthogonal projections where the coefficients of the linear combination are given by the eigenvalues.

Real spectral theorem: Assume $T$ is self-adjoint. Then the same statement holds.

Example: shift on $\mathbb{C}^n$, takes $(z_1, \ldots, z_n)$ to $(z_2, \ldots, z_n, z_1)$. Know the eigenvalues have to be of modulus 1 and $n$th power equals 1: roots of unity.

Definitions: spectrum (set of eigenvalues); resolution of the identity (the sum of the orthogonal projections onto each eigenspace); spectral decomposition of $T$.

Matrix representation of $T$; powers of $T$.

Corollary: if $V$ is a complex inner product space and $T$ is a normal linear operator, then $T^* = p(T)$ for some polynomial $p \in \mathbb{C}[x]$.

Corollary: if $V$ is a complex inner product space, $T$ is unitary iff $T$ is normal and every eigenvalue is of modulus 1.

Corollary: if $V$ is a complex inner product space and $T$ is normal, then $T$ is self-adjoint iff every eigenvalue is real.

Corollary: if $T$ has a spectral decomposition as in the spectral theorem, then each orthogonal projection onto an eigenspace is a polynomial of the operator $T$. 