Functions between sets: into, onto.

Linear transformation: want to talk about a function/map between two vector spaces. Not an arbitrary map. Each vector space has structure. Want the transformation to preserve this structure.

Consequences: \( T(0) = 0; \) \( T(cx + y) = cT(x) + T(y); \) \( T(x - y) = T(x) - T(y); \) linear combinations.

Non-linear examples (squaring, adding constant numbers), linear examples (derivative, definite integration, evaluating functions at specific points).

Linear combination of linear transformations is still linear.


Kernel. Image.

Compute for derivative example.

Kernel, Image are subspaces (where?). Kernel in \( V \), Image in \( W \).

Spanning sets for domain are mapped to spanning sets for range. Note that this does not say that the image of a spanning set is a spanning set for the codomain.

Dimension as size. Nullity is the dimension of the kernel. Rank is the dimension of the image.

(Dimension theorem) Rank-nullity for \( V \) is finite-dimension and \( T : V \to W \) linear. Intuition: conservation of dimension; nullity is what’s left behind; rank is what’s pushed forward by \( T \); sum should be equal to what you started with, which is \( \dim(V) \).

Into iff kernel is trivial. Onto iff \( \dim(R(T)) = \dim(W) \).

What do these have to do with each other? Assume \( \dim(V) = \dim(W) < \infty \) and \( T : V \to W \) is linear. Then

\[
T \text{ is into } \iff T \text{ is onto } \iff \dim(R(T)) = \dim(W)).
\]

Strange things can happen if things are not finite-dimensional. Indefinite integration on \( \mathbb{R}[x] \) where we assume the constant coming from the indefinite integral is always 0: this is into but not onto.

Linear transformation determined by action on a basis.

Define the map the way you know it has to be defined: check that it’s linear; check that it has the property you want; check that it is unique.

Corollary: If two linear transformations agree on a basis, then they agree on the whole vector space.

Example: integration versus evaluating at certain points. Definite integral from \(-1\) to 1 on \( \mathbb{R}_2[x] \). Then

\[
\int_{-1}^{1} p(x) \, dx = \frac{1}{3} p(-1) + \frac{4}{3} p(0) + \frac{1}{3} p(1).
\]