• Finish Section 2.1: recall linear transformations. Linear transformation determined by action on a basis: define the map the way you know it has to be defined: check that it’s linear; check that it has the property you want; check that it is unique.
• Corollary: If two linear transformations agree on a basis, then they agree on the whole vector space.
• Example: integration versus evaluating at certain points. Definite integral from $-1$ to 1 on $\mathbb{R}_2[x]$. Then  
\[
\int_{-1}^{1} p(x) \, dx = \frac{1}{3}p(-1) + \frac{4}{3}p(0) + \frac{1}{3}p(1)
\]
due to both transformations being linear and agreeing on the basis $\{1, x, x^2\}$.
• Example: derivative $D$ is the only linear operator on $\mathbb{R}[x]$ such that $D(x^n) = nx^{n-1}$ for $n \geq 1$ and $D(r) = 0$ for $r \in \mathbb{R}$.
• Concrete manipulation of linear transformations: use the matrix representation.
• Ordered basis $(x_1, \ldots, x_n) \neq (x_n, \ldots, x_1)$.
• Standard ordered basis for $\mathbb{F}_n$ is $\beta = (e_1, \ldots, e_n)$ where $e_i$ is the vector with 1 in the $i$th coordinate and 0 elsewhere.
• Standard ordered basis for $\mathbb{F}_n[x]$ is $(1, x, \ldots, x^n)$.
• Can use an ordered basis to identify abstract vectors with “usual” vectors, written as  
\[
v \mapsto [v]_\beta;
\]
however, this identification crucially depends on both the basis and the ordering.
• Example: In $\mathbb{F}_n[x]$, standard ordered basis versus shuffling versus $(1, 1 + x, 1 + x + x^2, \ldots, 1 + x + \cdots + x^n)$.
• The identification $I_\beta : V \mapsto \mathbb{F}^n$ that sends $v \mapsto [v]_\beta$ is linear.
• Assume $T : V \to W$ is linear with ordered bases $\beta = (v_1, \ldots, v_n)$ for $V$ and $\gamma = (w_1, \ldots, w_m)$ for $W$. Then we know that this map is determined by the vectors $(T(v_i))_{i=1}^n$, or, equivalently, by the vectors $([T(v_i)])_{i=1}^n$.
• $[T]_\beta$ is an $m \times n$ matrix: if $V = W$ and $\beta = \gamma$, then we write $[T]_\beta = [T]_\gamma$.
• Uniqueness: if $[T]_\beta = [U]_\beta$, then $T = U$.
• Example: derivative $D : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$.
• Notation: for functions $T, U : V \to W$, we can define their sum $T + U$ and scalar multiples $cT$ for $c \in \mathbb{F}$.
• Let $\mathcal{L}(V, W)$ denote the set of all linear transformations from $V$ to $W$. Then this is a vector space with the operations above. If $V = W$, then we write $\mathcal{L}(V, V) = \mathcal{L}(V)$.
• Matrix representations respect the vector space structure: $[cT + U]_\beta = c[T]_\beta + [U]_\beta$.
• If there’s time: composition of linear transformations; distributive properties; associative; multiplicative identity; scalar multiplication.
• Matrix multiplication.