Determinants of $2 \times 2$ matrices; not additive/linear transformation; multilinear

$$\det \left( \begin{pmatrix} u + kv \\ w \end{pmatrix} \right) = \det \left( \begin{pmatrix} u \\ w \end{pmatrix} \right) + k \det \left( \begin{pmatrix} v \\ w \end{pmatrix} \right).$$

i.e., a linear transformation on each of the rows when the other row is fixed. Should convince yourself that the same statement is true of the columns.

Determinant is nonzero iff $A$ is invertible:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Suppose $A$ is invertible. Two possibilities: zero or nonzero. If zero, apply $A$ to the vector $\begin{pmatrix} d \\ -c \end{pmatrix}$. So $\begin{pmatrix} d \\ -c \end{pmatrix}$ is in the kernel of $L_A$.

Determinants of $n \times n$ matrices. $\tilde{A}_{i,j}$ is the matrix you get from cutting out the $i$th row and the $j$th column.

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} A_{1,j} \cdot \det(\tilde{A}_{1,j}).$$

Sometimes also denoted $|A| = \det(A)$.

The $(i, j)$-th cofactor is the scalar $c_{i,j} = (-1)^{i+j} \det(\tilde{A}_{i,j})$. The determinant can then be written as $\det(A) = \sum_{j=1}^n A_{1,j} c_{1,j}$ (“cofactor expansion along the first row”)

This is consistent with the $2 \times 2$ case.

Compute

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

$\det(I_N) = 1$, by induction.

$\det$ is a multilinear function of the rows, by induction.

Corollary: if $A$ has an entire row of zeros, then $\det(A) = 0$.

If row $i$ of $B$ is equal to $e_k$ for some $k \in [n]$, then $\det(B) = (-1)^{i+k} \det(\tilde{B}_{i,k})$.

Using multilinearity, this allows us to conclude that we can take the cofactor expansion along any row to compute $\det(A)$.

Corollary: if $A$ has two identical rows, then $\det(A) = 0$.

Corollary: interchanging rows flips the sign of a determinant by the previous result and multilinearity.

Corollary: adding multiples of a given row to another row does not change the determinant.

$A$ less than full rank $\implies \det(A) = 0$.

Rules: can interchange two rows, but changes sign; can add rows to other rows without changing determinant; can multiply a row by a scalar, determinant gets multiplied by the same scalar.

Example: determinant calculation from before now much easier; determinant of special matrices.