SO AS TO NOT DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE.

ANSWERS TO THE TRUE/FALSE QUESTIONS DO NOT NEED TO BE JUSTIFIED; HOWEVER, INCORRECT ANSWERS TO THE TRUE/FALSE QUESTIONS ARE PENALIZED. IN PARTICULAR, A CORRECT ANSWER IS WORTH 5 POINTS, AN INCORRECT ANSWER IS WORTH -5 POINTS, AND A BLANK ANSWER IS WORTH 0 POINTS.

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED.

THE EXAM CONSISTS OF 5 TRUE/FALSE QUESTIONS AND 3 LONGER FORMAT QUESTIONS. YOUR ANSWERS TO THE LONGER FORMAT QUESTIONS SHOULD BE CAREFULLY JUSTIFIED. YOU ARE ALLOWED TO USE RESULTS FROM THE TEXTBOOK, HOMEWORK, AND LECTURE, BUT THEY SHOULD BE CLEARLY REFERENCED (FOR EXAMPLE, “BY THE CAYLEY-HAMILTON THEOREM, …”).
1. (25 points) Label the following statements as true or false. Any ambiguous answer (for example, resembling a hybrid of T and F) will be treated as an incorrect answer.

(a) ___________ If $T : V \to V$ is a linear operator and $W_1, W_2 \leq V$ are $T$-invariant subspaces, then the subspace $W_1 \cap W_2 = \{ v \in V : v \in W_1 \text{ and } v \in W_2 \}$ is also $T$-invariant.

(b) ___________ If $T : V \to V$ is a diagonalizable linear operator on a finite-dimensional vector space $V$, then $T^n : V \to V$ is also diagonalizable for every integer $n \geq 1$.

(c) ___________ If $T : V \to V$ is a linear operator on a finite-dimensional inner product space $V$ such that $TT^* = T_0$ (the zero operator), then $T = T_0$.

(d) ___________ Every orthogonal set is linearly independent.

(e) ___________ If $T : V \to V$ is a linear operator on a finite-dimensional inner product space $V$ such that $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for every $x, y \in V$, then $T$ is an isomorphism.
2. (25 points) Suppose that $T : V \to V$ is a linear operator on a finite-dimensional vector space $V$ with $\dim(V) = n$. Show that if $T$ has exactly three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$) with $\dim(E_{\lambda_i}) = n - 2$, then $T$ is diagonalizable. Be careful not to assume that the characteristic polynomial of $T$ splits (i.e., this must be proven).
3. (25 points) Assume that the matrix $A \in M_{n \times n}(F)$ has characteristic polynomial

$$p(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

for some scalars $a_0, \ldots, a_{n-1} \in F$.

(a) (5 pts) Prove that $A$ is invertible if and only if $a_0 \neq 0$.

(b) (10 pts) Prove that if $A$ is invertible, then

$$A^{-1} = \frac{-1}{a_0} \left[ (-1)^n A^{n-1} + a_{n-1} A^{n-2} + \cdots + a_1 I_n \right].$$
(c) (10 pts) Assume that $A$ is invertible. Prove that
\[ \text{span}\{\ldots, A^{-2}, A^{-1}, I_n, A, A^2, \ldots\} = \text{span}\{I_n, A, A^2, \ldots, A^{n-1}\}. \]
4. (30 points) Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional inner product space $V$ with inner product $\langle \cdot, \cdot \rangle_{old}$ and $\dim(V) = n$.

(a) (15 pts) Prove that $\ker(T) = \{0\}$ if and only if $\ker(T^*) = \{0\}$. Conclude that $T$ is an isomorphism if and only if $T^*$ is an isomorphism.
(b) (15 pts) Suppose that $\ker(T) = \{0\}$. Prove that the function $\langle \cdot_1, \cdot_2 \rangle_{\text{new}} : V \times V \to F$ given by
\[
\langle v, w \rangle_{\text{new}} = \langle TT^*(v), w \rangle_{\text{old}}
\]
defines an inner product $\langle \cdot_1, \cdot_2 \rangle_{\text{new}}$ on $V$. 
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)
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