MATH 180B - FINAL EXAM

WINTER 2020

Name (Last, First): ___________________________________________________________________

Student ID: _______________________________________________________________________

THIS IS THE TAKE-HOME FINAL EXAM FOR MATH 180B, WINTER 2020, AT UC SAN DIEGO, TO BE WRITTEN ON MARCH 20TH, 2020 FROM 3:00 PM TO 6:00 PM.

THIS EXAM IS PROPERTY OF THE REGENTS OF THE UNIVERSITY OF CALIFORNIA AND NOT MEANT FOR OUTSIDE DISTRIBUTION. IF YOU SEE THIS EXAM APPEARING ELSEWHERE, PLEASE NOTIFY THE INSTRUCTOR AT BAU@UCSD.EDU AND THE UCSD OFFICE OF ACADEMIC INTEGRITY AT AIO@UCSD.EDU.

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THE EXAM CONSISTS OF 8 QUESTIONS. YOU ARE ALLOWED TO USE RESULTS FROM THE TEXTBOOK, HOMEWORK, AND LECTURE AS LONG AS THEY ARE CLEARLY REFERENCED.
EXCEL WITH INTEGRITY PLEDGE

I pledge to be fair to my classmates and instructors by completing all of my academic work with integrity. This means that I will respect the standards set by the instructor and institution, be responsible for the consequences of my choices, honestly represent my knowledge and abilities, and be a community member that others can trust to do the right thing even when no one is watching. I will always put learning before grades, and integrity before performance. I pledge to excel with integrity.

In addition to the above, I pledge that I did not receive outside assistance with this exam. Outside assistance includes but is not limited to other people, the internet, and resources beyond the textbook, lecture notes, and homework assignments.

To acknowledge that you agree to this pledge, you must copy the sentence:

I choose to excel with integrity as a member of the University of California, San Diego.

Make sure to sign your name below it and date it as well (March 20th, 2020). Exams without this pledge will not be graded.
1. (25 points) Assume that \( \begin{pmatrix} Y \\ X \end{pmatrix} \sim \mathcal{N}(\hat{\mu}, \Sigma) \), where
\[
\hat{\mu} = \begin{pmatrix} \mu_Y \\ \mu_X \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_Y^2 & \sigma_{Y,X} \\ \sigma_{Y,X} & \sigma_X^2 \end{pmatrix}.
\]

(a) (10 points) Suppose that \( \Sigma \) is a diagonal matrix such that \( \det(\Sigma) \neq 0 \). Compute \( \mathbb{E}[Y^2 X^2] \) and \( \mathbb{E}[Y X] \).
Extra space for problem 1(a):
(b) (15 points) Assume that $S \sim \mathcal{N}(0, 1)$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 
0 & \text{if } x = 0; \\
x & \text{if } |x| \in \bigcup_{n=0}^{\infty} (2n, 2n + 1]; \\
-x & \text{if } |x| \in \bigcup_{n=0}^{\infty} (2n + 1, 2n + 2].
\end{cases}$$

Prove that

$$\mathbb{E}[e^{tf(S)}] = \mathbb{E}[e^{tS}]$$

for any $t \in \mathbb{R}$. Determine the distribution of the random variable $f(S)$. For full credit, you must identify the name of the distribution as well.

Hint: you do not need to compute the value of the integrals defined by $\mathbb{E}[e^{tf(S)}]$ and $\mathbb{E}[e^{tS}]$. You simply need to work with the integrals to prove that they are equal.
Extra space for problem 1(b):
2. (15 points) Suppose that the number of customers arriving at a grocery store in a given
day can be modeled by a Poisson distribution with rate $\lambda > 0$. Assume that the cashier
needs exactly 1 minute per customer. After all of the customers are taken care of, the cashier
must then close the store, which takes $X$ minutes, where $X$ is independent of the number of
customers and $X \sim \text{Unif}[0, 1]$. Find the density of the number of minutes the cashier spends
working in a given day.
Extra space for problem 2:
3. (25 points) Consider a branching process with common offspring distribution

\[ \xi \sim \text{Unif}\{0, 1, \ldots, n\}. \]

In other words,

\[ P(\xi = m) = \frac{1}{n + 1} \text{ for } m \in \{0, 1, \ldots, n\}. \]

(a) (15 points) What is the expectation of \( \xi \)? For which values of \( n \) does the population go extinct with probability 1?
Extra space for problem 3(a):
(b) (10 points) Are there any values of $n$ such that the population goes extinct with probability 0? Explain why or why not.
Extra space for problem 3(b):
4. (25 points) Suppose that you have a thick coin that returns heads (H) with probability \( p \in (0, 1) \), tails (T) with probability \( q \in (0, 1) \), and lands on its edge (E) with probability \( r \in (0, 1) \), where \( p + q + r = 1 \). After each flip, you record the result; however, if the coin lands on its edge, you do not record a result. Suppose that you flip until you either record two heads in a row or two tails in a row. For example, you would stop flipping if you flip (T, E, T), which counts as 3 flips. You would also stop flipping if you flip (H, E, E, E, H), which counts as 5 flips. In other words, you want a streak of two heads or a streak of two tails; if the coin lands on its edge, it does not break the streak, but it does count as a flip.

Suppose that \( p = q = r \). Find the expected number of flips until you stop.

Hint: draw a diagram.
Extra space for problem 4:
5. (15 points) Let \((X_n)_{n \geq 0}\) be a Markov chain on the finite state space \([N] = \{1, \ldots, N\}\) with transition matrix \(P\). In other words,

\[
P(i, j) = \mathbb{P}(X_{n+1} = j | X_n = i), \quad \forall i, j \in [N].
\]

Recall that the entries of each row of \(P\) sum to 1, i.e., \(\sum_{j=1}^n P(i, j) = 1\) for each \(i \in [N]\). Assume that \(P\) is regular and that the limiting distribution \(\hat{\pi}\) is uniform, i.e.,

\[
\hat{\pi} = \left( \frac{1}{N}, \ldots, \frac{1}{N} \right).
\]

Prove that the entries of each column of \(P\) sum to 1, i.e., \(\sum_{i=1}^n P(i, j) = 1\) for each \(j \in [N]\).
Extra space for problem 5:
6. (25 points) Consider a Markov chain \((X_n)_{n \geq 0}\) on the infinite state space \(\{0, 1, 2, \ldots\}\) with transition matrix

\[
P(i, j) = p_j,
\]

where \(\sum_{j=0}^{\infty} p_j = 1\) and \(p_j > 0\) for every \(j\).

(a) (15 points) Compute the first return probabilities \(f_j^{(n)}\) for each \(j \in \{0, 1, 2, \ldots\}\) and \(n \in \{1, 2, 3, \ldots\}\). Recall that

\[
f_j^{(n)} = \mathbb{P}(X_n = j \text{ and } X_k \neq j \text{ for } k = 1, \ldots, n - 1|X_0 = j).
\]

You just need a single formula for \(f_j^{(n)}\) that takes into account both \(j\) and \(n\).

Hint: use the formula \(P(i, j) = p_j\) to write the transition matrix as an infinite matrix.
Extra space for problem 6(a):
(b) (10 points) Compute the limiting distribution \( \hat{\pi} = (\pi_0, \pi_1, \pi_2, \ldots) \) of the Markov chain. You may assume that the assumptions of the basic limit theorem of Markov chains are met.
Extra space for problem 6(b):
7. (25 points) The total number of monkeys arriving at an airport by time $t$ (in hours) can be modeled by a homogeneous Poisson process $(M_t)_{t \geq 0}$ with rate $e^{\sqrt{2\pi}}$. If two monkeys arrive within 30 minutes of each other, then a fight breaks out between the monkeys. Assume that after an hour, two monkeys have arrived (in other words, $M_1 = 2$). What is the probability that these two monkeys fought?

Hint: it might help to draw a unit square in $\mathbb{R}^2$. 
Extra space for problem 7:
8. (45 points) The total number of patients admitted to a hospital by time $t$ can be modeled by a homogeneous Poisson process $(P_t)_{t \geq 0}$ with rate $\kappa > 0$ (in particular, assume that $P_0 = 0$). Assume that every patient admitted to the hospital is immediately tested for COVID-19 and that the result of the test is immediate as well (in particular, no time passes from the arrival of a patient to the result of the test). Each patient tests positive with probability $r \in (0, 1)$ and negative with probability $1 - r$ independently of all other patients.

(a) (10 points) Let $(C_t)_{t \geq 0}$ be the number of patients at the hospital at time $t$ who have tested positive for COVID-19. Assume that $u > s$. Compute

$$P(C_u - C_s = a | P_u - P_s = b)$$

for any non-negative integers $a, b$. 
Extra space for problem 8(a):
(b) (10 points) Compute

$$P(C_u - C_s = a)$$

for any non-negative integer $a$ by writing $C_u - C_s$ as a random sum. You may cite calculations from the lecture/textbook to justify your answer, but this should be done clearly. Hint: use part (a).
Extra space for problem 8(b):
(c) (5 points) Are the increments of the process \((C_t)_{t \geq 0}\) independent? Justify your answer.
Extra space for problem 8(c):
(d) (5 points) What is the distribution of the waiting time until the first patient arrives?
Extra space for problem 8(d):
(e) (15 points) What is the distribution of the waiting time until the first positive test in the hospital?

Hint: take a look at your work from all the previous parts.
Extra space for problem 8(e):