REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED.

THE EXAM CONSISTS OF 4 QUESTIONS. YOUR ANSWERS SHOULD BE CAREFULLY JUSTIFIED. YOU ARE ALLOWED TO USE RESULTS FROM THE TEXTBOOK, HOMEWORK, AND LECTURE, BUT THEY SHOULD BE CLEARLY REFERENCED. FOR EXAMPLE,

“BY THE PIGEONHOLE PRINCIPLE, ...”
1. Suppose that $A$ is a subset of $[5n] = \{1, 2, \ldots, 5n\}$ for some $n \geq 1$. If $\#(A) \geq 4n + 1$, prove that $A$ necessarily has a subset of five consecutive integers, i.e., there exists an integer $m \in [5n - 4]$ such that $\{m, m + 1, m + 2, m + 3, m + 4\} \subseteq A$. 
2. 
(a) Find the number of nested $m$-tuples $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq [n]$. In other words, find $|S|$, where 

$$S = \{(A_1, A_2, \ldots, A_m) : A_i \subseteq A_{i+1} \text{ for each } i \in [m-1] \text{ and } A_m \subseteq [n]\}.$$
(b) We want to select an ordered pair \((A, B)\) of subsets of \([n]\) so that \(A \cap B \neq \emptyset\). In how many different ways can we do this?
3. 

(a) A restaurant offers five different soups, ten main courses, and six desserts. Joe decided to order at most one soup, at most one main course, and at most one dessert. In how many ways can he do this?
(b) A student has to take twelve hours of classes a week. Due to her extracurricular activities, she must take at least three hours of classes on Monday, at least two on Tuesday, and at least one on Friday. In how many ways can she do this?
4. Recall that $S(n, k)$ denotes a Stirling number of the second kind. Find a closed-form expression for $S(n, n - 2)$ for $n \geq 3$. By a closed-form expression, we mean a simple formula that can be evaluated to obtain the desired quantity. For example, recall that $S(n, 2) = 2^{n-1} - 1$. In particular, simply stating the recursive identity for the Stirling numbers of the second kind will not receive any credit.
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)
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