Math 150B Homework Problems: Winter 2020

Problems are our text book, "Differential Forms," by Guillemin and Haine or from the lecture notes as indicated. The problems from the lecture notes are restated here.

1.1 Homework 0, Due Wednesday, January 8, 2020 (Not to be collected)

• Lecture note Exercises: 2.1, 2.2, 2.3, 2.4, and 2.5.

1.2 Homework 1. Due Thursday, January 16, 2020

- Lecture note Exercises: 2.6, 2.7, 2.8, 2.9
- Book Exercises: 1.2.vi.

Lecture Note Problems

Exercise 2.1. Find the value of the following integral;

$$I := \int_1^9 dy \int_{\sqrt{y}}^3 dx \ x e^y.$$

Hint: use Tonelli's theorem to change the order of integrations.

Exercise 2.2. Write the following iterated integral

$$I := \int_0^1 dx \int_{y=x^{2/3}}^1 dy \ x e^{y^4}$$

as a multiple integral and use this to change the order of integrations and then compute I.

Exercise 2.3. Suppose that d = 2, show $m_2(B(0,r)) = \pi r^2$.

Exercise 2.4. Suppose that d = 3, show $m_3(B(0,r)) = \frac{4\pi}{3}r^3$.

Exercise 2.5. Let $V_d(r) := m_d(B(0,r))$. Show for $d \ge 1$ that

$$V_{d+1}(r) = \int_{-r}^{r} dz \cdot V_d\left(\sqrt{r^2 - z^2}\right) = r \int_{-\pi/2}^{\pi/2} V_d\left(r\cos\theta\right)\cos\theta d\theta$$

Exercise 2.6 (Change of variables for elementary matrices). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support. Show by direct calculation that;

$$\left|\det T\right| \int_{\mathbb{R}^d} f\left(T\left(x\right)\right) dx = \int_{\mathbb{R}^d} f\left(y\right) dy \tag{2.1}$$

for each of the following linear transformations;

1. Suppose that i < k and

$$T(x_1, x_2, \dots, x_d) = (x_1, \dots, x_{i-1}, x_k, x_{i+1}, \dots, x_{k-1}, x_i, x_{k+1}, \dots, x_d),$$

i.e. T swaps the i and k coordinates of x. [In matrix notation T is the identity matrix with the i and k column interchanged.]

2. $T(x_1, \ldots, x_k, \ldots, x_d) = (x_1, \ldots, cx_k, \ldots, x_d)$ where $c \in \mathbb{R} \setminus \{0\}$. [In matrix notation, $T = [e_1| \ldots |e_{k-1}| ce_k |e_{k+1}| \ldots |e_d]$.]

3. $T(x_1, x_2, \ldots, x_d) = (x_1, \ldots, x_i + cx_k, \ldots x_k, \ldots x_d)$ where $c \in \mathbb{R}$. [In matrix notation $T = [e_1| \ldots |e_i| \ldots |e_k + ce_i|e_{k+1}| \ldots |e_d]$.

Hint: you should use Fubini's theorem along with the one dimensional change of variables theorem.

[To be more concrete here are examples of each of the T appearing above in the special case d = 4,

1. If
$$i = 2$$
 and $k = 3$ then $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
2. If $k = 3$ then $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,
3. If $i = 2$ and $k = 4$ then

$$T\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix} = \begin{pmatrix} x_1\\ x_2 + cx_4\\ x_3\\ x_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & c\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix}$$

while if i = 4 and k = 2,

$$T\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix} = \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 + cx_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & c & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix}.$$

Exercise 2.7. Let $V = \mathbb{R}^n$ and $\beta = \{u_j\}_{j=1}^n$ be a basis for \mathbb{R}^n . Recall that every $\ell \in (\mathbb{R}^n)^*$ is of the form $\ell_a(x) = a \cdot x$ for some $a \in \mathbb{R}^n$. Thus the dual basis, β^* , to β can be written as $\{u_j^* = \ell_{a_j}\}_{j=1}^n$ for some $\{a_j\}_{j=1}^n \subset \mathbb{R}^n$. In this problem you are asked to show how to find the $\{a_j\}_{j=1}^n$ by the following steps.

1. Show that for $j \in [n]$, a_j must solve the following k-linear equation;

$$\delta_{j,k} = \ell_{a_j} \left(u_k \right) = a_j \cdot u_k = u_k^{\text{tr}} a_j \text{ for } k \in [n].$$

$$(2.2)$$

- 2. Let $U := [u_1| \dots |u_n]$ (i.e. the columns of U are the vectors from β). Show that the equations in (2.2) may be written in matrix form as, $U^{\text{tr}}a_j = e_j$, where $\{e_j\}_{j=1}^n$ is the standard basis for \mathbb{R}^n .
- 3. Conclude that $a_j = [U^{\text{tr}}]^{-1} e_j$ or equivalently;

$$[a_1|\ldots|a_n] = \left[U^{\mathrm{tr}}\right]^{-1}$$

Exercise 2.8. Let $V = \mathbb{R}^2$ and $\beta = \{u_1, u_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
 and $u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Find $a_1, a_2 \in \mathbb{R}^2$ explicitly so that explicitly the dual basis $\beta^* := \{u_1^* = \ell_{a_1}, u_2^* = \ell_{a_2}\}$ is the dual basis to β . Please explicitly verify your answer is correct by showing $u_j^*(u_k) = \delta_{jk}$.

Exercise 2.9. In this problem, let

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$
 and $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$.

Which of the following functions formulas for T define a 2-tensors on \mathbb{R}^3 . Please justify your answers.

1. $T(v, w) = v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1.$ 2. $T(v, w) = v_1 + 7 v_1 + v_2.$ 3. $T(v, w) = v_1^2 w_3 + v_2 w_1,$ 4. $T(v, w) = \sin(v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1).$