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Math 150B Homework Problems: Winter 2020

Problems are our text book, "Differential Forms," by Guillemin and Haine or from the lecture notes as indicated. The problems from the lecture notes are restated here.

1.1 Homework 0, Due Wednesday, January 8, 2020 (Not to be collected)

- Lecture note Exercises: 2.1, 2.2, 2.3, 2.4, and 2.5.

1.2 Homework 1. Due Thursday, January 16, 2020

- Lecture note Exercises: 2.6, 2.7, 2.8, 2.11
- Book Exercises: 1.2.vi.

1.3 Homework 2. Due Thursday, January 23, 2020

- Lecture note Exercises: 2.9, 2.10, 2.12, 2.13, 2.14, 2.15, 2.16
- Book Exercises: 1.3.iii., 1.3.v., 1.3.vii, 1.4ix

Lecture Note Problems

Exercise 2.1. Find the value of the following integral;

$$I := \int_1^9 dy \int_{\sqrt{y}}^3 dx\ xe^y.$$

Hint: use Tonelli's theorem to change the order of integrations.

Exercise 2.2. Write the following iterated integral

$$I := \int_0^1 dx \int_{y=x^{2/3}}^1 dy\ xe^{y^4}.$$

as a multiple integral and use this to change the order of integrations and then compute I.

Exercise 2.3. Suppose that $d=2$, show $m_2(B(0,r)) = \pi r^2$.

Exercise 2.4. Suppose that $d=3$, show $m_3(B(0,r)) = \frac{4\pi}{3} r^3$.

Exercise 2.5. Let $V_d(r) := m_d(B(0,r))$. Show for $d \geq 1$ that

$$V_{d+1}(r) = \int_{-r}^{r} dz \cdot V_d\left(\sqrt{r^2 - z^2}\right) = r \int_{-\pi/2}^{\pi/2} V_d(r\cos\theta)\cos\theta d\theta.$$

Exercise 2.6 (Change of variables for elementary matrices). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support. Show by direct calculation that;

$$|\det T| \int_{\mathbb{R}^d} f(T(x))\, dx = \int_{\mathbb{R}^d} f(y)\, dy \tag{2.1}$$

for each of the following linear transformations;

1. Suppose that $i < k$ and

   $$T(x_1, x_2 \dots, x_d) = (x_1, \dots, x_{i-1}, x_k, x_{i+1} \dots, x_{k-1}, x_i, x_{k+1}, \dots x_d),$$

   i.e. T swaps the i and k coordinates of x. [In matrix notation T is the identity matrix with the i and k column interchanged.]
2. $T(x_1, \dots x_k, \dots, x_d) = (x_1, \dots, cx_k, \dots x_d)$ where $c \in \mathbb{R} \setminus \{0\}$. [In matrix notation, $T = [e_1| \dots |e_{k-1}|ce_k|e_{k+1}| \dots |e_d]$.]
3. $T(x_1, x_2 \dots, x_d) = (x_1, \dots, \overbrace{x_i + cx_k}^{\text{i'th spot}}, \dots x_k, \dots x_d)$ where $c \in \mathbb{R}$. [In matrix notation $T = [e_1| \dots |e_i| \dots |e_k + ce_i|e_{k+1}| \dots |e_d]$.

Hint: you should use Fubini's theorem along with the one dimensional change of variables theorem.

[To be more concrete here are examples of each of the T appearing above in the special case $d=4$,

1. If $i=2$ and $k=3$ then $T = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}$.
2. If $k=3$ then $T = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&c&0 \\ 0&0&0&1 \end{bmatrix}$,
3. If $i=2$ and $k=4$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + cx_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&c \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

while if $i=4$ and $k=2$,

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + cx_2 \end{pmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&c&0&1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Exercise 2.7. Let $V = \mathbb{R}^n$ and $\beta = \{u_j\}_{j=1}^n$ be a basis for $\mathbb{R}^n$. Recall that every $\ell \in (\mathbb{R}^n)^*$ is of the form $\ell_a(x) = a \cdot x$ for some $a \in \mathbb{R}^n$. Thus the dual basis, β^*, to β can be written as $\left\{u_j^* = \ell_{a_j}\right\}_{j=1}^n$ for some $\{a_j\}_{j=1}^n \subset \mathbb{R}^n$. In this problem you are asked to show how to find the $\{a_j\}_{j=1}^n$ by the following steps.

1. Show that for $j \in [n]$, a_j must solve the following k-linear equations;

$$\delta_{j,k} = \ell_{a_j}(u_k) = a_j \cdot u_k = u_k^{\text{tr}} a_j \text{ for } k \in [n]. \tag{2.2}$$

2. Let $U := [u_1|\dots|u_n]$ (i.e. the columns of U are the vectors from β). Show that the equations in (2.2) may be written in matrix form as, $U^{\mathrm{tr}} a_j = e_j$, where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.
3. Conclude that $a_j = [U^{\mathrm{tr}}]^{-1} e_j$ or equivalently;

$$[a_1|\dots|a_n] = \left[U^{\mathrm{tr}}\right]^{-1}$$

Exercise 2.8. Let $V = \mathbb{R}^2$ and $\beta = \{u_1, u_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find $a_1, a_2 \in \mathbb{R}^2$ explicitly so that explicitly the dual basis $\beta^* := \{u_1^* = \ell_{a_1}, u_2^* = \ell_{a_2}\}$ is the dual basis to β. Please explicitly verify your answer is correct by showing $u_j^*(u_k) = \delta_{jk}$.

Exercise 2.9. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. Show $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set if and only if $\{a_j\}_{j=1}^k \subset V$ is a linearly independent set.

Exercise 2.10. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. If $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set, show there exists $\{u_j\}_{j=1}^k \subset V$ so that $\ell_i(u_j) = \delta_{ij}$ for $i, j \in [k]$. Here is a possible outline.

1. Using Exercise 2.9 and citing a basic fact from Linear algebra, you may choose $\{a_j\}_{j=k+1}^n \subset V$ so that $\{a_j\}_{j=1}^n$ is a basis for V.
2. Argue that it suffices to find $u_j \in V$ so that
$$a_i \cdot u_j = \delta_{ij} \text{ for all } i, j \in [n]. \tag{2.3}$$
3. Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$ and $A := [a_1|\dots|a_n]$ be the $n \times n$ matrix with columns given by that $\{a_j\}_{j=1}^n$. Show that the Eqs. (2.3) may be written as
$$A^{\mathrm{tr}} u_j = e_j \text{ for } j \in [n]. \tag{2.4}$$
4. Cite basic facts from linear algebra to explain why $A := [a_1|\dots|a_n]$ and A^{tr} are both invertible $n \times n$ matrices.
5. Argue that Eq. (2.4) has a unique solution, $u_j \in \mathbb{R}^n$, for each j.

Exercise 2.11. In this problem, let

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ and } w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Which of the following functions formulas for T define a 2-tensors on $\mathbb{R}^3$. Please justify your answers.

1. $T(v, w) = v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1$.
2. $T(v, w) = v_1 + 7 v_1 + v_2$.
3. $T(v, w) = v_1^2 w_3 + v_2 w_1$,
4. $T(v, w) = \sin(v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1)$.

Exercise 2.12. If $T \in \Lambda^k(V^*)$, show $T(v_1, \dots, v_k) = 0$ whenever $\{v_i\}_{i=1}^k \subset V$ are linearly dependent.

Exercise 2.13. Let V, W, and Z be three finite dimensional vector spaces and suppose that $V \overset{T}{\to} W \overset{S}{\to} Z$ are linear transformations. Noting that $V \overset{ST}{\to} Z$, show $(ST)^* = T^* S^*$.

Exercise 2.14. If $\psi \in \Lambda^n(V^*) \setminus \{0\}$, show $\psi(v_1, \dots, v_n) \neq 0$ whenever $\{v_i\}_{i=1}^n \subset V$ are linearly independent. [Coupled with Exercise 2.12, it follows that $\psi(v_1, \dots, v_n) \neq 0$ iff $\{v_i\}_{i=1}^n \subset V$ are linearly independent.]

Exercise 2.15. Let $\{e_i\}_{i=1}^4$ be the standard basis for $\mathbb{R}^4$ and $\{\varepsilon_i = e_i^*\}_{i=1}^4$ be the associated dual basis (i.e. $\varepsilon_i(v) = v_i$ for all $v \in \mathbb{R}^4$.) Compute;

1. $\varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_4 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \right)$,
2. $\varepsilon_3 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$,
3. $\varepsilon_1 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$,
4. $(\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$, and
5. $\varepsilon_4 \wedge \varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_1 (e_1, e_2, e_3, e_4)$.

Exercise 2.16. Show, using basic knowledge of determinants, that for $\ell_0, \ell_1, \ell_2, \ell_3 \in V^*$, that

$$(\ell_0 + \ell_1) \wedge \ell_2 \wedge \ell_3 = \ell_0 \wedge \ell_2 \wedge \ell_3 + \ell_1 \wedge \ell_2 \wedge \ell_3.$$