Math 150B Homework Problems: Winter 2020

Problems are our text book, "Differential Forms," by Guillemin and Haine or from the lecture notes as indicated. The problems from the lecture notes are restated here, however there may be broken references. If this is the case, please find the corresponding problem in the lecture notes for the proper references and for more context of the problem.

1.1 Homework 0, Due Wednesday, January 8, 2020 (Not to be collected)

- Lecture note Exercises: 2.1, 2.2, 2.3, 2.4, and 2.5.

1.2 Homework 1. Due Thursday, January 16, 2020

- Lecture note Exercises: 2.6, 2.7, 2.8, 2.11
- Book Exercises: 1.2.vi.

1.3 Homework 2. Due Thursday, January 23, 2020

- Lecture note Exercises: 2.9, 2.10, 2.12, 2.13, 2.14, 2.15, 2.16
- Book Exercises: 1.3.iii., 1.3.v., 1.3.vii, 1.4ix

1.4 Homework 3. Due Thursday, January 30, 2020

- Lecture note Exercises: 2.17, 2.18, 2.21, 2.22, 2.23, 2.24, 2.25
- Look at (but **don't hand in**) Exercises 2.19, 2.20 and the Book Exercises: 1.7.iv., 1.8vi.

Lecture Note Problems

Exercise 2.1. Find the value of the following integral;

$$I := \int_1^9 dy \int_{\sqrt{y}}^3 dx \ xe^y.$$

Hint: use Tonelli's theorem to change the order of integrations.

Exercise 2.2. Write the following iterated integral

$$I := \int_0^1 dx \int_{y=x^{2/3}}^1 dy \ xe^{y^4}.$$

as a multiple integral and use this to change the order of integrations and then compute I.

Exercise 2.3. Suppose that $d = 2$, show $m_2 \left(B \left(0, r \right) \right) = \pi r^2$.

Exercise 2.4. Suppose that $d = 3$, show $m_3 \left(B \left(0, r \right) \right) = \frac{4\pi}{3} r^3$.

Exercise 2.5. Let $V_d \left(r \right) := m_d \left(B \left(0, r \right) \right)$. Show for $d \geq 1$ that

$$V_{d+1} \left(r \right) = \int_{-r}^r dz \cdot V_d \left(\sqrt{r^2 - z^2} \right) = r \int_{-\pi/2}^{\pi/2} V_d \left(r \cos \theta \right) \cos \theta d\theta.$$

Exercise 2.6 (Change of variables for elementary matrices). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support. Show by direct calculation that;

$$|\det T| \int_{\mathbb{R}^d} f \left(T \left(x \right) \right) dx = \int_{\mathbb{R}^d} f \left(y \right) dy \tag{2.1}$$

for each of the following linear transformations;

1. Suppose that $i < k$ and

$$T(x_1, x_2 \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_k, x_{i+1} \ldots, x_{k-1}, x_i, x_{k+1}, \ldots x_d),$$

   i.e. T swaps the i and k coordinates of x. [In matrix notation T is the identity matrix with the i and k column interchanged.]
2. $T(x_1, \ldots x_k, \ldots, x_d) = (x_1, \ldots, cx_k, \ldots x_d)$ where $c \in \mathbb{R} \setminus \{0\}$. [In matrix notation, $T = [e_1| \ldots |e_{k-1}|ce_k|e_{k+1}| \ldots |e_d|.]$

3. $T(x_1, x_2 \ldots, x_d) = (x_1, \ldots, \overset{\text{i'th spot}}{x_i + cx_k}, \ldots x_k, \ldots x_d)$ where $c \in \mathbb{R}$. [In matrix notation $T = [e_1| \ldots |e_i| \ldots |e_k + ce_i|e_{k+1}| \ldots |e_d].$

   Hint: you should use Fubini's theorem along with the one dimensional change of variables theorem.
   [To be more concrete here are examples of each of the T appearing above in the special case $d = 4$,

1. If $i = 2$ and $k = 3$ then $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

2. If $k = 3$ then $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$

3. If $i = 2$ and $k = 4$ then

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + cx_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

while if $i = 4$ and $k = 2$,

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + cx_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Exercise 2.7. Let $V = \mathbb{R}^n$ and $\beta = \{u_j\}_{j=1}^n$ be a basis for $\mathbb{R}^n$. Recall that every $\ell \in (\mathbb{R}^n)^*$ is of the form $\ell_a (x) = a \cdot x$ for some $a \in \mathbb{R}^n$. Thus the dual basis, β^*, to β can be written as $\{u_j^* = \ell_{a_j}\}_{j=1}^n$ for some $\{a_j\}_{j=1}^n \subset \mathbb{R}^n$. In this problem you are asked to show how to find the $\{a_j\}_{j=1}^n$ by the following steps.

1. Show that for $j \in [n]$, a_j must solve the following k-linear equations;

$$\delta_{j,k} = \ell_{a_j} (u_k) = a_j \cdot u_k = u_k^{\text{tr}} a_j \text{ for } k \in [n]. \tag{2.2}$$

2. Let $U := [u_1| \ldots |u_n]$ (i.e. the columns of U are the vectors from β). Show that the equations in (2.2) may be written in matrix form as, $U^{\mathrm{tr}} a_j = e_j$, where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.

3. Conclude that $a_j = [U^{\mathrm{tr}}]^{-1} e_j$ or equivalently;

$$[a_1| \ldots |a_n] = [U^{\mathrm{tr}}]^{-1}$$

Exercise 2.8. Let $V = \mathbb{R}^2$ and $\beta = \{u_1, u_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find $a_1, a_2 \in \mathbb{R}^2$ explicitly so that explicitly the dual basis $\beta^* := \{u_1^* = \ell_{a_1}, u_2^* = \ell_{a_2}\}$ is the dual basis to β. Please explicitly verify your answer is correct by showing $u_j^*(u_k) = \delta_{jk}$.

Exercise 2.9. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. Show $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set if and only if $\{a_j\}_{j=1}^k \subset V$ is a linearly independent set.

Exercise 2.10. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. If $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set, show there exists $\{u_j\}_{j=1}^k \subset V$ so that $\ell_i(u_j) = \delta_{ij}$ for $i, j \in [k]$. Here is a possible outline.

1. Using Exercise 2.9 and citing a basic fact from Linear algebra, you may choose $\{a_j\}_{j=k+1}^n \subset V$ so that $\{a_j\}_{j=1}^n$ is a basis for V.
2. Argue that it suffices to find $u_j \in V$ so that

$$a_i \cdot u_j = \delta_{ij} \text{ for all } i, j \in [n]. \tag{2.3}$$

3. Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$ and $A := [a_1| \ldots |a_n]$ be the $n \times n$ matrix with columns given by that $\{a_j\}_{j=1}^n$. Show that the Eqs. (2.3) may be written as

$$A^{\mathrm{tr}} u_j = e_j \text{ for } j \in [n]. \tag{2.4}$$

4. Cite basic facts from linear algebra to explain why $A := [a_1| \ldots |a_n]$ and A^{tr} are both invertible $n \times n$ matrices.
5. Argue that Eq. (2.4) has a unique solution, $u_j \in \mathbb{R}^n$, for each j.

Exercise 2.11. In this problem, let

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ and } w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Which of the following functions formulas for T define a 2-tensors on $\mathbb{R}^3$. Please justify your answers.

1. $T(v, w) = v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1$.
2. $T(v, w) = v_1 + 7 v_1 + v_2$.
3. $T(v, w) = v_1^2 w_3 + v_2 w_1$,
4. $T(v, w) = \sin(v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1)$.

Exercise 2.12. If $T \in \Lambda^k(V^*)$, show $T(v_1, \ldots, v_k) = 0$ whenever $\{v_i\}_{i=1}^k \subset V$ are linearly dependent.

Exercise 2.13. Let V, W, and Z be three finite dimensional vector spaces and suppose that $V \xrightarrow{T} W \xrightarrow{S} Z$ are linear transformations. Noting that $V \xrightarrow{ST} Z$, show $(ST)^* = T^* S^*$.

Exercise 2.14. If $\psi \in \Lambda^n(V^*) \setminus \{0\}$, show $\psi(v_1, \ldots, v_n) \neq 0$ whenever $\{v_i\}_{i=1}^n \subset V$ are linearly independent. [Coupled with Exercise 2.12, it follows that $\psi(v_1, \ldots, v_n) \neq 0$ iff $\{v_i\}_{i=1}^n \subset V$ are linearly independent.]

Exercise 2.15. Let $\{e_i\}_{i=1}^4$ be the standard basis for $\mathbb{R}^4$ and $\{\varepsilon_i = e_i^*\}_{i=1}^4$ be the associated dual basis (i.e. $\varepsilon_i(v) = v_i$ for all $v \in \mathbb{R}^4$.) Compute;

1. $\quad \varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_4 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \right),$

2. $\quad \varepsilon_3 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right),$

3. $\quad \varepsilon_1 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right),$

4. $\quad (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$, and

5. $\quad \varepsilon_4 \wedge \varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_1(e_1, e_2, e_3, e_4)$.

Exercise 2.16. Show, using basic knowledge of determinants, that for $\ell_0, \ell_1, \ell_2, \ell_3 \in V^*$, that

$$(\ell_0 + \ell_1) \wedge \ell_2 \wedge \ell_3 = \ell_0 \wedge \ell_2 \wedge \ell_3 + \ell_1 \wedge \ell_2 \wedge \ell_3.$$

Exercise 2.17 (Cross I). For $a \in \mathbb{R}^3$, let $\ell_a(v) = a \cdot v = a^{\mathrm{tr}} v$, so that $\ell_a \in (\mathbb{R}^3)^*$. In particular we have $\varepsilon_i = \ell_{e_i}$ for $i \in [3]$ is the dual basis to the standard basis $\{e_i\}_{i=1}^3$. Show for $a, b \in \mathbb{R}^3$,

$$\ell_a \wedge \ell_b = i_{a \times b} [\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3] \tag{2.5}$$

Hints: 1) write $\ell_a = \sum_{i=1}^3 a_i \varepsilon_i$ and 2) make use of Eq. (**??**)

Exercise 2.18 (Cross II). Use Exercise 2.17 to prove the standard vector calculus identity;

$$(a \times b) \cdot (x \times y) = (a \cdot x)(b \cdot y) - (b \cdot x)(a \cdot y)$$

which is valid for all $a, b, x, y \in \mathbb{R}^3$. Hint: evaluate Eq. (2.5) at (x, y) while using Lemma **??**.

Exercise 2.19 (Surface Integrals). In this exercise, let $\omega \in \mathcal{A}_3(\mathbb{R}^3)$ be the standard volume form, $\omega(v_1, v_2, v_3) := \det[v_1|v_2|v_3]$, suppose D is an open subset of $\mathbb{R}^2$, and $\Sigma : D \to S \subset \mathbb{R}^3$ is a "parametrized surface," refer to Figure 2.1. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field on $\mathbb{R}^3$, then from your vector calculus class,
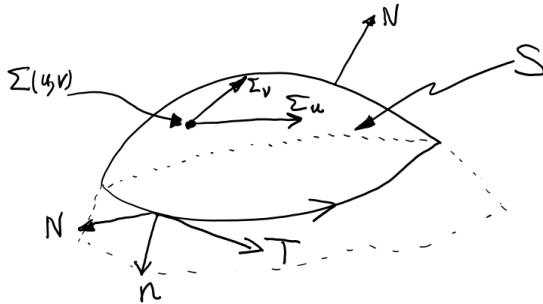
Fig. 2.1. In this figure N is a smoothly varying normal to S, n is a normal to the boundary of S, and T is a tangential vector to the boundary of S. Moreover, $D \ni (u, v) \to \Sigma(u, v) \in S$ is a parametrization of S where $D \subset \mathbb{R}^2$.

$$\iint_S F \cdot N dA = \varepsilon \cdot \iint_D F(\Sigma(u, v)) \cdot [\Sigma_u(u, v) \times \Sigma_v(u, v)] \, du dv \tag{2.6}$$

where $\varepsilon = 1$ ($\varepsilon = -1$) if $N(\Sigma(u, v))$ points in the same (opposite) direction as $\Sigma_u(u, v) \times \Sigma_v(u, v)$. We assume that ε is independent of $(u, v) \in D$.

Show the formula in Eq. (2.6) may be rewritten as

$$\iint_S F \cdot N dA = \varepsilon \iint_D \left(i_{F(\Sigma(u,v))}\omega\right)(\Sigma_u(u, v), \Sigma_v(u, v)) \, du dv \tag{2.7}$$

where

$$\varepsilon := \mathrm{sgn}(\omega(N \circ \Sigma, \Sigma_u, \Sigma_v)) = \begin{cases} 1 & \text{if } \omega(N \circ \Sigma, \Sigma_u, \Sigma_v) > 0 \\ -1 & \text{if } \omega(N \circ \Sigma, \Sigma_u, \Sigma_v) < 0. \end{cases}$$

Remarks: Once we introduce the proper notation, we will be able to write Eq. (2.7) more succinctly as

$$\iint_S F \cdot N dA = \iint_S i_F \omega := \varepsilon \iint_D \Sigma^*(i_F \omega).$$

Exercise 2.20 (Boundary Orientation). Referring to the set up in Exercise 2.19, the tangent vector T has been chosen by using the "right-hand" rule in order to determine the orientation on the boundary, ∂S, of S so that Stoke's theorem holds, i.e.

$$\iint_S [\nabla \times F] \cdot N dA = \int_{\partial S} F \cdot T ds. \tag{2.8}$$

Show by using the "right hand rule" that $T = c \cdot N \times n$ with $c > 0$ and then also show

$$c = \omega(N, n, T) = (i_n i_N \omega)(T).$$

Also note by Exercise 2.19, that Eq. (2.8) may be written as

$$\iint_S i_{\nabla \times F} \omega = \int_{\partial S} F \cdot T ds \tag{2.9}$$

Remark: We will introduce the "one form", $F \cdot dx$ and an "exterior derivative" operator, d, so that

$$d[F \cdot dx] = i_{\nabla \times F} \omega$$

and Eq. (2.9) may be written in the pleasant form,

$$\iint_S d[F \cdot dx] = \int_{\partial S} F \cdot dx.$$

Exercise 2.21. Let

$$f \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix} \text{ for } \begin{pmatrix} r \\ \theta \end{pmatrix} \in \mathbb{R}^2.$$

Find;

$$f' \begin{pmatrix} r \\ \theta \end{pmatrix} \text{ and } \det \left[f' \begin{pmatrix} r \\ \theta \end{pmatrix} \right].$$

Exercise 2.22. Let

$$f \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} = \begin{bmatrix} r \sin \varphi \cdot \cos \theta \\ r \sin \varphi \cdot \sin \theta \\ r \cos \varphi \end{bmatrix} \text{ for } \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \in \mathbb{R}^3.$$

Find;

$$f' \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \text{ and } \det \left[f' \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \right].$$

Exercise 2.23. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} = [a_1| \dots |a_n]$$

be an $n \times n$ matrix with i^{th}-column

$$a_i = \begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{ni} \end{bmatrix}.$$

Given another $n \times n$ matrix B with analogous notation, show

$$(\partial_B \det)(A) = \sum_{j=1}^{n} \det [a_1| \dots |a_{j-1}|b_j|a_{j+1}| \dots |b_n]. \qquad (2.10)$$

For example if $n = 3$, this formula reads,

$$(\partial_B \det)(A) = \det [b_1|a_2|a_3] + \det [a_1|b_2|a_3] + \det [a_1|a_2|b_3].$$

Suggestions; by definition,

$$(\partial_B \det)(A) := \frac{d}{dt}\Big|_0 \det (A + tB) = \frac{d}{dt}\Big|_0 \det [a_1 + tb_1| \dots |a_n + tb_n].$$

Now apply Lemma **??** with

$$f(x_1, \dots, x_n) = \det [a_1 + x_1 b_1| \dots |a_n + x_n b_n].$$

Exercise 2.24 (Exercise 2.23 continued). Continuing the notation and results from Exercise 2.23, show;

1. If $A = I$ is the $n \times n$ identity matrix in Eq. (2.10), then

$$(\partial_B \det)(I) = \text{tr}(B) = \sum_{j=1}^{n} B_{j,j}.$$

2. If A is an $n \times n$ invertible matrix, shows

$$(\partial_B \det)(A) = \det(A) \cdot \text{tr}(A^{-1}B).$$

Hint: Verify the identity,

$$\det(A + tB) = \det(A) \cdot \det(I + tA^{-1}B)$$

which you should then use along with first item of this exercise.

Exercise 2.25. Using Proposition **??**, find df when

$$f(x_1, x_2, x_3) = x_1^2 \sin(e^{x_2}) + \cos(x_3).$$