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Math 150B Differential Geometry

January 17, 2020 *File:150BNotes.tex*

Contents

Part

Homework Problems

Math 150B Homework Problems: Winter 2020

Problems are our text book, "Differential Forms," by Guillemin and Haine or from the lecture notes as indicated. The problems from the lecture notes are restated here.

0.1 Homework 0, Due Wednesday, January 8, 2020 (Not to be collected)

- Lecture note Exercises: 3.1, 3.2, 3.3, 3.4, and 3.5.

0.2 Homework 1. Due Thursday, January 16, 2020

- Lecture note Exercises: 3.6, 5.1, 5.2, 5.5
- Book Exercises: 1.2.vi.

0.3 Homework 2. Due Thursday, January 23, 2020

- Lecture note Exercises: 5.3, 5.4, 6.1, 6.2, 6.3, 6.4, 6.5
- Book Exercises: 1.3.iii., 1.3.v., 1.3.vii, 1.4ix

Part I

Background Material

1

Introduction

This class is devoted to understanding, proving, and exploring the multi-dimensional and "manifold" analogues of the classic one dimensional fundamental theorem of calculus and change of variables theorem. These theorems take on the following form;

$$\int_M d\omega = \int_{\partial M} \omega \quad \longleftrightarrow \quad \int_a^b g'(x)\, dx = g(x)|_a^b \text{ and} \tag{1.1}$$

$$\int_M f^* \omega = \deg(f) \cdot \int_N \omega \quad \longleftrightarrow \quad \int_a^b g(f(x))\, f'(x)\, dx = \int_{f(a)}^{f(b)} g(y)\, dy. \tag{1.2}$$

In meeting our goals we will need to understand all the ingredients in the above formula including;

1. M is a **manifold.**
2. ∂M is the **boundary of** M.
3. ω is a **differential form** an $d\omega$ is its **differential.**
4. $f^*\omega$ is the **pull back** of ω by a "smooth map" $f : M \to N$.
5. $\deg(f) \in \mathbb{Z}$ is the **degree** of f.
6. There is also a hidden notion of **orientation** needed to make sense of the above integrals.

Remark 1.1. We will see that Eq. (1.1) encodes (all wrapped into one neat formula) the key integration formulas from 20E: Green's theorem, Divergence theorem, and Stoke's theorem.

2

Permutations Basics

The following proposition should be verified by the reader.

Proposition 2.1 (Permutation Groups). *Let Λ be a set and*

$$\Sigma(\Lambda) := \{\sigma : \Lambda \to \Lambda |\ \sigma \text{ is bijective}\}.$$

If we equip G with the binary operation of function composition, then G is a group. The identity element in G is the identity function, ε, and the inverse, σ^{-1}, to $\sigma \in G$ is the inverse function to σ.

Definition 2.2 (Finite permutation groups). *For $n \in \mathbb{Z}_+$, let $[n] := \{1, 2, \ldots, n\}$, and $\Sigma_n := \Sigma([n])$ be the group described in Proposition 2.1. We will identify elements, $\sigma \in \Sigma_n$, with the following $2 \times n$ array,*

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}.$$

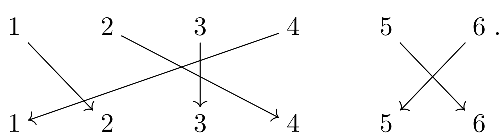
(Notice that $|\Sigma_n| = n!$ since there are n choices for $\sigma(1)$, $n-1$ for $\sigma(2)$, $n-2$ for $\sigma(3)$, ..., 1 for $\sigma(n)$.)

For examples, suppose that $n = 6$ and let

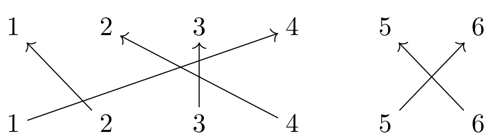
$$\varepsilon = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,2\,3\,4\,5\,6 \end{pmatrix} \text{ – the identity, and}$$

$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,4\,3\,1\,6\,5 \end{pmatrix}.$$

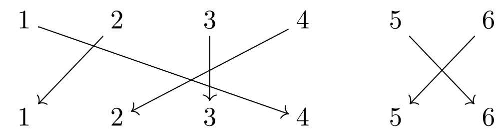
We identify σ with the following picture,



The inverse to σ is gotten pictorially by reversing all of the arrows above to find,



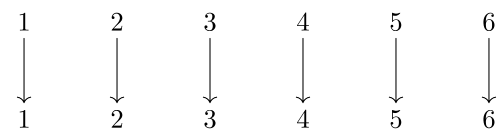
or equivalently,



and hence,

$$\sigma^{-1} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,1\,3\,2\,6\,5 \end{pmatrix}.$$

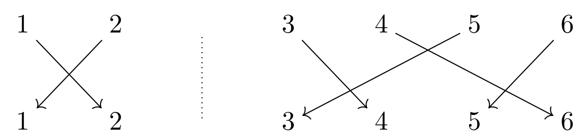
Of course the identity in this graphical picture is simply given by



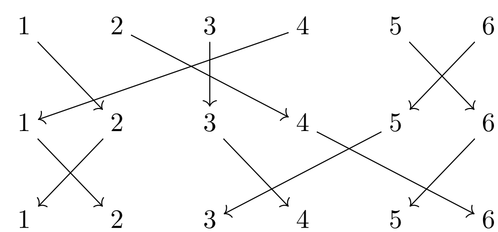
Now let $\beta \in S_6$ be given by

$$\beta = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,4\,6\,3\,5 \end{pmatrix},$$

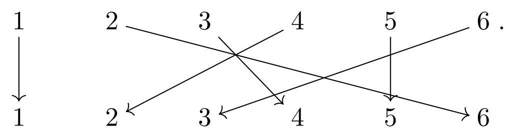
or in pictures;



We can now compose the two permutations $\beta \circ \sigma$ graphically to find,



which after erasing the intermediate arrows gives,



In terms of our array notation we have,

$$\begin{aligned}\beta\circ\sigma &= \begin{pmatrix}1\,2\,3\,4\,5\,6\\2\,1\,4\,6\,3\,5\end{pmatrix}\circ\begin{pmatrix}1\,2\,3\,4\,5\,6\\2\,4\,3\,1\,6\,5\end{pmatrix}\\ &= \begin{pmatrix}1\,2\,3\,4\,5\,6\\1\,6\,4\,2\,5\,3\end{pmatrix}.\end{aligned}$$

Remark 2.3 (Optional). It is interesting to observe that β splits into a product of two permutations,

$$\begin{aligned}\beta &= \begin{pmatrix}1\,2\,3\,4\,5\,6\\2\,1\,3\,4\,5\,6\end{pmatrix}\circ\begin{pmatrix}1\,2\,3\,4\,5\,6\\1\,2\,4\,6\,3\,5\end{pmatrix}\\ &= \begin{pmatrix}1\,2\,3\,4\,5\,6\\1\,2\,4\,6\,3\,5\end{pmatrix}\circ\begin{pmatrix}1\,2\,3\,4\,5\,6\\2\,1\,3\,4\,5\,6\end{pmatrix},\end{aligned}$$

corresponding to the non-crossing parts in the graphical picture for β. Each of these permutations is called a “cycle.”

Definition 2.4 (Transpositions). *A permutation, $\sigma\in\Sigma_k$, is a **transposition** if*

$$\#\{l\in[k]:\sigma(l)\neq l\}=2.$$

*We further say that σ is an **adjacent transposition** if*

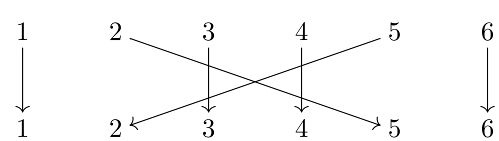
$$\{l\in[k]:\sigma(l)\neq l\}=\{i,i+1\}$$

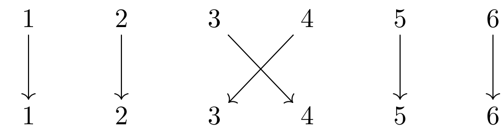
for some $1\le i<k$.

Example 2.5. If

$$\sigma=\begin{pmatrix}1\,2\,3\,4\,5\,6\\1\,5\,3\,4\,2\,6\end{pmatrix}\text{ and }\tau=\begin{pmatrix}1\,2\,3\,4\,5\,6\\1\,2\,4\,3\,5\,6\end{pmatrix}$$

then σ is a transposition and τ is an adjacent transposition. Here are the pictorial representation of σ and τ;





In terms of these pictures it is easy to recognize transpositions and adjacent transpositions.

Integration Theory Outline

In this course we are going to be considering integrals over open subsets of $\mathbb{R}^d$ and more generally over "manifolds." As the prerequisites for this class do not include real analysis, I will begin by summarizing a reasonable working knowledge of integration theory over $\mathbb{R}^d$. We will thus be neglecting some technical details involving measures and $\sigma-$ algebras. The knowledgeable reader should be able to fill in the missing hypothesis while the less knowledgeable readers should not be too harmed by the omissions to follow.

Definition 3.1. *The* ***indicator function of*** *a subset, $A \subset \mathbb{R}^d$, is defined by*

$$1_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Remark 3.2 (Optional). Every function, $f : \mathbb{R}^d \to \mathbb{R}$, may be approximated by a linear combination of indicator functions as follows. If $\varepsilon > 0$ is given we let

$$f_\varepsilon := \sum_{n \in \mathbb{N}} n\varepsilon \cdot 1_{\{n\varepsilon \le f < (n+1)\varepsilon\}}, \tag{3.1}$$

where $\{n\varepsilon \le f < (n+1)\varepsilon\}$ is shorthand for the set,

$$\left\{x \in \mathbb{R}^d : n\varepsilon \le f(x) < (n+1)\varepsilon\right\}.$$

We now summarize "modern" Lebesgue integration theory over $\mathbb{R}^d$.

1. For each d, there is a uniquely determined **volume measure,** m_d on all[1] subsets of $\mathbb{R}^d$ (subsets of $\mathbb{R}^d$) with the following properties;
   a) $m_d(A) \in [0, \infty]$ for all $A \subset \mathbb{R}^d$ with $m_d(\emptyset) = 0$.
   b) $m_d(A \cup B) = m_d(A) + m_d(B)$ is $A \cap B = \emptyset$. More generally, if $A_n \subset \mathbb{R}^d$ for all n with $A_n \cap A_m = \emptyset$ for $m \ne n$ we have
   $$m_d(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty m_d(A_n).$$
   c) $m_d(x + A) = m_d(A)$ for all $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, where
   $$x + A := \left\{x + y \in \mathbb{R}^d : y \in A\right\}.$$
   d) $m_d\left([0,1]^d\right) = 1$.
   [The reader is supposed to view $m_d(A)$ as the d-dimensional volume of a subset, $A \subset \mathbb{R}^d$.]
2. Associated to this volume measure is an integral which takes (not all) functions, $f : \mathbb{R}^d \to \mathbb{R}$, and assigns to them a number denoted by
   $$\int_{\mathbb{R}^d} f dm_d = \int_{\mathbb{R}^d} f(x)\, dm_d(x) \in \mathbb{R}.$$
   This integral has the following properties;
   a) When $d = 1$ and f is continuous function with compact support, $\int_{\mathbb{R}} f dm_1$ is the ordinary integral you studied in your first few calculus courses.
   b) The integral is defined for "all" $f \ge 0$ and in this case
   $$\int_{\mathbb{R}^d} f dm_d \in [0, \infty] \text{ and } \int_{\mathbb{R}^d} 1_A dm_d = m_d(A) \text{ for all } A \subset \mathbb{R}^d.$$
   c) The integral is **"positive" linear**, i.e. if $f, g \ge 0$ and $c \in [0, \infty)$, then
   $$\int_{\mathbb{R}^d} (f + cg)\, dm_d = \int_{\mathbb{R}^d} f dm_d + c \int_{\mathbb{R}^d} g dm_d. \tag{3.2}$$
   d) The integral is **monotonic, i.e.** if $0 \le f \le g$, then
   $$\int_{\mathbb{R}^d} f dm_d \le \int_{\mathbb{R}^d} g dm_d. \tag{3.3}$$
   e) Let $L^1(m_d)$ denote those functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} |f|\, dm_d < \infty$. Then for $f \in L^1(m_d)$ we define
   $$\int_{\mathbb{R}^d} f dm_d := \int_{\mathbb{R}^d} f_+ dm_d - \int_{\mathbb{R}^d} f_- dm_d$$
   where
   $$f_\pm(x) = \max(\pm f(x), 0) \text{ and so that } f(x) = f_+(x) - f_-(x).$$

[1] This is a lie! Nevertheless, for our purposes it will be reasonably safe to ignore this lie.

f) The integral, $L^1(m_d) \ni f \to \int_{\mathbb{R}^d} f dm_d$ is linear, i.e. Eq. (3.2) holds for all $f, g \in L^1(m_d)$ and $c \in \mathbb{R}$.
g) If $f, g \in L^1(m_d)$ and $f \le g$ then Eq. (3.3) still holds.

3. The integral enjoys the following continuity properties.
 a) **MCT**: the **monotone convergence theorem** holds; if $0 \le f_n \uparrow f$ then
$$\uparrow \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n dm_d = \int_{\mathbb{R}^d} f dm_d \text{ (with } \infty \text{ allowed as a possible value).}$$
Example 1: If $\{A_n\}_{n=1}^\infty$ is a sequence of subsets of $\mathbb{R}^d$ such that $A_n \uparrow A$ (i.e. $A_n \subset A_{n+1}$ for all n and $A = \cup_{n=1}^\infty A_n$), then
$$m_d(A_n) = \int_{\mathbb{R}^d} 1_{A_n} dm_d \uparrow \int_{\mathbb{R}^d} 1_A dm_d = m_d(A) \text{ as } n \to \infty$$
Example 2: If $g_n : \mathbb{R}^d \to [0, \infty]$ for $n \in \mathbb{N}$ then
$$\int_{\mathbb{R}^d} \sum_{n=1}^\infty g_n = \int_{\mathbb{R}^d} \lim_{N\to\infty} \sum_{n=1}^N g_n = \lim_{N\to\infty} \int_{\mathbb{R}^d} \sum_{n=1}^N g_n$$
$$= \lim_{N\to\infty} \sum_{n=1}^N \int_{\mathbb{R}^d} g_n = \sum_{n=1}^\infty \int_{\mathbb{R}^d} g_n.$$
 b) **DCT**: the **dominated convergence theorem** holds**,** if $f_n : \mathbb{R}^d \to \mathbb{R}$ are functions **dominating** by a function $G \in L^1(m_d)$ is the sense that $|f_n(x)| \le G(x)$ for all $x \in \mathbb{R}^d$. Then assuming that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. $x \in \mathbb{R}^d$, we may conclude that
$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n dm_d = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n dm_d = \int_{\mathbb{R}^d} f dm_d.$$
Example: If $\{g_n\}_{n=1}^\infty$ is a sequence of real valued random variables such that
$$\int_{\mathbb{R}^d} \sum_{n=1}^\infty |g_n| = \sum_{n=1}^\infty \int_{\mathbb{R}^d} |g_n| < \infty,$$
then; 1) $G := \sum_{n=1}^\infty |g_n| < \infty$ a.e. and hence $\sum_{n=1}^\infty g_n = \lim_{N\to\infty} \sum_{n=1}^N g_n$ exist a.e., 2) $\left|\sum_{n=1}^N g_n\right| \le G$ and $\int_{\mathbb{R}^d} G < \infty$, and so 3) by DCT,
$$\int_{\mathbb{R}^d} \sum_{n=1}^\infty g_n = \int_{\mathbb{R}^d} \lim_{N\to\infty} \sum_{n=1}^N g_n = \lim_{N\to\infty} \int_{\mathbb{R}^d} \sum_{n=1}^N g_n$$
$$= \lim_{N\to\infty} \sum_{n=1}^N \int_{\mathbb{R}^d} g_n = \sum_{n=1}^\infty \int_{\mathbb{R}^d} g_n.$$
 c) **Fatou's Lemma (*Optional):** if $0 \le f_n \le \infty$, then
$$\int_{\mathbb{R}^d} \left[\liminf_{n\to\infty} f_n\right] \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n dm_d.$$
This may be proved as an application of MCT.

4. **Tonelli's theorem;** if $f : \mathbb{R}^d \to [0, \infty]$, then for any $i \in [d]$,
$$\int_{\mathbb{R}^d} f dm_d = \int_{\mathbb{R}^{d-1}} \bar{f} dm_{d-1} \text{ where}$$
$$\bar{f}(x_1, \dots, \hat{x}_i, \dots x_d) := \int_{\mathbb{R}} f(x_1, \dots, x_i, \dots x_d) dx_i.$$
5. **Fubini's theorem;** if $f \in L^1(m_d)$ then the previous formula still hold.
6. For our purposes, by repeated use of use of items 4. and 5. we may compute $\int_{\mathbb{R}^d} f dm_d$ in terms of iterated integrals in any order we prefer. In more detail if $\sigma \in \Sigma_d$ is any permutation of $[d]$, then
$$\int_{\mathbb{R}^d} f dm_d = \int_{\mathbb{R}} dx_{\sigma(1)} \cdots \int_{\mathbb{R}} dx_{\sigma(d)} f(x_1, \dots, x_d)$$
provided either that $f \ge 0$ or
$$\int_{\mathbb{R}} dx_{\sigma(1)} \cdots \int_{\mathbb{R}} dx_{\sigma(d)} |f(x_1, \dots, x_d)| = \int_{\mathbb{R}^d} |f| dm_d < \infty.$$
This fact coupled with item 2a. will basically allow us to understand most integrals appearing in this course.

Notation 3.3 *For $A \subset \mathbb{R}^d$, we let*
$$\int_A f dm_d := \int_{\mathbb{R}^d} 1_A f \, dm_d.$$
Also when $d = 1$ and $-\infty \le s < t \le \infty$, we write
$$\int_s^t f dm_1 = \int_{(s,t)} f dm_1 = \int_{\mathbb{R}} 1_{(s,t)} f dm_1$$
and (as usual in Riemann integration theory)
$$\int_t^s f dm_1 := -\int_s^t f dm_1.$$

Example 3.4. Here is a MCT example,

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{1+t^2}dt &= \int_{-\infty}^{\infty}\lim_{n\to\infty}1_{[-n,n]}(t)\frac{1}{1+t^2}dt \\ &\overset{MCT}{=}\lim_{n\to\infty}\int_{-\infty}^{\infty}1_{[-n,n]}(t)\frac{1}{1+t^2}dt = \lim_{n\to\infty}\int_{-n}^{n}\frac{1}{1+t^2}dt \\ &= \lim_{n\to\infty}\left[\tan^{-1}(n)-\tan^{-1}(-n)\right] = \frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi.\end{aligned}$$

Example 3.5. Similarly for any $x>0$,

$$\begin{aligned}\int_0^{\infty}e^{-tx}dt &= \int_{-\infty}^{\infty}\lim_{n\to\infty}1_{[0,n]}(t)e^{-tx}dt \overset{MCT}{=}\lim_{n\to\infty}\int_{-\infty}^{\infty}1_{[0,n]}(t)e^{-tx}dt \\ &= \lim_{n\to\infty}\int_0^n e^{-tx}dt = \lim_{n\to\infty}\frac{-1}{x}e^{-tx}|_{t=0}^{n} = \frac{1}{x}.\end{aligned} \tag{3.4}$$

Example 3.6. Here is a DCT example,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\sin\left(\frac{t}{n}\right)dt = \int_{-\infty}^{\infty}\lim_{n\to\infty}\frac{1}{1+t^2}\sin\left(\frac{t}{n}\right)dt = \int_{\mathbb{R}}0dm = 0$$

since

$$\lim_{n\to\infty}\frac{1}{1+t^2}\sin\left(\frac{t}{n}\right) = 0 \text{ for all } t\in\mathbb{R}$$

and

$$\left|\frac{1}{1+t^2}\sin\left(\frac{t}{n}\right)\right| \le \frac{1}{1+t^2} \text{ with } \int_{\mathbb{R}}\frac{1}{1+t^2}dt<\infty.$$

Example 3.7. In this example we will show

$$\lim_{M\to\infty}\int_0^M\frac{\sin x}{x}dx = \pi/2 \tag{3.5}$$

Let us first note that $\left|\frac{\sin x}{x}\right|\le 1$ for all x and hence by DCT,

$$\int_0^M\frac{\sin x}{x}dx = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^M\frac{\sin x}{x}dx.$$

Moreover making use of Eq. (3.4), if $0<\varepsilon<M<\infty$, then by Fubini's theorem, DCT, and FTC (Fundamental Theorem of Calculus) that

$$\begin{aligned}\int_\varepsilon^M\frac{\sin x}{x}dx &= \int_\varepsilon^M\left[\lim_{N\to\infty}\int_0^N e^{-tx}\sin x\, dt\right]dx \quad \text{(DCT)} \\ &= \lim_{N\to\infty}\int_\varepsilon^M dx\int_0^N dte^{-tx}\sin x \quad \text{(DCT)} \\ &= \lim_{N\to\infty}\int_0^N dt\int_\varepsilon^M dxe^{-tx}\sin x \quad \text{(Fubini)} \\ &= \lim_{N\to\infty}\int_0^N dt\left[\frac{1}{1+t^2}(-\cos x - t\sin x)e^{-tx}\right]_{x=\varepsilon}^{x=M} \quad \text{(FTC)} \\ &= \int_0^\infty dt\left[\frac{1}{1+t^2}(-\cos x - t\sin x)e^{-tx}\right]_{x=\varepsilon}^{x=M}. \quad \text{(DCT)}\end{aligned}$$

Since

$$\left[\frac{1}{1+t^2}(-\cos x - t\sin x)e^{-tx}\right]_{x=\varepsilon}^{x=M} \to \frac{1}{1+t^2} \text{ as } M\uparrow\infty \text{ and } \varepsilon\downarrow 0,$$

we may again apply DCT with $G(t)=\frac{1}{1+t^2}$ being the dominating function in order to show

$$\begin{aligned}\int_0^M\frac{\sin x}{x}dx &= \lim_{\varepsilon\downarrow 0}\int_\varepsilon^M\frac{\sin x}{x}dx = \lim_{\varepsilon\downarrow 0}\int_0^\infty dt\left[\frac{1}{1+t^2}(-\cos x - t\sin x)e^{-tx}\right]_{x=\varepsilon}^{x=M} \\ &\overset{DCT}{=}\int_0^\infty dt\left[\frac{1}{1+t^2}(-\cos x - t\sin x)e^{-tx}\right]_{x=0}^{x=M} \\ &\underset{M\to\infty}{\overset{DCT}{\longrightarrow}}\int_0^\infty\frac{1}{1+t^2}dt = \frac{\pi}{2}.\end{aligned}$$

Theorem 3.8 (Linear Change of Variables Theorem). *If $T\in GL(d,\mathbb{R}) = GL(\mathbb{R}^d)$ – the space of $d\times d$ invertible matrices, then the change of variables formula,*

$$\int_{\mathbb{R}^d}fdm_d = |\det T|\int_{\mathbb{R}^d}f\circ T\ dm_d, \tag{3.6}$$

holds for all Riemann integrable functions $f:\mathbb{R}^d\to\mathbb{R}$.

Proof. From Exercise 3.6 below, we know that Eq. (3.6) is valid whenever T is an elementary matrix. From the elementary theory of row reduction in linear algebra, every matrix $T\in GL(\mathbb{R}^d)$ may be expressed as a finite product of the "elementary matrices", i.e. $T=T_1\circ T_2\circ\cdots\circ T_n$ where the T_i are elementary matrices. From these assertions we may conclude that

$$\int_{\mathbb{R}^d}f\circ T\ dm_d = \int_{\mathbb{R}^d}f\circ T_1\circ T_2\circ\cdots\circ T_n\ dm_d = \frac{1}{|\det T_n|}\int_{\mathbb{R}^d}f\circ T_1\circ T_2\circ\cdots\circ T_{n-1}\ dm_d.$$

Repeating this procedure $n-1$ more times (i.e. by induction), we find,

$$\int_{\mathbb{R}^d} f\circ T\ dm_d = \frac{1}{|\det T_n|\dots|\det T_1|}\int_{\mathbb{R}^d} f\ dm_d.$$

Finally we use,

$$|\det T_n|\dots|\det T_1| = |\det T_n \dots \det T_1| = |\det(T_1 T_2\dots T_n)| = |\det T|$$

in order to complete the proof. ∎

3.1 Exercises

Exercise 3.1. Find the value of the following integral;

$$I := \int_1^9 dy \int_{\sqrt{y}}^3 dx\ xe^y.$$

Hint: use Tonelli's theorem to change the order of integrations.

Exercise 3.2. Write the following iterated integral

$$I := \int_0^1 dx \int_{y=x^{2/3}}^1 dy\ xe^{y^4}.$$

as a multiple integral and use this to change the order of integrations and then compute I.

For the next three exercises let

$$B(0,r) := \left\{x\in\mathbb{R}^d : \|x\| = \sqrt{\sum_{i=1}^d x_i^2} < r\right\}$$

be the d – dimensional ball of radius r and let

$$V_d(r) := m_d(B(0,r)) = \int_{\mathbb{R}^d} 1_{B(0,r)} dm_d$$

be its volume. For example,

$$V_1(r) = m_1((-r,r)) = \int_{-r}^r dx = 2r.$$

Exercise 3.3. Suppose that $d=2$, show $m_2(B(0,r)) = \pi r^2$.

Exercise 3.4. Suppose that $d=3$, show $m_3(B(0,r)) = \frac{4\pi}{3}r^3$.

Exercise 3.5. Let $V_d(r) := m_d(B(0,r))$. Show for $d\geq 1$ that

$$V_{d+1}(r) = \int_{-r}^r dz\cdot V_d\left(\sqrt{r^2-z^2}\right) = r\int_{-\pi/2}^{\pi/2} V_d(r\cos\theta)\cos\theta d\theta.$$

Remark 3.9. Using Exercise 3.5 we may deduce again that

$$\begin{aligned}
V_1(r) &= m_1((-r,r)) = 2r,\\
V_2(r) &= r\int_{-\pi/2}^{\pi/2} 2r\cos\theta\cos\theta d\theta = \pi r^2,\\
V_3(r) &= \int_{-r}^r dz\cdot V_2\left(\sqrt{r^2-z^2}\right) = \int_{-r}^r dz\cdot\pi\left(r^2-z^2\right) = \frac{4\pi}{3}r^3.
\end{aligned}$$

In principle we may now compute the volume of balls in all dimensions inductively this way.

Exercise 3.6 (Change of variables for elementary matrices). Let $f:\mathbb{R}^d\to\mathbb{R}$ be a continuous function with compact support. Show by direct calculation that;

$$|\det T|\int_{\mathbb{R}^d} f(T(x))\,dx = \int_{\mathbb{R}^d} f(y)\,dy \tag{3.7}$$

for each of the following linear transformations;

1. Suppose that $i<k$ and
$$T(x_1,x_2\dots,x_d) = (x_1,\dots,x_{i-1},x_k,x_{i+1}\dots,x_{k-1},x_i,x_{k+1},\dots x_d),$$
i.e. T swaps the i and k coordinates of x. [In matrix notation T is the identity matrix with the i and k column interchanged.]
2. $T(x_1,\dots x_k,\dots,x_d) = (x_1,\dots,cx_k,\dots x_d)$ where $c\in\mathbb{R}\setminus\{0\}$. [In matrix notation, $T = [e_1|\dots|e_{k-1}|ce_k|e_{k+1}|\dots|e_d]$.]
3. $T(x_1,x_2\dots,x_d) = (x_1,\dots,\overset{\text{i'th spot}}{x_i+cx_k},\dots x_k,\dots x_d)$ where $c\in\mathbb{R}$. [In matrix notation $T = [e_1|\dots|e_i|\dots|e_k+ce_i|e_{k+1}|\dots|e_d]$.

Hint: you should use Fubini's theorem along with the one dimensional change of variables theorem.

[To be more concrete here are examples of each of the T appearing above in the special case $d=4$,

1. If $i=2$ and $k=3$ then $T = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$.

2. If $k=3$ then $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,
3. If $i=2$ and $k=4$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + cx_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

while if $i=4$ and $k=2$,

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + cx_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

3.2 *Appendix: Another approach to the linear change of variables theorem

Let $\langle x, y\rangle$ or $x \cdot y$ denote the standard dot product on $\mathbb{R}^d$, i.e.

$$\langle x, y\rangle = x \cdot y = \sum_{j=1}^{d} x_j y_j.$$

Recall that if A is a $d \times d$ real matrix then the transpose matrix, A^{tr}, may be characterized as the unique real $d \times d$ matrix such that

$$\langle Ax, y\rangle = \left\langle x, A^{\mathrm{tr}} y\right\rangle \text{ for all } x, y \in \mathbb{R}^d.$$

Definition 3.10. *A $d \times d$ real matrix, S, is orthogonal iff $S^{\mathrm{tr}}S = I$ or equivalently stated $S^{\mathrm{tr}} = S^{-1}$.*

Here are a few basic facts about orthogonal matrices.

1. A $d \times d$ real matrix, S, is orthogonal iff $\langle Sx, Sy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^d$.
2. If $\{u_j\}_{j=1}^d$ is any orthonormal basis for $\mathbb{R}^d$ and S is the $d \times d$ matrix determined by $Se_j = u_j$ for $1 \le j \le d$, then S is orthogonal.[2] Here is a proof for your convenience; if $x, y \in \mathbb{R}^d$, then

[2] This is a standard result from linear algebra often stated as a matrix, S, is orthogonal iff the columns of S form an orthonormal basis.

$$\left\langle x, S^{\mathrm{tr}} y\right\rangle = \langle Sx, y\rangle = \sum_{j=1}^{d} \langle x, e_j\rangle \langle Se_j, y\rangle = \sum_{j=1}^{d} \langle x, e_j\rangle \langle u_j, y\rangle$$
$$= \sum_{j=1}^{d} \left\langle x, S^{-1} u_j\right\rangle \langle u_j, y\rangle = \left\langle x, S^{-1} y\right\rangle$$

   from which it follows that $S^{\mathrm{tr}} = S^{-1}$.
3. If S is orthogonal, then $1 = \det I = \det\left(S^{\mathrm{tr}}S\right) = \det S^{\mathrm{tr}} \cdot \det S = (\det S)^2$ and hence $\det S = \pm 1$.

The following lemma is a special case the well known **singular value decomposition** or SVD for short..

Lemma 3.11 (SVD). *If T is a real $d \times d$ matrix, then there exists $D = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_d \ge 0$ and two orthogonal matrices R and S such that $T = RDS$. Further observe that $|\det T| = \det D = \lambda_1 \dots \lambda_d$.*

Proof. Since $T^{\mathrm{tr}}T$ is symmetric, by the spectral theorem there exists an orthonormal basis $\{u_j\}_{j=1}^d$ of $\mathbb{R}^d$ and $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_d \ge 0$ such that $T^{\mathrm{tr}}Tu_j = \lambda_j^2 u_j$ for all j. In particular we have

$$\langle Tu_j, Tu_k\rangle = \left\langle T^{\mathrm{tr}}Tu_j, u_k\right\rangle = \lambda_j^2 \delta_{jk} \ \forall\ 1 \le j, k \le d.$$

Case where $\det T \ne 0$. In this case $\lambda_1 \dots \lambda_d = \det T^{\mathrm{tr}}T = (\det T)^2 > 0$ and so $\lambda_d > 0$. It then follows that $\left\{v_j := \frac{1}{\lambda_j} Tu_j\right\}_{j=1}^d$ is an orthonormal basis for $\mathbb{R}^d$. Let us further let $D = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ (i.e. $De_j = \lambda_j e_j$ for $1 \le j \le d$) and R and S be the orthogonal matrices defined by

$$Re_j = v_j \text{ and } S^{\mathrm{tr}}e_j = S^{-1}e_j = u_j \text{ for all } 1 \le j \le d.$$

Combining these definitions with the identity, $Tu_j = \lambda_j v_j$, implies

$$TS^{-1}e_j = \lambda_j Re_j = R\lambda_j e_j = RDe_j \text{ for all } 1 \le j \le d,$$

i.e. $TS^{-1} = RD$ or equivalently $T = RDS$.

Case where $\det T = 0$. In this case there exists $1 \le k < d$ such that $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_k > 0 = \lambda_{k+1} = \dots = \lambda_d$. The only modification needed for the above proof is to define $v_j := \frac{1}{\lambda_j} Tu_j$ for $j \le k$ and then extend choose $v_{k+1}, \dots, v_d \in \mathbb{R}^d$ so that $\{v_j\}_{j=1}^d$ is an orthonormal basis for $\mathbb{R}^d$. We still have $Tu_j = \lambda_j v_j$ for all j and so the proof in the first case goes through without change. ∎

In the next theorem we will make use the characterization of m_d that it is the unique measure on $\left(\mathbb{R}^d\right)$ which is translation invariant assigns unit measure to $[0,1]^d$.

Theorem 3.12. *If T is a real $d \times d$ matrix, then $m_d \circ T = |\det T|\, m_d$.*

Proof. Recall that we know $m_d T = \delta(T)\, m_d$ for some $\delta(T) \in (0, \infty)$ and so we must show $\delta(T) = |\det T|$. We first consider two special cases.

1. If $T = R$ is orthogonal and B is the unit ball in $\mathbb{R}^d$,[3] then $\delta(R)\, m_d(B) = m_d(RB) = m_d(B)$ from which it follows $\delta(R) = 1 = |\det R|$.
2. If $T = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_i \geq 0$, then $D[0,1]^d = [0, \lambda_1] \times \cdots \times [0, \lambda_d]$ so that

$$\delta(D) = \delta(D)\, m_d\left([0,1]^d\right) = m_d\left(D[0,1]^d\right) = \lambda_1 \ldots \lambda_d = \det D.$$

3. For the general case we use singular value decomposition (Lemma 3.11) to write $T = RDS$ and then find

$$\delta(T) = \delta(R)\,\delta(D)\,\delta(S) = 1 \cdot \det D \cdot 1 = |\det T|.$$

■

[3] $B = \left\{x \in \mathbb{R}^d : \|x\| < 1\right\}.$

Part II

Multi-Linear Algebra

4

Properties of Volumes

The goal of this short chapter is to show how computing volumes naturally gives rise to the idea of the key objects of this book, namely differential forms, i.e. alternating tensors. The point is that these objects are intimately related to computing areas and volumes.

Let $Q^n := \{x \in \mathbb{R}^n : 0 \le t_j \le 1 \ \forall\ j\} = [0,1]^n$ be the unit cube in $\mathbb{R}^n$ which we I think all agree should have volume equal to 1. For n-vectors, $a_1, \dots, a_n \in \mathbb{R}^n$, let

$$P(a_1, \dots, a_n) = [a_1| \dots |a_n]\, Q = \left\{ \sum_{j=1}^{n} t_j a_j : 0 \le t_j \le 1 \ \forall\ j \right\}$$

be the parallelepiped spanned by $(a_1, \dots, a_n)$ and let

$$\delta(a_1, \dots, a_n) = \text{“signed”}\ \ \mathrm{Vol}\,(P(v_1, \dots, v_n)).$$

be the **signed volume** of the parallelepiped. To find the properties of this volume, let us fix $\{a_i\}_{i=1}^{n-1}$ and consider the function, $F(a_n) = \delta(a_1, \dots, a_n)$. This is easily computed using the formula of a slant cylinder, see Figure 4.1, as

$$F(a_n) = \delta(a_1, \dots, a_n) = \pm\,(\text{Area of base })\cdot \mathbf{n}\cdot a_n \tag{4.1}$$

where $\mathbf{n}$ is a unit vector orthogonal to $\{a_1, \dots, a_{n-1}\}$.

Example 4.1. When $n = 2$, let us first verify Eq. (4.1) in this case by considering

$$\delta(ae_1, b) = \int_0^{b_2} [\text{slice width}]_h\, dh = \int_0^{b_2} a\,dh = a\,(b \cdot e_2).$$

The sign in Eq. (4.1) is positive if $(a_1, \dots, a_{n-1}, \mathbf{n})$ is “positively oriented,” think of the right hand rule in dimensions 2 and 3. This show $a_n \to \delta(a_1, \dots, a_{n-1}, a_n)$ is a linear function. A similar argument shows

$$a_j \to \delta(a_1, \dots, a_j, \dots, a_n)$$

is linear as well. That is δ is a “**multi-linear function**” of its arguments. We further have that $\delta(a_1, \dots, a_n) = 0$ if $a_i = a_j$ for any $i \ne j$ as the parallelepiped generated by $(a_1, \dots, a_n)$ is degenerate and zero volume. We summarize these two properties by saying δ is an **alternating multi-linear n-function** on $\mathbb{R}^n$. Lastly as $P(e_1, \dots e_n) = Q$ we further have that

Fig. 4.1. The volume of a slant cylinder is it’s height, $\mathbf{n} \cdot a_n$.

$$\delta(e_1, \dots, e_n) = 1. \tag{4.2}$$

Fact 4.2 *We are going to show there is precisely one alternating multi-linear n-function, δ, on $\mathbb{R}^n$ such that Eq. (4.2) holds. This function is in fact the function you know and the determinant.*

Example 4.3 ($n = 1$ Det). When $n = 1$ we must have $\delta([a]) = \pm a$, we choose a by convention.

Example 4.4 ($n = 2$ Det). When $n = 2$, we find

$$\begin{aligned}
\delta(a, b) &= \delta(a_1e_1 + a_2e_2, b) = a_1\delta(e_1, b) + a_2\delta(e_2, b)\\
&= a_1\delta(e_1, b_1e_1 + b_2e_2) + a_2\delta(e_2, b_1e_1 + b_2e_2)\\
&= a_1b_2\delta(e_1, e_2) + a_2b_1\delta(e_2, e_1) = a_1b_2 - a_2b_1\\
&= \det[a|b].
\end{aligned}$$

We now proceed to develop the theory of alternating multilinear functions in general.

Multi-linear Functions (Tensors)

For the rest of these notes, V will denote a real vector space. Typically we will assume that $n = \dim V < \infty$.

Example 5.1. $V = \mathbb{R}^n$, subspaces of $\mathbb{R}^n$, polynomials of degree $< n$. The most general overarching vector space is typically

$$V = \mathcal{F}(X, \mathbb{R}) = \{\text{all functions from } X \text{ to } \mathbb{R}\}.$$

An interesting subspace is the space of **finitely supported functions,**

$$\mathcal{F}_f(X, \mathbb{R}) = \{f \in \mathcal{F}(X, \mathbb{R}) : \#(\{f \neq 0\}) < \infty\},$$

where

$$\{f \neq 0\} = \{x \in X : f(x) \neq 0\}.$$

5.1 Basis and Dual Basis

Definition 5.2. *Let V^* denote the **dual space** of V, i.e. the vector space of all linear functions, $\ell : V \to \mathbb{R}$.*

Example 5.3. Here are some examples;

1. If $V = \mathbb{R}^n$, then $\ell(v) = w \cdot v = w^{\mathrm{tr}} v$ for $w \in V$ is in V^*.
2. V = polynomials of deg $< n$ is a vector space and $\ell_0(p) = p(0)$ or $\ell(p) = \int_{-1}^{1} p(x)\, dx$ given $\ell \in V^*$.
3. For $\{a_j\}_{j=1}^p \subset \mathbb{R}$ and $\{x_j\}_{j=1}^p \subset X$, let $\ell(f) = \sum_{j=1}^p a_j f(x_j)$, then $\ell \in \mathcal{F}(X, \mathbb{R})^*$.

Notation 5.4 *Let $\beta := \{e_j\}_{j=1}^n$ be a basis for V and $\beta^* := \{\varepsilon_j\}_{j=1}^n$ be its **dual basis**, i.e.*

$$\varepsilon_j\left(\sum_{i=1}^n a_i e_i\right) := a_j \text{ for all } j.$$

The book denote ε_j as e_j^*. In case, $V = \mathbb{R}^n$ and $\{e_j\}_{j=1}^n$ is the standard basis, we may write dx_j for $\varepsilon_j = e_j^*$.

Example 5.5. If $V = \mathbb{R}^n$ and $\beta = \{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$, then $\varepsilon_i(v) = e_i \cdot v = e_i^{\mathrm{tr}} v$ for $1 \leq i \leq n$ is the dual basis to β.

Example 5.6. If V denotes polynomials of degree $< n$, with basis $e_j(x) = x^j$ for $0 \leq j < n$, then $\varepsilon_j(p) := \frac{1}{j!} p^{(j)}(0)$ is the associated dual basis.

Example 5.7. For $x \in X$, let $\delta_x \in \mathcal{F}_f(X, \mathbb{R})$ be defined by

$$\delta_x(y) = 1_{\{x\}}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}.$$

One may easily show that $\{\delta_x\}_{x \in X}$ is a basis for $\mathcal{F}_f(X, \mathbb{R})$ and for $f \in \mathcal{F}_f(X, \mathbb{R})$,

$$f = \sum_{x : f(x) \neq 0} f(x)\, \delta_x.$$

The dual basis ideas are complicated in this case when X is an infinite set as Vaki mentioned in section. We will not consider such "infinite dimensional" problems in these notes.

Proposition 5.8. *Continuing the notation above, then*

$$v = \sum_{j=1}^n \varepsilon_j(v)\, e_j \quad \text{for all } v \in V, \text{ and} \tag{5.1}$$

$$\ell = \sum_{j=1}^n \ell(e_j)\, \varepsilon_j \quad \text{for all } \ell \in V^*. \tag{5.2}$$

Moreover, β^*, is indeed a basis for V^*.

Proof. Because $\{e_j\}$ is a basis, we know that $v = \sum_{j=1}^n a_j e_j$. Applying ε_k to this formula shows

$$\varepsilon_k(v) = \sum_{j=1}^n a_j \varepsilon_k(e_j) = a_k$$

and hence Eq. (5.1) holds. Now apply ℓ to Eq. (5.1) to find,

$$\ell(v) = \sum_{j=1}^{n} \varepsilon_j(v)\,\ell(e_j) = \sum_{j=1}^{n} \ell(e_j)\,\varepsilon_j(v) = \left(\sum_{j=1}^{n} \ell(e_j)\,\varepsilon_j\right)(v)$$

which proves Eq. (5.2). From Eq. (5.2) we know that $\{\varepsilon_j\}_{j=1}^n$ spans V^*. Moreover if

$$\mathbf{0} = \sum_{j=1}^{n} a_j \varepsilon_j \implies 0 = \mathbf{0}(e_k) = \sum_{j=1}^{n} a_j \varepsilon_j(e_k) = a_k$$

which shows $\{\varepsilon_j\}_{j=1}^n$ is linearly independent. ■

Exercise 5.1. Let $V = \mathbb{R}^n$ and $\beta = \{u_j\}_{j=1}^n$ be a basis for $\mathbb{R}^n$. Recall that every $\ell \in (\mathbb{R}^n)^*$ is of the form $\ell_a(x) = a \cdot x$ for some $a \in \mathbb{R}^n$. Thus the dual basis, β^*, to β can be written as $\left\{u_j^* = \ell_{a_j}\right\}_{j=1}^n$ for some $\{a_j\}_{j=1}^n \subset \mathbb{R}^n$. In this problem you are asked to show how to find the $\{a_j\}_{j=1}^n$ by the following steps.

1. Show that for $j \in [n]$, a_j must solve the following k-linear equations;
$$\delta_{j,k} = \ell_{a_j}(u_k) = a_j \cdot u_k = u_k^{\mathrm{tr}} a_j \text{ for } k \in [n]. \tag{5.3}$$
2. Let $U := [u_1| \dots |u_n]$ (i.e. the columns of U are the vectors from β). Show that the equations in (5.3) may be written in matrix form as, $U^{\mathrm{tr}} a_j = e_j$, where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.
3. Conclude that $a_j = [U^{\mathrm{tr}}]^{-1} e_j$ or equivalently;
$$[a_1| \dots |a_n] = \left[U^{\mathrm{tr}}\right]^{-1}$$

Exercise 5.2. Let $V = \mathbb{R}^2$ and $\beta = \{u_1, u_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find $a_1, a_2 \in \mathbb{R}^2$ explicitly so that explicitly the dual basis $\beta^* := \{u_1^* = \ell_{a_1}, u_2^* = \ell_{a_2}\}$ is the dual basis to β. Please explicitly verify your answer is correct by showing $u_j^*(u_k) = \delta_{jk}$.

Exercise 5.3. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. Show $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set if and only if $\{a_j\}_{j=1}^k \subset V$ is a linearly independent set.

Exercise 5.4. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. If $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set, show there exists $\{u_j\}_{j=1}^k \subset V$ so that $\ell_i(u_j) = \delta_{ij}$ for $i, j \in [k]$. Here is a possible outline.

1. Using Exercise 5.3 and citing a basic fact from Linear algebra, you may choose $\{a_j\}_{j=k+1}^n \subset V$ so that $\{a_j\}_{j=1}^n$ is a basis for V.
2. Argue that it suffices to find $u_j \in V$ so that
$$a_i \cdot u_j = \delta_{ij} \text{ for all } i, j \in [n]. \tag{5.4}$$
3. Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$ and $A := [a_1| \dots |a_n]$ be the $n \times n$ matrix with columns given by that $\{a_j\}_{j=1}^n$. Show that the Eqs. (5.4) may be written as
$$A^{\mathrm{tr}} u_j = e_j \text{ for } j \in [n]. \tag{5.5}$$
4. Cite basic facts from linear algebra to explain why $A := [a_1| \dots |a_n]$ and A^{tr} are both invertible $n \times n$ matrices.
5. Argue that Eq. (5.5) has a unique solution, $u_j \in \mathbb{R}^n$, for each j.

5.2 Multi-linear Forms

Definition 5.9. *A function $T : V^k \to \mathbb{R}$ is **multi-linear** (k-linear to be precise) if for each $1 \le i \le k$, the map*

$$V \ni v_i \to T(v_1, \dots, v_i, \dots v_k) \in \mathbb{R}$$

*is linear. We denote the space of k-linear maps by $\mathcal{L}^k(V)$ and element of this space is a k**-tensor on (in)** V.*

Lemma 5.10. *Note that $\mathcal{L}^k(V)$ is a **vector subspace** of all functions from $V^k \to \mathbb{R}$.*

Example 5.11. If $\ell_1, \dots, \ell_k \in V^*$, we let $\ell_1 \otimes \dots \otimes \ell_k \in \mathcal{L}^k(V)$ be defined

$$(\ell_1 \otimes \dots \otimes \ell_k)(v_1, \dots, v_k) = \prod_{j=1}^{k} \ell_j(v_j)$$

for all $(v_1, \dots, v_k) \in V^k$.

Exercise 5.5. In this problem, let

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ and } w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Which of the following functions formulas for T define a 2-tensors on $\mathbb{R}^3$. Please justify your answers.

1. $T(v,w) = v_1w_3 + v_1w_2 + v_2w_1 + 7v_1w_1$.
2. $T(v,w) = v_1 + 7v_1 + v_2$.
3. $T(v,w) = v_1^2w_3 + v_2w_1$,
4. $T(v,w) = \sin(v_1w_3 + v_1w_2 + v_2w_1 + 7v_1w_1)$.

Theorem 5.12. *If $\{e_j\}_{j=1}^n$ is a basis for V, then $\{\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} : j_i \in [n]\}$ is a basis for $\mathcal{L}^k(V)$ and moreover if $T \in \mathcal{L}^k(V)$, then*

$$T = \sum_{j_1,\dots,j_k \in [n]} T(e_{j_1},\dots,e_{j_k}) \cdot \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} \tag{5.6}$$

and this decomposition is unique. [One might identify 2-tensors with matrices via $T \to A_{ij} := T(e_i,e_j)$.]

Proof. Given $v_1,\dots,v_k \in V$, we know that

$$v_i = \sum_{j_i=1}^n \varepsilon_{j_i}(v_i)\, e_{j_i}$$

and hence

$$\begin{aligned} T(v_1,\dots,v_k) &= T\left(\sum_{j_1=1}^n \varepsilon_{j_1}(v_1)\, e_{j_1},\dots,\sum_{j_k=1}^n \varepsilon_{j_k}(v_k)\, e_{j_k}\right) \\ &= \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n T(\varepsilon_{j_1}(v_1)\, e_{j_1},\dots,\varepsilon_{j_k}(v_k)\, e_{j_k}) \\ &= \sum_{j_1,\dots,j_k\in[n]} T(e_{j_1},\dots,e_{j_k})\, \varepsilon_{j_1}(v_1)\dots\varepsilon_{j_k}(v_k) \\ &= \sum_{j_1,\dots,j_k\in[n]} T(e_{j_1},\dots,e_{j_k})\, \varepsilon_{j_1}\otimes\cdots\otimes\varepsilon_{j_k}(v_1,\dots,v_k). \end{aligned}$$

This verifies that Eq. (5.6) holds and also that

$$\{\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} : j_i \in [n]\} \text{ spans } \mathcal{L}^k(V).$$

For linearly independence, if $\{a_{j_1,\dots,j_k}\} \subset \mathbb{R}$ are such that

$$\mathbf{0} = \sum_{j_1,\dots,j_k\in[n]} a_{j_1,\dots,j_k} \cdot \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k},$$

then evaluating this expression at $(e_{i_1},\dots e_{i_k})$ shows

$$\begin{aligned} 0 = \mathbf{0}(e_{i_1},\dots e_{i_k}) &= \sum_{j_1,\dots,j_k\in[n]} a_{j_1,\dots,j_k} \cdot \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}(e_{i_1},\dots e_{i_k}) \\ &= \sum_{j_1,\dots,j_k\in[n]} a_{j_1,\dots,j_k} \cdot \varepsilon_{j_1}(e_{i_1})\dots\varepsilon_{j_k}(e_{i_k}) \\ &= \sum_{j_1,\dots,j_k\in[n]} a_{j_1,\dots,j_k} \cdot \delta_{j_1,i_1}\dots\delta_{j_k,i_k} = a_{i_1\dots,i_k} \end{aligned}$$

which shows $a_{i_1\dots,i_k} = 0$ for all indices and completes the proof. ∎

Corollary 5.13. $\dim \mathcal{L}^k(V) = n^k$.

Definition 5.14. *If $S \in \mathcal{L}^p(V)$ and $T \in \mathcal{L}^q(V)$, then we define $S \otimes T \in \mathcal{L}^{p+q}(V)$ by,*

$$S \otimes T(v_1,\dots,v_p,w_1,\dots,w_q) = S(v_1,\dots,v_p)\, T(w_1,\dots,w_q).$$

Definition 5.15. *If $A : V \to W$ is a linear transformation, and $T \in \mathcal{L}^k(W)$, then we define the **pull back** $A^*T \in \mathcal{L}^k(V)$ by*

$$(A^*T)(v_1,\dots,v_k) = A(Tv_1,\dots,Tv_k).$$

$$\begin{array}{ccccc} V \times \cdots \times V & \longrightarrow & W \times \cdots \times W & \longrightarrow & \mathbb{R} \\ (v_1,\dots,v_k) & \longrightarrow & (Av_1,\dots,Av_k) & \longrightarrow & T(Av_1,\dots,Av_k). \end{array}$$

It is called pull back since $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ maps the opposite direction of A.

Remark 5.16. As shown in the book the tensor product satisfies

$$\begin{aligned} (R \otimes S) \otimes T &= R \otimes (S \otimes T), \\ T \otimes (S_1 + S_2) &= T \otimes S_1 + T \otimes S_2, \\ (S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T, \\ &\vdots \end{aligned}$$

Remark 5.17. The definition of $T_1 \otimes T_2$ and the associated "tensor algebra." [Typically the tensor symbol, $\otimes$, in mathematics is used to denote the product of two functions which have distinct arguments. Thus if $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are two functions on the sets X and Y respectively, then $f \otimes g : X \times Y \to \mathbb{R}$ is defined by

$$(f \otimes g)(x,y) = f(x)\, g(y).$$

In contrast, if $Y = X$ we may also define the more familiar product, $f \cdot g : X \to \mathbb{R}$, by

$$(f \cdot g)(x) = f(x)\, g(x).$$

Incidentally, the relationship between these two products is

$$(f \cdot g)(x) = (f \otimes g)(x,x).$$

Lemma 5.18. *The product, $\otimes$, defined in the previous remark is associative and distributive over addition. We also have for $\lambda \in \mathbb{R}$, that*

$$(\lambda f) \otimes g = f \otimes (\lambda g) = \lambda \cdot f \otimes g. \tag{5.7}$$

That is $\otimes$ satisfies the rules we expect of a "product," i.e. plays nicely with the vector space operations.

Proof. If $h : Z \to \mathbb{R}$ is another function, then

$$\begin{aligned}((f \otimes g) \otimes h)(x, y, z) &= (f \otimes g)(x, y) \cdot h(z) = (f(x) g(y)) h(z) \\ &= f(x)(g(y) h(z)) = (f \otimes (g \otimes h))(x, y, z).\end{aligned}$$

This shows in general that $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. $\otimes$ is associative.

Similarly if $Z = Y$, then

$$\begin{aligned}(f \otimes (g + h))(x, y) &= f(x) \cdot (g + h)(y) = f(x) \cdot (g(y) + h(y)) \\ &= f(x) \cdot g(y) + f(x) \cdot h(y) \\ &= (f \otimes g)(x, y) + (f \otimes h)(x, y) \\ &= (f \otimes g + f \otimes h)(x, y)\end{aligned}$$

from which we conclude that

$$f \otimes (g + h) = f \otimes g + f \otimes h$$

Similarly one shows $(f + h) \otimes g = f \otimes g + h \otimes g$ when $Z = X$. These are the distributive rules. The easy proof of Eq. (5.7) is left to the reader. ■

Alternating Multi-linear Functions

Definition 6.1. *$T \in \mathcal{L}^k(V)$ is said to be* ***alternating*** *if $T(v_1, \ldots, v_k) = -T(w_1, \ldots, w_k)$ whenever $(w_1, \ldots, w_k)$ is the list $(v_1, \ldots, v_k)$ with any two entries interchanged. We denote the subspace[1] of alternating functions by $\mathcal{A}^k(V)$ or by $\Lambda^k(V^*)$ with the convention that $\mathcal{A}^0(V) = \Lambda^0(V^*) = \mathbb{R}$. An element, $T \in \mathcal{A}^k(V) = \Lambda^k(V^*)$ will be called a k-**form.***

Remark 6.2. If $f(v, w)$ is a multi-linear function such that $f(v, v) = 0$ then for all $v, w \in V$, then

$$\begin{aligned} 0 &= f(v+w, v+w) = f(v,v) + f(w,w) + f(v,w) + f(w,v) \\ &= f(w,v) + f(v,w) \implies f(v,w) = -f(w,v). \end{aligned}$$

Conversely, if $f(v, w) = -f(w, v)$ for all v, and w, then $f(v, v) = -f(v, v)$ which shows $f(v, v) = 0$.

Lemma 6.3. *If $T \in \mathcal{L}^k(V)$, then the following are equivalent;*

1. *T is alternating, i.e. $T \in \Lambda^k(V^*)$.*
2. *$T(v_1, \ldots, v_k) = 0$ whenever any two distinct entries are equal.*
3. *$T(v_1, \ldots, v_k) = 0$ whenever any two consecutive entries are equal.*

Proof. 1. $\implies$ 2. If $v_i = v_j$ for some $i < j$ and $T \in \Lambda^k(V^*)$, then by interchanging the i and j entries we learn that $T(v_1, \ldots, v_k) = -T(v_1, \ldots, v_k)$ which implies $T(v_1, \ldots, v_k) = 0$.

2. $\implies$ 3. This is obvious.

3. $\implies$ 1. Applying Remark 6.2 with

$$f(v, w) := T(v_1, \ldots, v_{j-1}, v, w, v_{j+2}, \ldots, v_k)$$

shows that $T(v_1, \ldots, v_k) = -T(w_1, \ldots, w_k)$ if $(w_1, \ldots, w_k)$ is the list $(v_1, \ldots, v_k)$ with the j and $j+1$ entries interchanged. If $(w_1, \ldots, w_k)$ is the list $(v_1, \ldots, v_k)$ with the $i < j$ entries interchanged, then $(w_1, \ldots, w_k)$ can be transformed back to $(v_1, \ldots, v_k)$ by an odd number of nearest neighbor interchanges and therefore it follows by what we just proved that

$$T(v_1, \ldots, v_k) = -T(w_1, \ldots, w_k).$$

For example, to transform

$$(v_1, v_5, v_3, v_4, v_2, v_6) \text{ back to } (v_1, v_2, v_3, v_4, v_5, v_6),$$

we transpose v_5 with its nearest neighbor to the right 2 times to arrive at the list $(v_1, v_3, v_4, v_5, v_2, v_6)$. We then we transpose v_2 with its nearest neighbor to the left 3 times to arrive (after a sum total of 5 adjacent transpositions) back to the list $(v_1, v_2, v_3, v_4, v_5, v_6)$. For the general $i < j$ the number of adjacent transposition needed needed is $2(j-i) - 1$ which is always odd. ■

Exercise 6.1. If $T \in \Lambda^k(V^*)$, show $T(v_1, \ldots, v_k) = 0$ whenever $\{v_i\}_{i=1}^k \subset V$ are linearly dependent.

A simple consequence of this exercise is the following basic lemma.

Lemma 6.4. *If $T \in \Lambda^k(V^*)$ with $k > \dim V$, then $T \equiv 0$, i.e. $\Lambda^k(V^*) = \{0\}$ for all $k > \dim V$.*

At this point we have not given any non-zero examples of alternating forms. The next definition and proposition gives a mechanism for constructing many (in fact a full basis of) alternating forms.

Definition 6.5. *For $\ell \in V^*$ and $\varphi \in \Lambda^k(V^*)$, let $L_\ell \varphi$ be the multi-linear $k+1$ – form on V defined by*

$$(L_\ell \varphi)(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \ell(v_i) \varphi(v_0, \ldots, \hat{v}_i, \ldots, v_k).$$

for all $(v_0, \ldots, v_k) \in V^{k+1}$.

Proposition 6.6. *If $\ell \in V^*$ and $\varphi \in \Lambda^k(V^*)$, then $(L_\ell \varphi) \in \Lambda^{k+1}(V^*)$.*

Proof. We must show $L_\ell \varphi$ is alternating. According to Lemma 6.3, it suffices to show $(L_\ell \varphi)(v_0, \ldots, v_k) = 0$ whenever $v_j = v_{j+1}$ for some $0 \le j < k$. So suppose that $v_j = v_{j+1}$, then since φ is alternating

[1] The alternating conditions are linear equations that $T \in \mathcal{L}^k(V)$ must satisfy and hence $\mathcal{A}^k(V)$ is a subspace of $\mathcal{L}^k(V)$.

$$\begin{aligned}(L_\ell\varphi)(v_0,\dots,v_k) &= \sum_{i=0}^{k}(-1)^i\,\ell(v_i)\,\varphi(v_0,\dots,\hat{v}_i,\dots,v_k)\\ &= \sum_{i=j}^{j+1}(-1)^i\,\ell(v_i)\,\varphi(v_0,\dots,\hat{v}_i,\dots,v_k)\\ &= \left[(-1)^j+(-1)^{j+1}\right]\ell(v_j)\,\varphi(v_0,\dots,\hat{v}_j,\dots,v_k)=0.\end{aligned}$$

■

Proposition 6.7. *Let $\{e_i\}_{i=1}^n$ be a basis for V and $\{\varepsilon_i\}_{i=1}^n$ be its dual basis for V^*. Then*

$$\varphi_j := L_{\varepsilon_j}L_{\varepsilon_{j+1}}\dots L_{\varepsilon_{n-1}}\varepsilon_n \in \Lambda^{n-j+1}(V^*)\setminus\{0\}$$

for all $j\in[n]$ and in particular, $\dim \Lambda^k(V^*)\ge 1$ for all $0\le k\le n$. [We will see in Theorem 6.30 below that $\dim \Lambda^k(V^*)=\binom{n}{k}$ for all $0\le k\le n$.]

Proof. We will show that φ_j is not zero by showing that

$$\varphi_j(e_j,\dots,e_n)=1 \text{ for all } j\in[n].$$

This is easily proved by (reverse induction) on j. Indeed, for $j=n$ we have $\varphi_n(e_n)=\varepsilon_n(e_n)=1$ and for $1\le j<n$ we have $\varphi_j := L_{\varepsilon_j}\varphi_{j+1}$ so that

$$\begin{aligned}\varphi_j(e_j,\dots,e_n) &= \sum_{k=j}^{n}(-1)^{k-j}\,\varepsilon_j(e_k)\,\varphi_{j+1}(e_j,\dots,\widehat{e_k},\dots,e_n)\\ &= \varphi_{j+1}(\widehat{e_j},e_{j+1},\dots,e_n)=\varphi_{j+1}(e_{j+1},\dots,e_n)=1\end{aligned}$$

wherein we used the induction hypothesis for the last equality. This completes the proof for $j\in[n]$. Finally for $k=0$, we have $\Lambda^0(V^*)=\mathbb{R}$ by convention and hence $\dim\Lambda^0(V^*)=1$. ■

Notation 6.8 *Fix a basis $\{e_i\}_{i=1}^n$ of V with dual basis, $\{\varepsilon_i\}_{i=1}^n\subset V^*$, and then let*

$$\varphi=\varphi_1=L_{\varepsilon_1}L_{\varepsilon_2}\dots L_{\varepsilon_{n-1}}\varepsilon_n. \tag{6.1}$$

Definition 6.9 (Signature of σ). *For $\sigma\in\Sigma_n$, let*

$$(-1)^\sigma := \varphi(e_{\sigma 1},\dots,e_{\sigma n}),$$

*where φ is as in Notation 6.8. We call $(-1)^\sigma$ the **sign of the permutation**, σ.*

Lemma 6.10. *If $\sigma\in\Sigma_n$, then $(-1)^\sigma$ may be computed as $(-1)^N$ where N is the number of transpositions[2] needed to bring $(\sigma 1,\dots,\sigma n)$ back to $(1,2,\dots,n)$ and so $(-1)^\sigma$ does not depend on the choices made in defining $(-1)^\sigma$. Moreover, if $\{v_j\}_{j=1}^n\subset V$, then*

$$\varphi(v_{\sigma 1},\dots,v_{\sigma n})=(-1)^\sigma\varphi(v_1,\dots,v_d)\ \forall\ \sigma\in\Sigma_n.$$

Proof. Straightforward and left to the reader. ■

Corollary 6.11. *If $\sigma\in\Sigma_n$ is a transposition, then $(-1)^\sigma=-1$.*

Proof. This has already been proved in the course of proving Lemma 6.3. ■

Lemma 6.12. *If $\sigma,\tau\in\Sigma_n$, then $(-1)^{\sigma\tau}=(-1)^\sigma(-1)^\tau$ and in particular it follows that $(-1)^{\sigma^{-1}}=(-1)^\sigma$.*

Proof. Let $v_j := e_{\sigma j}$ for each j, then

$$\begin{aligned}(-1)^{\sigma\tau} &:= \varphi(e_{\sigma\tau 1},\dots,e_{\sigma\tau n})=\varphi(v_{\tau 1},\dots,v_{\tau n})\\ &= (-1)^\tau\varphi(v_1,\dots,v_d)=(-1)^\tau\varphi(e_{\sigma 1},\dots,e_{\sigma d})\\ &= (-1)^\tau(-1)^\sigma.\end{aligned}$$

■

Lemma 6.13. *A multi-linear map, $T\in\mathcal{L}^k(V)$, is alternating (i.e. $T\in\mathcal{A}^k(V)=\Lambda^k(V^*)$) iff*

$$T(v_{\sigma 1},\dots,v_{\sigma k})=(-1)^\sigma T(v_1,\dots,v_k)\ \text{ for all } \sigma\in\Sigma_k.$$

Proof. $(-1)^\sigma=(-1)^N$ where N is the number of transpositions need to transform σ to the identity permutation. For each of these transpositions produce an interchange of entries of the T function and hence introduce a (-1) factor. Thus in total,

$$T(v_{\sigma 1},\dots,v_{\sigma k})=(-1)^N T(v_1,\dots,v_k)=(-1)^\sigma T(v_1,\dots,v_k).$$

The converse direction follows from the simple fact that the sign of a transposition is -1. ■

Notation 6.14 (Pull Backs) *Let V and W be finite dimensional vector spaces. To each linear transformation, $T:V\to W$, there is linear transformation, $T^*:\Lambda^k(W^*)\to\Lambda^k(V^*)$ defined by*

$$(T^*\varphi)(v_1,\dots,v_k):=\varphi(Tv_1,\dots,Tv_k)$$

for all $\varphi\in\Lambda^k(W^*)$ and $(v_1,\dots,v_k)\in V^k$. [We leave to the reader the easy proof that $T^*\varphi$ is indeed in $\Lambda^k(V^*)$.]

[2] N is not unique but $(-1)^N=(-1)^\sigma$ is unique.

Exercise 6.2. Let V, W, and Z be three finite dimensional vector spaces and suppose that $V \xrightarrow{T} W \xrightarrow{S} Z$ are linear transformations. Noting that $V \xrightarrow{ST} Z$, show $(ST)^* = T^* S^*$.

6.1 Structure of $\Lambda^n(V^*)$ and Determinants

In what follows we will continue to use the notation introduced in Notation 6.8.

Proposition 6.15 (Structure of $\Lambda^n(V^*)$). *If $\psi \in \Lambda^n(V^*)$, then $\psi = \psi(e_1, \dots, e_n)\varphi$ and in particular,* $\dim \Lambda^n(V^*) = 1$. *Moreover for any $\{v_j\}_{j=1}^n \subset V$,*

$$\varphi(v_1, \dots, v_n) = \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1}(v_1) \dots \varepsilon_{\sigma n}(v_n)$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_1(v_{\sigma 1}) \dots \varepsilon_n(v_{\sigma n}).$$

The first equality may be rewritten as

$$\varphi = \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1} \otimes \dots \otimes \varepsilon_{\sigma n}.$$

Proof. Let $\{v_j\}_{j=1}^n \subset V$ and recall that

$$v_j = \sum_{k_j=1}^n \varepsilon_{k_j}(v_j) e_{k_j}.$$

Using the fact that ψ is multi-linear and alternating we find,

$$\psi(v_1, \dots, v_n) = \sum_{k_1, \dots, k_n = 1}^n \left[\prod_{j=1}^n \varepsilon_{k_j}(v_j)\right] \psi(e_{k_1}, \dots, e_{k_n})$$
$$= \sum_{\sigma \in \Sigma_n} \left[\prod_{j=1}^n \varepsilon_{\sigma j}(v_j)\right] \psi(e_{\sigma_1}, \dots, e_{\sigma n})$$
$$= \sum_{\sigma \in \Sigma_n} \left[\prod_{j=1}^n \varepsilon_{\sigma j}(v_j)\right] (-1)^\sigma \psi(e_1, \dots, e_n)$$

while the same computation shows

$$\varphi(v_1, \dots, v_n) = \sum_{\sigma \in \Sigma_n} \left[\prod_{j=1}^n \varepsilon_{\sigma j}(v_j)\right] (-1)^\sigma \varphi(e_1, \dots, e_n)$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1}(v_1) \dots \varepsilon_{\sigma n}(v_n).$$

Lastly let us note that

$$\prod_{j=1}^n \varepsilon_{\sigma j}(v_j) = \prod_{j=1}^n \varepsilon_{\sigma \sigma^{-1} j}\left(v_{\sigma^{-1} j}\right) = \prod_{j=1}^n \varepsilon_j\left(v_{\sigma^{-1} j}\right)$$

so that

$$\varphi(v_1, \dots, v_n) = \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n \varepsilon_j\left(v_{\sigma^{-1} j}\right) (-1)^\sigma$$
$$= \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n \varepsilon_j\left(v_{\sigma^{-1} j}\right) (-1)^{\sigma^{-1}}$$
$$= \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n \varepsilon_j(v_{\sigma j}) (-1)^\sigma$$

wherein we have used $\Sigma_n \ni \sigma \to \sigma^{-1} \in \Sigma_n$ is a bijection for the last equality. ∎

Exercise 6.3. If $\psi \in \Lambda^n(V^*) \setminus \{0\}$, show $\psi(v_1, \dots, v_n) \neq 0$ whenever $\{v_i\}_{i=1}^n \subset V$ are linearly independent. [Coupled with Exercise 6.1, it follows that $\psi(v_1, \dots, v_n) \neq 0$ iff $\{v_i\}_{i=1}^n \subset V$ are linearly independent.]

Definition 6.16. *Suppose that $T : V \to V$ is a linear map between a finite dimensional vector space, then we define* $\det T \in \mathbb{R}$ *by the relationship,* $T^*\psi = \det T \cdot \psi$ *where ψ is any non-zero element in $\Lambda^n(V^*)$. [The reader should verify that* $\det T$ *is independent of the choice of $\psi \in \Lambda^n(V^*) \setminus \{0\}$.]*

The next lemma gives a slight variant of the definition of the determinant.

Lemma 6.17. *If $\psi \in \Lambda^n(V^*) \setminus \{0\}$, $\{e_j\}_{j=1}^n$ is a basis for V, and $T : V \to V$ is a linear transformation, then*

$$\det T = \frac{\psi(Te_1, \dots, Te_n)}{\psi(e_1, \dots, e_n)}. \tag{6.2}$$

Proof. Evaluation the identity, $\det T \cdot \psi = T^*\psi$, at $(e_1, \dots, e_n)$ shows

$$\det T \cdot \psi(e_1, \dots, e_n) = (T^*\psi)(e_1, \dots, e_n) = \psi(Te_1, \dots, Te_n)$$

from which the lemma directly follows. ∎

Corollary 6.18. *Let T be as in Definition 6.16 and suppose $\{e_j\}_{j=1}^n$ is a basis for V and $\{\varepsilon_j\}_{j=1}^n$ is its dual basis, then*

$$\begin{aligned}\det T &= \sum_{\sigma\in\Sigma_n} (-1)^\sigma \varepsilon_1 (Te_{\sigma 1}) \dots \varepsilon_n (Te_{\sigma n}) \\ &= \sum_{\sigma\in\Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1} (Te_1) \dots \varepsilon_{\sigma n} (Te_n) .\end{aligned}$$

Proof. We take $\varphi \in \Lambda^n (V^*)$ so that $\varphi (e_1, \dots, e_n) = 1$. Since $T^*\varphi \in \Lambda^n (V^*)$ we have seen that $T^*\varphi = \lambda\varphi$ where

$$\begin{aligned}\lambda = (T^*\varphi) (e_1, \dots, e_n) &= \varphi (Te_1, \dots, Te_n) \\ &= \sum_{\sigma\in\Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1} (Te_1) \dots \varepsilon_{\sigma n} (Te_n) \\ &= \sum_{\sigma\in\Sigma_n} (-1)^\sigma \varepsilon_1 (Te_{\sigma 1}) \dots \varepsilon_n (Te_{\sigma n}) .\end{aligned}$$

■

Corollary 6.19. *Suppose that $S, T : V \to V$ are linear maps between a finite dimensional vector space, V, then*

$$\det (ST) = \det (S) \cdot \det (T) .$$

Proof. On one hand

$$(ST)^* \varphi = \det (ST) \varphi.$$

On the other using Exercise 6.2 we have

$$(ST)^* \varphi = T^* (S^*\varphi) = T^* (\det S \cdot \varphi) = \det S \cdot T^* (\varphi) = \det S \cdot \det T \cdot \varphi.$$

Comparing the last two equations completes the proof. ■

6.2 Determinants of Matrices

In this section we will restrict our attention to linear transformations on $V = \mathbb{R}^n$ which we identify with $n \times n$ matrices. Also, for the purposes of this section let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$. Finally recall that the i^{th} column of A is $v_i = Ae_i$ and so we may express A as

$$A = [v_1| \dots |v_n] = [Ae_1| \dots |Ae_n] .$$

Proposition 6.20. *The function, $A \to \det (A)$ is the unique alternating multi-linear function of the columns of A such that $\det (I) = \det [e_1| \dots |e_n] = 1$.*

Proof. Let $\psi \in \mathcal{A}^n (\mathbb{R}^n) \setminus \{0\}$. Then by Lemma 6.17,

$$\det A = \frac{\psi (Ae_1, \dots, Ae_n)}{\psi (e_1, \dots, e_n)}$$

which shows that $\det A$ is and alternating multi-linear function of the columns of A. We have already seen in Proposition 6.15 that there is only one such function.

■

Theorem 6.21. *If A is a $n\times n$ matrix which we view as a linear transformation on $\mathbb{R}^n$, then;*

1. $\det A = \sum_{\sigma\in\Sigma_n} (-1)^\sigma a_{\sigma 1,1} \dots a_{\sigma n,n}$,
2. $\det A = \sum_{\sigma\in\Sigma_n} (-1)^\sigma a_{1,\sigma 1} \dots a_{n,\sigma n}$, *and*
3. $\det A = \det A^{\text{tr}}$.
4. *The map $A \to \det A$ is the unique alternating multilinear function of the rows of A such that $\det I = 1$.*

Proof. We take $\{e_i\}_{i=1}^n$ to be the standard basis for $\mathbb{R}^n$ and $\{\varepsilon_i\}_{i=1}^n$ be its dual basis. Then by Corollary 6.18,

$$\begin{aligned}\det A &= \sum_{\sigma\in\Sigma_n} (-1)^\sigma \varepsilon_1 (Ae_{\sigma 1}) \dots \varepsilon_n (Ae_{\sigma n}) \\ &= \sum_{\sigma\in\Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1} (Ae_1) \dots \varepsilon_{\sigma n} (Ae_n)\end{aligned}$$

which completes the proof of item 1. and 2. since $\varepsilon_i (Ae_j) = a_{i,j}$. For item 3 we use item 1. with A replaced by A^{tr} to find,

$$\det A^{\text{tr}} = \sum_{\sigma\in\Sigma_n} (-1)^\sigma \left(A^{\text{tr}}\right)_{\sigma 1,1} \dots \left(A^{\text{tr}}\right)_{\sigma n,n} = \sum_{\sigma\in\Sigma_n} (-1)^\sigma a_{1,\sigma 1} \dots a_{n,\sigma n}.$$

This completes the proof item 3. since the latter expression is equality to $\det A$ by item 2. Finally item 4. follows from item 3. and Proposition 6.20. ■

Proposition 6.22. *Suppose that $n = n_1 + n_2$ with $n_i \in \mathbb{N}$ and T is a $n \times n$ matrix which has the block form,*

$$T = \begin{bmatrix} A & B \\ 0_{n_2\times n_1} & C \end{bmatrix},$$

where A is a $n_1 \times n_1$ – matrix, C is a $n_2 \times n_2$ – matrix and B is a $n_1 \times n_2$ – matrix. Then

$$\det T = \det A \cdot \det C.$$

Proof. Fix B and C and consider $\delta(A) := \det \begin{bmatrix} A & B \\ 0_{n_2\times n_1} & C \end{bmatrix}$. Then $\delta \in \Lambda^{n_1}(\mathbb{R}^{n_1})$ and hence

$$\delta(A) = \delta(I)\cdot\det(A) = \det(A)\cdot\det\begin{bmatrix} I & B \\ 0_{n_2\times n_1} & C \end{bmatrix}.$$

By doing standard column operations it follows that

$$\det\begin{bmatrix} I & B \\ 0_{n_2\times n_1} & C \end{bmatrix} = \det\begin{bmatrix} I & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & C \end{bmatrix} =: \tilde{\delta}(C).$$

Working as we did with δ we conclude that $\tilde{\delta}(C) = \det[C]\cdot\tilde{\delta}(I) = \det C$. Putting this all together completes the proof. ■

Next we want to prove the standard cofactor expansion of $\det A$.

Notation 6.23 *If A is a $n\times n$ matrix and $1 \le i,j \le n$, let $A(i,j)$ denotes A with its i^{th} row and j^{th} – column being deleted.*

Proposition 6.24 (Co-factor Expansion). *If A is a $n\times n$ matrix and $1 \le j \le n$, then*

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij}\det[A(i,j)] \tag{6.3}$$

and similarly if $1 \le i \le n$, then

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij}\det[A(i,j)]. \tag{6.4}$$

We refer to Eq. (6.3) as the ***cofactor expansion along the*** *j^{th} –* ***column*** *and Eq. (6.4) as the* ***cofactor expansion along the*** *i^{th} –* ***row.***

Proof. Equation (6.4) follows from Eq. (6.3) with that aid of item 3. of Theorem 6.21. To prove Eq. (6.3), let $A = [v_1|\dots|v_n]$ and for $b \in \mathbb{R}^n$ let $b^{(i)} := b - b_i e_i$ and then write $v_j = \sum_{i=1}^n a_{ij}e_i$. We then find,

$$\begin{aligned}
\det A &= \sum_{i=1}^n a_{ij}\det[v_1|\dots|v_{j-1}|e_i|v_{j+1}|\dots|v_n] \\
&= \sum_{i=1}^n a_{ij}\det\left[v_1^{(i)}|\dots|v_{j-1}^{(i)}|e_i|v_{j+1}^{(i)}|\dots|v_n^{(i)}\right] \\
&= \sum_{i=1}^n a_{ij}(-1)^{j-1}\det\left[e_i|v_1^{(i)}|\dots|v_{j-1}^{(i)}|v_{j+1}^{(i)}|\dots|v_n^{(i)}\right] \\
&= \sum_{i=1}^n a_{ij}(-1)^{j-1}(-1)^{i-1}\det\begin{bmatrix} 1 & 0 \\ 0 & A(i,j)\end{bmatrix} \\
&= \sum_{i=1}^n (-1)^{i+j} a_{ij}\det[A(i,j)]
\end{aligned}$$

wherein we have used the determinant changes sign any time one interchanges two columns or two rows. ■

Example 6.25. Let us illustrate the above proof in the 3×3 case by expanding along the second column. To shorten the notation we we write $\det A = |A|$;

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = a_{12}\begin{vmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33}\end{vmatrix} + a_{12}\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & 0 & a_{33}\end{vmatrix} + a_{12}\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 1 & a_{33}\end{vmatrix}$$

where

$$\begin{vmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33}\end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33}\end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33}\end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} = -\det A(1,2),$$

$$\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & 0 & a_{33}\end{vmatrix} = \begin{vmatrix} a_{11} & 0 & a_{13} \\ 0 & 1 & 0 \\ a_{31} & 0 & a_{33}\end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ a_{11} & 0 & a_{13} \\ a_{31} & 0 & a_{33}\end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{13} \\ 0 & a_{31} & a_{33}\end{vmatrix} = \det[A(2,2)],$$

and

$$\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 1 & a_{33}\end{vmatrix} = -\begin{vmatrix} 0 & a_{11} & a_{13} \\ 0 & a_{21} & a_{23} \\ 1 & 0 & 0\end{vmatrix} = \begin{vmatrix} 0 & a_{11} & a_{13} \\ 1 & 0 & 0 \\ 0 & a_{21} & a_{23}\end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{13} \\ 0 & a_{21} & a_{23}\end{vmatrix} = -\det[A(3,1)].$$

6.3 The structure of $\Lambda^k(V^*)$

Definition 6.26. *Let $m \in \mathbb{N}$ and $\{\ell_j\}_{j=1}^m \subset V^*$, we define $\ell_1\wedge\dots\wedge\ell_m \in \mathcal{A}^m(V)$ by*

$$\left(\ell_1 \wedge \dots \wedge \ell_m\right)(v_1, \dots, v_m) = \det \begin{bmatrix} \ell_1(v_1) & \dots & \ell_1(v_m) \\ \ell_2(v_1) & \dots & \ell_2(v_m) \\ \vdots & \vdots & \vdots \\ \ell_m(v_1) & \dots & \ell_m(v_m) \end{bmatrix} \tag{6.5}$$

$$= \sum_{\sigma \in \Sigma_m} (-1)^\sigma \prod_{i=1}^m \ell_i(v_{\sigma i}) = \sum_{\sigma \in \Sigma_m} (-1)^\sigma \ell_1(v_{\sigma 1}) \dots \ell_m(v_{\sigma m}). \tag{6.6}$$

or alternatively using $\det A^{\mathrm{tr}} = \det A$,

$$\left(\ell_1 \wedge \dots \wedge \ell_m\right)(v_1, \dots, v_m) = \det \begin{bmatrix} \ell_1(v_1) & \ell_2(v_1) & \dots & \ell_m(v_1) \\ \ell_1(v_2) & \ell_2(v_2) & \dots & \ell_m(v_2) \\ \vdots & \vdots & \vdots & \vdots \\ \ell_1(v_m) & \ell_2(v_m) & \dots & \ell_m(v_m) \end{bmatrix} \tag{6.7}$$

$$= \sum_{\sigma \in \Sigma_m} (-1)^\sigma \prod_{i=1}^m \ell_{\sigma i}(v_i) = \sum_{\sigma \in \Sigma_m} (-1)^\sigma \ell_{\sigma 1}(v_1) \dots \ell_{\sigma m}(v_m). \tag{6.8}$$

which may be written as,

$$\ell_1 \wedge \dots \wedge \ell_m = \sum_{\sigma \in \Sigma_m} (-1)^\sigma \ell_{\sigma 1} \otimes \dots \otimes \ell_{\sigma m}. \tag{6.9}$$

Exercise 6.4. Let $\{e_i\}_{i=1}^4$ be the standard basis for $\mathbb{R}^4$ and $\{\varepsilon_i = e_i^*\}_{i=1}^4$ be the associated dual basis (i.e. $\varepsilon_i(v) = v_i$ for all $v \in \mathbb{R}^4$.) Compute;

1. $$\varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_4 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \right),$$
2. $$\varepsilon_3 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right),$$
3. $$\varepsilon_1 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right),$$
4. $$(\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right),$$ and
5. $\varepsilon_4 \wedge \varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_1 (e_1, e_2, e_3, e_4)$.

The next problem is a special case of Theorem 6.28 below.

Exercise 6.5. Show, using basic knowledge of determinants, that for $\ell_0, \ell_1, \ell_2, \ell_3 \in V^*$, that

$$(\ell_0 + \ell_1) \wedge \ell_2 \wedge \ell_3 = \ell_0 \wedge \ell_2 \wedge \ell_3 + \ell_1 \wedge \ell_2 \wedge \ell_3.$$

Remark 6.27. Note that

$$\ell_{\sigma 1} \wedge \dots \wedge \ell_{\sigma m} = (-1)^\sigma \ell_1 \wedge \dots \wedge \ell_m$$

for all $\sigma \in \Sigma_m$ and in particular if $m = p + q$ with $p, q \in \mathbb{N}$, then

$$\ell_{p+1} \wedge \dots \wedge \ell_m \wedge \ell_1 \wedge \dots \wedge \ell_p = (-1)^{pq} \ell_1 \wedge \dots \wedge \ell_p \wedge \ell_{p+1} \wedge \dots \wedge \ell_m.$$

Theorem 6.28. *For any fixed* $\ell_2, \dots, \ell_k \in V^*$, *the map,*

$$V^* \ni \ell_1 \to \ell_1 \wedge \dots \wedge \ell_k \in \Lambda^k(V^*)$$

is linear.

Proof. From Eq. (6.6) we find,

$$\begin{aligned} &\left(\left(\ell_1 + c\tilde{\ell}_1\right) \wedge \dots \wedge \ell_k\right)(v_1, \dots, v_k) \\ &\quad = \sum_{\sigma \in \Sigma_k} (-1)^\sigma \left(\ell_1 + c\tilde{\ell}_1\right)(v_{\sigma 1}) \dots \ell_k(v_{\sigma k}) \\ &\quad = \sum_{\sigma \in \Sigma_k} (-1)^\sigma \ell_1(v_{\sigma 1}) \dots \ell_k(v_{\sigma k}) + c \sum_{\sigma \in \Sigma_k} (-1)^\sigma \tilde{\ell}_1(v_{\sigma 1}) \dots \ell_k(v_{\sigma k}) \\ &\quad = \ell_1 \wedge \dots \wedge \ell_k(v_1, \dots, v_k) + c \cdot \tilde{\ell}_1 \wedge \dots \wedge \ell_k(v_1, \dots, v_k) \\ &\quad = \left(\ell_1 \wedge \dots \wedge \ell_k + c \cdot \tilde{\ell}_1 \wedge \dots \wedge \ell_k\right)(v_1, \dots, v_k). \end{aligned}$$

As this holds for all $(v_1, \dots, v_k)$, it follows that

$$\left(\ell_1 + c\tilde{\ell}_1\right) \wedge \dots \wedge \ell_k = \ell_1 \wedge \dots \wedge \ell_k + c \cdot \tilde{\ell}_1 \wedge \dots \wedge \ell_k$$

which is the desired linearity. ■

Remark 6.29. If W is another finite dimensional vector space and $T : W \to V$ is a linear transformation, then $T^*(\ell_1 \wedge \dots \wedge \ell_m) = (T^*\ell_1) \wedge \dots \wedge (T^*\ell_m)$. To see this is the case, let $w_i \in W$ for $i \in [m]$, then

$$\begin{aligned} &T^*(\ell_1 \wedge \dots \wedge \ell_m)(w_1, \dots, w_m) \\ &\quad = (\ell_1 \wedge \dots \wedge \ell_m)(Tw_1, \dots, Tw_m) \\ &\quad = \sum_{\sigma \in \Sigma_m} (-1)^\sigma \prod_{i=1}^m \ell_i(Tw_{\sigma i}) = \sum_{\sigma \in \Sigma_m} (-1)^\sigma \prod_{i=1}^m (T^*\ell_i)(w_{\sigma i}) \\ &\quad = (T^*\ell_1) \wedge \dots \wedge (T^*\ell_m)(w_1, \dots, w_m) \end{aligned}$$

Theorem 6.30. *Let $\{e_i\}_{i=1}^N$ be a basis for V and $\{\varepsilon_i\}_{i=1}^N$ be its' dual basis and for*

$$J = \{1 \le a_1 < a_2 < \dots < a_p \le N\} \subset [N],$$

let with $\#(J) = p$,

$$e_J := \left(e_{a_1}, \dots, e_{a_p}\right), \text{ and } \varepsilon_J := \varepsilon_{a_1} \wedge \dots \wedge \varepsilon_{a_p}. \tag{6.10}$$

Then;

1. *$\beta_p := \{\varepsilon_J : J \subset [N] \text{ with } \#(J) = p\}$ is a basis for $\Lambda^p(V^*)$ and so $\dim(\Lambda^p(V^*)) = \binom{N}{p}$, and*
2. *any $A \in \Lambda^p(V^*)$ admits the following expansions,*

$$A = \frac{1}{p!}\sum_{j_1,\dots,j_p=1}^N A\left(e_{j_1},\dots,e_{j_p}\right)\varepsilon_{j_1}\wedge\dots\wedge\varepsilon_{j_p} \tag{6.11}$$

$$= \sum_{J\subset[N]} A(e_J)\,\varepsilon_J. \tag{6.12}$$

Proof. We begin by proving Eqs. (6.11) and (6.12). To this end let $v_1,\dots,v_p \in V$ and then compute using the multi-linear and alternating properties of A that

$$A(v_1,\dots,v_p) = \sum_{j_1,\dots,j_p=1}^N \varepsilon_{j_1}(v_1)\dots\varepsilon_{j_p}(v_p)\,A\left(e_{j_1},\dots,e_{j_p}\right) \tag{6.13}$$

$$= \sum_{j_1,\dots,j_p=1}^N \frac{1}{p!}\sum_{\sigma\in\Sigma_p}\varepsilon_{j_{\sigma 1}}(v_1)\dots\varepsilon_{j_{\sigma p}}(v_p)\,A\left(e_{j_{\sigma 1}},\dots,e_{j_{\sigma p}}\right)$$

$$= \sum_{j_1,\dots,j_p=1}^N \frac{1}{p!}\sum_{\sigma\in\Sigma_p}(-1)^\sigma\varepsilon_{j_1}(v_{\sigma^{-1}1})\dots\varepsilon_{j_p}\left(v_{\sigma^{-1}p}\right)A\left(e_{j_1},\dots,e_{j_p}\right)$$

$$= \frac{1}{p!}\sum_{j_1,\dots,j_p=1}^N A\left(e_{j_1},\dots,e_{j_p}\right)\varepsilon_{j_1}\wedge\dots\wedge\varepsilon_{j_p}(v_1,\dots,v_p),$$

which is Eq. (6.11). Alternatively we may write Eq. (6.13) as

$$A(v_1,\dots,v_p) = \sum_{j_1,\dots,j_p=1}^N 1_{\#(\{j_1,\dots,j_p\})=p}\,\varepsilon_{j_1}(v_1)\dots\varepsilon_{j_p}(v_p)\,A\left(e_{j_1},\dots,e_{j_p}\right)$$

$$= \sum_{1\le a_1<a_2<\dots<a_p\le N}\ \sum_{\sigma\in\Sigma_n}\varepsilon_{a_{\sigma 1}}(v_1)\dots\varepsilon_{a_{\sigma p}}(v_p)\,A\left(e_{a_{\sigma 1}},\dots,e_{a_{\sigma p}}\right)$$

$$= \sum_{1\le a_1<a_2<\dots<a_p\le N} A\left(e_{a_1},\dots,e_{a_p}\right)\sum_{\sigma\in\Sigma_n}(-1)^\sigma\varepsilon_{a_{\sigma 1}}(v_1)\dots\varepsilon_{a_{\sigma p}}(v_p)$$

$$= \sum_{1\le a_1<a_2<\dots<a_p\le N} A\left(e_{a_1},\dots,e_{a_p}\right)\varepsilon_{a_1}\wedge\dots\wedge\varepsilon_{a_p}(v_1,\dots,v_p)$$

$$= \sum_{J\subset[N]} A(e_J)\,\varepsilon_J(v_1,\dots,v_p).$$

which verifies Eq. (6.12) and hence item 2. is proved.

To prove item 1., since (by Eq. (6.12) we know that β_p spans $\Lambda^p(V^*)$, it suffices to show β_p is linearly independent. The key point is that for

$$J = \{1 \le a_1 < a_2 < \dots < a_p \le N\} \text{ and}$$
$$K = \{1 \le b_1 < b_2 < \dots < b_p \le N\}$$

we have

$$\varepsilon_J(e_K) = \det\begin{bmatrix} \varepsilon_{a_1}(e_{b_1}) & \dots & \varepsilon_{a_1}(e_{b_p}) \\ \varepsilon_{a_2}(e_{b_1}) & \dots & \varepsilon_{a_2}(e_{b_p}) \\ \vdots & \vdots & \vdots \\ \varepsilon_{a_p}(e_{b_1}) & \dots & \varepsilon_{a_p}(e_{b_p}) \end{bmatrix} = \delta_{J,K}.$$

Thus if $\sum_{J\subset[N]} a_J\varepsilon_J = 0$, then

$$0 = 0(e_K) = \sum_{J\subset[N]} a_J\varepsilon_J(e_K) = \sum_{J\subset[N]} a_J\delta_{J,K} = a_K$$

which shows that $a_K = 0$ for all K as above. ∎

7

Exterior/Wedge and Interior Products

The main goal of this chapter is to define a good notion of how to multiply two alternating multi-linear forms. The multiplication will be referred to as the "wedge product." Here is the result we wish to prove whose proof will be delayed until Section 7.4.

Theorem 7.1. *Let V be a finite dimensional vector space, $n = \dim(V)$, $p, q \in [n]$, and let $m = p + q$. Then there is a unique bilinear map,*

$$M_{p,q} : \Lambda^p(V^*) \times \Lambda^q(V^*) \to \Lambda^m(V^*),$$

such that for any $\{f_i\}_{i=1}^p \subset V^*$ and $\{g_j\}_{j=1}^q \subset V^*$, we have,

$$M_{p,q}(f_1 \wedge \cdots \wedge f_p, g_1 \wedge \cdots \wedge g_q) = f_1 \wedge \cdots \wedge f_p \wedge g_1 \wedge \cdots \wedge g_q. \tag{7.1}$$

The notation, $M_{p,q}$, in the previous theorem is a bit bulky and so we introduce the following (also temporary) notation.

Notation 7.2 (Preliminary) *For $A \in \Lambda^p(V^*)$ and $B \in \Lambda^q(V^*)$, let us simply denote $M_{p,q}(A, B)$ by $A \cdot B$.*[1]

Remark 7.3. If $m = p + q > n$, then $\Lambda^m(V^*) = \{0\}$ and hence $A \cdot B = 0$.

7.1 Consequences of Theorem 7.1

Before going to the proof of Theorem 7.1 (see Section 7.4) let us work out some of its consequences. By Theorem 6.30 it is always possible to write $A \in \Lambda^p(V^*)$ in the form

$$A = \sum_{i=1}^{\alpha} a_i f_1^i \wedge \cdots \wedge f_p^i \tag{7.2}$$

for some $\alpha \in \mathbb{N}$, $\{a_i\}_{i=1}^N \subset \mathbb{R}$, and $\left\{f_j^i : j \in [p] \text{ and } i \in [\alpha]\right\} \subset V^*$. Similarly we may write $B \in \Lambda^q(V^*)$ in the form,

[1] We will see shortly that it is reasonable and more suggestive to write $A \wedge B$ rather than $A \cdot B$. We will make this change after it is justified, see Notation 7.7 below.

$$B = \sum_{j=1}^{\beta} b_j g_1^j \wedge \cdots \wedge g_q^j \tag{7.3}$$

for some $\beta \in \mathbb{N}$, $\{b_j\}_{j=1}^N \subset \mathbb{R}$, and $\left\{g_j^j : j \in [q] \text{ and } j \in [\beta]\right\} \subset V^*$. Thus by Theorem 7.1 we must have

$$\begin{aligned} A \cdot B = M_{p,q}(A, B) &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j M_{p,q}\left(f_1^i \wedge \cdots \wedge f_p^i, g_1^j \wedge \cdots \wedge g_q^j\right) \\ &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j. \end{aligned} \tag{7.4}$$

Proposition 7.4 (Associativity). *If $A \in \Lambda^p(V^*)$, $B \in \Lambda^q(V^*)$, and $C \in \Lambda^r(V^*)$ for some $r \in [n]$, then*

$$(A \cdot B) \cdot C = A \cdot (B \cdot C). \tag{7.5}$$

Proof. Let us express C as

$$C = \sum_{k=1}^{\gamma} c_k h_1^k \wedge \cdots \wedge h_r^k.$$

Then working as above we find with the aid of Eq. (7.4) that

$$(A \cdot B) \cdot C = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} a_i b_j c_k f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j \wedge h_1^k \wedge \cdots \wedge h_r^k.$$

A completely analogous computation then shows that $A \cdot (B \cdot C)$ is also given by the right side of the previously displayed equation and so Eq. (7.5) is proved. ■

Remark 7.5. Since our multiplication rule is associative it now makes sense to simply write $A \cdot B \cdot C$ rather than $(A \cdot B) \cdot C$ or $A \cdot (B \cdot C)$. More generally if $A_j \in \Lambda^{p_j}(V^*)$ we may now simply write $A_1 \cdot \cdots \cdot A_k$. For example by the above associativity we may easily show,

$$A\cdot(B\cdot(C\cdot D))=(A\cdot B)\cdot(C\cdot D)=((A\cdot B)\cdot C)\cdot D$$

and so it makes sense to simply write $A\cdot B\cdot C\cdot D$ for any one of these expressions.

Corollary 7.6. *If $\{\ell_j\}_{j=1}^p\subset V^*$, then*

$$\ell_1\cdot\dots\cdot\ell_p=\ell_1\wedge\dots\wedge\ell_p.$$

Proof. For clarity of the argument let us suppose that $p=5$ in which case we have

$$\begin{aligned}\ell_1\cdot\ell_2\cdot\ell_3\cdot\ell_4\cdot\ell_5&=\ell_1\cdot(\ell_2\cdot(\ell_3\cdot(\ell_4\cdot\ell_5)))\\&=\ell_1\cdot(\ell_2\cdot(\ell_3\cdot(\ell_4\wedge\ell_5)))\\&=\ell_1\cdot(\ell_2\cdot(\ell_3\wedge\ell_4\wedge\ell_5))\\&=\ell_1\cdot(\ell_2\wedge\ell_3\wedge\ell_4\wedge\ell_5)\\&=\ell_1\wedge\ell_2\wedge\ell_3\wedge\ell_4\wedge\ell_5.\end{aligned}$$

■

Because of Corollary 7.6 there is no longer any danger in denoting $A\cdot B=M_{p,q}(A,B)$ by $A\wedge B$. Moreover, this notation suggestively leads one to the correct multiplication formulas.

Notation 7.7 (Wedge=Exterior Product) *For $A\in\Lambda^p(V^*)$ and $B\in\Lambda^q(V^*)$, we will from now on denote $M_{p,q}(A,B)$ by $A\wedge B$.*

Although the wedge product is associative, one must be careful to observe that the wedge product is not commutative, i.e. groupings do not matter but order may matter.

Lemma 7.8 (Non-commutativity). *For $A\in\Lambda^p(V^*)$ and $B\in\Lambda^q(V^*)$ we have*

$$A\wedge B=(-1)^{pq}B\wedge A.$$

Proof. See Remark 6.27. ■

7.2 Interior product

There is yet one more product structure on $\Lambda^m(V^*)$ that we will used throughout these notes given in the following definition.

Definition 7.9 (Interior product). *For $v\in V$ and $T\in\Lambda^m(V^*)$, let $i_vT\in\Lambda^{m-1}(V^*)$ be defined by $i_vT=T(v,\cdots)$.*

Lemma 7.10. *If $\{\ell_i\}_{i=1}^m\subset V^*$, $T=\ell_1\wedge\dots\wedge\ell_m$, and $v\in V$, then*

$$i_v(\ell_1\wedge\dots\wedge\ell_m)=\sum_{j=1}^m(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m. \tag{7.6}$$

Proof. Expanding the determinant along its first column we find,

$$\begin{aligned}T(v_1,\dots,v_m)&=\begin{vmatrix}\ell_1(v_1)&\dots&\ell_1(v_m)\\\ell_2(v_1)&\dots&\ell_2(v_m)\\\vdots&\vdots&\vdots\\\ell_m(v_1)&\dots&\ell_m(v_m)\end{vmatrix}\\&=\sum_{j=1}^m(-1)^{j-1}\ell_j(v_1)\cdot\begin{vmatrix}\ell_1(v_2)&\dots&\ell_1(v_m)\\\ell_2(v_2)&\dots&\ell_2(v_m)\\\vdots&\vdots&\vdots\\\ell_{j-1}(v_2)&\dots&\ell_{j-1}(v_m)\\\ell_{j+1}(v_2)&\dots&\ell_{j+1}(v_m)\\\vdots&\vdots&\vdots\\\ell_m(v_2)&\dots&\ell_m(v_m)\end{vmatrix}\\&=\sum_{j=1}^m(-1)^{j-1}\ell_j(v_1)\left(\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m\right)(v_2,\dots,v_m)\end{aligned}$$

from which Eq. (7.6) follows. ■

Corollary 7.11. *For $A\in\Lambda^p(V^*)$ and $B\in\Lambda^q(V^*)$ and $v\in V$, we have*

$$i_v[A\wedge B]=(i_vA)\wedge B+(-1)^pA\wedge(i_vB).$$

Proof. It suffices to verify this identity on decomposable forms, $A=\ell_1\wedge\dots\wedge\ell_p$ and $B=\ell_{p+1}\wedge\dots\wedge\ell_m$ so that $A\wedge B=\ell_1\wedge\dots\wedge\ell_m$ and we have

$$\begin{aligned}&i_v(A\wedge B)\\&\quad=\sum_{j=1}^m(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m\\&\quad=\sum_{j=1}^p(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\ell_p\wedge\ell_{p+1}\wedge\dots\wedge\ell_m\\&\qquad+\sum_{j=p+1}^m(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\ell_p\wedge\ell_{p+1}\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m\\&\quad=:T_1+T_2\end{aligned}$$

where

$$T_1 = \left[\sum_{j=1}^{p} (-1)^{j-1} \ell_j(v) \cdot \ell_1 \wedge \dots \wedge \widehat{\ell_j} \wedge \dots \ell_p\right] \wedge B = (i_v A) \wedge B$$

and

$$\begin{aligned} T_2 &= A \wedge \left[\sum_{j=p+1}^{m} (-1)^{j-1} \ell_j(v) \, \ell_{p+1} \wedge \dots \wedge \widehat{\ell_j} \wedge \dots \wedge \ell_m\right] \\ &= (-1)^p A \wedge \left[\sum_{j=p+1}^{m} (-1)^{j-(p+1)} \ell_j(v) \, \ell_{p+1} \wedge \dots \wedge \widehat{\ell_j} \wedge \dots \wedge \ell_m\right] \\ &= (-1)^p A \wedge (i_v B). \end{aligned}$$

■

Lemma 7.12. *If $v, w \in V$, then $i_v^2 = 0$ and $i_v i_w = -i_w i_v$.*

Proof. Let $T \in \Lambda^k(V^*)$, then

$$i_v i_w T = T(w, v, —) = T(v, w, —) = i_w i_v T.$$

■

Definition 7.13 (Cross product on $\mathbb{R}^3$). *For $a, b \in \mathbb{R}^3$, let $a \times b$ be the unique vector in $\mathbb{R}^3$ so that*

$$\det[c|a|b] = c \cdot (a \times b) \text{ for all } c \in \mathbb{R}^3.$$

Such a unique vector exists since we know that $c \to \det[c|a|b]$ is a linear functional on $\mathbb{R}^3$ for each $a, b \in \mathbb{R}^3$.

Lemma 7.14 (Cross product). *The cross product in Definition 7.13 agrees with the "usual definition,*

$$\begin{aligned} a \times b &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &=: \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \end{aligned}$$

where $\mathbf{i} = e_1$, $\mathbf{j} = e_2$, and $\mathbf{k} = e_3$ is the standard basis for $\mathbb{R}^3$.

Proof. Suppose that $a \times b$ is defined by the formula in the lemma, then for all $c \in \mathbb{R}$,

$$\begin{aligned} (a \times b) \cdot c &= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \det[c|a|b], \end{aligned}$$

wherein we have used the cofactor expansion along the top row for the second equality and the fact that $\det A = \det A^{\mathrm{tr}}$ for the last equality.

■

Remark 7.15 (Generalized Cross product). If $a_1, a_2 \dots, a_{n-1} \in \mathbb{R}^n$, let $a_1 \times a_2 \times \dots \times a_{n-1}$ denote the unique vector in $\mathbb{R}^n$ such that

$$\det[c|a_1|a_2|\dots|a_{n-1}] = c \cdot a_1 \times a_2 \times \dots \times a_{n-1} \ \forall\, c \in \mathbb{R}^n.$$

This "multi-product" is the $n > 3$ analogue of the cross product in $\mathbb{R}^3$. I don't anticipate using this generalized cross product.

7.3 Exercises

Exercise 7.1 (Cross I). For $a \in \mathbb{R}^3$, let $\ell_a(v) = a \cdot v = a^{\mathrm{tr}} v$, so that $\ell_a \in (\mathbb{R}^3)^*$. In particular we have $\varepsilon_i = \ell_{e_i}$ for $i \in [3]$ is the dual basis to the standard basis $\{e_i\}_{i=1}^3$. Show for $a, b \in \mathbb{R}^3$,

$$\ell_a \wedge \ell_b = i_{a \times b} [\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3] \tag{7.7}$$

Hints: 1) write $\ell_a = \sum_{i=1}^3 a_i \varepsilon_i$ and 2) make use of Eq. (7.6)

Exercise 7.2 (Cross II). Use Exercise 7.1 to prove the standard vector calculus identity;

$$(a \times b) \cdot (x \times y) = (a \cdot x)(b \cdot y) - (b \cdot x)(a \cdot y)$$

which is valid for all $a, b, x, y \in \mathbb{R}^3$.Hint: evaluate Eq. (7.7) at (x, y) while using Lemma 7.14.

7.4 *Proof of Theorem 7.1

[This section may safely be skipped if you are willing to believe the results as stated!]

If Theorem 7.1 is going to be true we must have $M_{p,q}(A,B) = A \cdot B = D$ where, as written in Eq. (7.4),

$$D = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j. \tag{7.8}$$

The problem with this presumed definition is that the formula for D in Eq. (7.8) seems to depend on the expansions of A and B in Eqs. (7.2) and (7.3) rather than on only A and B. [The expansions for A and B in Eqs. (7.2) and (7.3) are highly non-unique!] In order to see that D is independent of the possible choices of expansions of A and B, we are going to show in Proposition 7.19 below that $D(v_1, \dots, v_m)$ (with D as in Eq. (7.8)) may be expressed by a formula which only involves A and B and **not** their expansions. Before getting to this proposition we need some more notation and a preliminary lemma.

Notation 7.16 *Let $m = p + q$ be as in Theorem 7.1 and let $\{v_i\}_{i=1}^m \subset V$ be fixed. For each $J \subset [m]$ with $\#J = p$ write*

$$\begin{aligned} J &= \{1 \le a_1 < a_2 < \cdots < a_p \le m\}, \\ J^c &= \{1 \le b_1 < b_2 < \cdots < b_q \le m\}, \\ v_J &:= (v_{a_1}, \dots, v_{a_p}), \text{ and } v_{J^c} := (v_{b_1}, \dots, v_{b_q}). \end{aligned}$$

Also for any $\alpha \in \Sigma_p$ and $\beta \in \Sigma_q$, let

$$\sigma_{J,\alpha,\beta} = \begin{pmatrix} 1 & \dots & p & p+1 & \dots & m \\ a_{\alpha 1} & \dots & a_{\alpha p} & b_{\beta 1} & \dots & b_{\beta q} \end{pmatrix}.$$

When α and β are the identity permutations in Σ_p and Σ_q respectively we will simply denote $\sigma_{J,\alpha,\beta}$ by σ_J, i.e.

$$\sigma_J = \begin{pmatrix} 1 & \dots & p & p+1 & \dots & m \\ a_1 & \dots & a_p & b_1 & \dots & b_q \end{pmatrix}.$$

The point of this notation is contained in the following lemma.

Lemma 7.17. *Assuming Notation 7.16,*

1. *the map,*
$$\mathcal{P}_{p,m} \times \Sigma_p \times \Sigma_q \ni (J, \alpha, \beta) \to \sigma_{J,\alpha,\beta} \in \Sigma_m,$$
is a bijection, and
2. $(-1)^{\sigma_{J,\alpha,\beta}} = (-1)^{\sigma_J} (-1)^{\alpha} (-1)^{\beta}$.

Proof. We leave proof of these assertions to the reader. ■

Lemma 7.18 (Wedge Product I). *Let $n = \dim V$, $p, q \in [n]$, $m := p + q$, $\{f_i\}_{i=1}^p \subset V^*$, $\{g_j\}_{j=1}^q \subset V^*$, and $\{v_j\}_{j=1}^m \subset V$, then*

$$\begin{aligned} &(f_1 \wedge \cdots \wedge f_p \wedge g_1 \wedge \cdots \wedge g_q)(v_1, \dots, v_m) \\ &\quad = \sum_{\#J=p} (-1)^{\sigma_J} (f_1 \wedge \cdots \wedge f_p)(v_J)(g_1 \wedge \cdots \wedge g_q)(v_{J^c}). \end{aligned} \tag{7.9}$$

Proof. In order to simplify notation in the proof let, $\ell_i = f_i$ for $1 \le i \le p$ and $\ell_{j+p} = g_j$ for $1 \le j \le q$ so that

$$f_1 \wedge \cdots \wedge f_p \wedge g_1 \wedge \cdots \wedge g_q = \ell_1 \wedge \cdots \wedge \ell_m.$$

Then by Definition 6.26 of $\ell_1 \wedge \cdots \wedge \ell_m$ along with Lemma 7.17, we find,

$$\begin{aligned} &(\ell_1 \wedge \cdots \wedge \ell_m)(v_1, \dots, v_m) \\ &= \det\left[\{\ell_i(v_j)\}_{i,j=1}^m\right] = \sum_{\sigma \in \Sigma_m} (-1)^{\sigma} \prod_{i=1}^m \ell_i(v_{\sigma i}). \\ &= \sum_J \sum_{\alpha \in \Sigma_p} \sum_{\beta \in \Sigma_q} (-1)^{\sigma_{J,\alpha,\beta}} \prod_{i=1}^m \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) \\ &= \sum_J (-1)^{\sigma_J} \sum_{\alpha \in \Sigma_p} \sum_{\beta \in \Sigma_q} (-1)^{\sigma_\alpha} \prod_{i=1}^p \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) (-1)^{\sigma_\beta} \prod_{i=p+1}^m \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right). \end{aligned}$$

Combining this with the following identity,

$$\begin{aligned} &\sum_{\alpha \in \Sigma_p} \sum_{\beta \in \Sigma_q} (-1)^{\sigma_\alpha} \prod_{i=1}^p \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) (-1)^{\sigma_\beta} \prod_{i=p+1}^m \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) \\ &= \sum_{\alpha \in \Sigma_p} (-1)^{\sigma_\alpha} \prod_{i=1}^p \ell_i\left(v_{a_{\alpha i}}\right) \sum_{\beta \in \Sigma_q} (-1)^{\sigma_\beta} \prod_{i=p+1}^m \ell_i\left(v_{b_{\beta i}}\right) \\ &= (\ell_1 \wedge \cdots \wedge \ell_p)(v_J)(\ell_{p+1} \wedge \cdots \wedge \ell_m)(v_{J^c}) \\ &= (f_1 \wedge \cdots \wedge f_p)(v_J)(g_1 \wedge \cdots \wedge g_q)(v_{J^c}) \end{aligned}$$

completes the proof. ■

Proposition 7.19 (Wedge Product II). *If $A \in \Lambda^p(V^*)$ and $B \in \Lambda^q(V^*)$ are written as in Eqs. (7.2) and (7.3) and $D \in \Lambda^m(V^*)$ is defined as in Eq. (7.8), then*

$$D(v_1, \dots, v_m) = \sum_{\#J=p} (-1)^{\sigma_J} A(v_J) B(v_{J^c}) \quad \forall \ \{v_j\}_{j=1}^m \subset V. \tag{7.10}$$

This shows defining $A \wedge B$ by Eq. (7.4)is well defined and in fact could have been defined intrinsically using the formula,

$$A \wedge B\left(v_1, \ldots, v_m\right)=\sum_{\# J=p}(-1)^{\sigma_J} A\left(v_J\right) B\left(v_{J^c}\right). \tag{7.11}$$

Proof. By Lemma 7.18,

$$\begin{aligned} & f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j\left(v_1, \ldots, v_m\right) \\ & =\sum_{\# J=p}(-1)^{\sigma_J}\left(f_1^i \wedge \cdots \wedge f_p^i\right)\left(v_J\right) \cdot\left(g_1^j \wedge \cdots \wedge g_q^j\right)\left(v_{J^c}\right) \end{aligned}$$

and therefore,

$$\begin{aligned} & D\left(v_1, \ldots, v_m\right) \\ & =\sum_{i=1}^\alpha \sum_{j=1}^\beta a_i b_j\left(f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j\right)\left(v_1, \ldots, v_m\right) \\ & =\sum_{i=1}^\alpha \sum_{j=1}^\beta a_i b_j \sum_{\# J=p}(-1)^{\sigma_J}\left(f_1^i \wedge \cdots \wedge f_p^i\right)\left(v_J\right) \cdot\left(g_1^j \wedge \cdots \wedge g_q^j\right)\left(v_{J^c}\right) \\ & =\sum_{\# J=p}(-1)^{\sigma_J} \sum_{i=1}^\alpha \sum_{j=1}^\beta a_i\left(f_1^i \wedge \cdots \wedge f_p^i\right)\left(v_J\right) \cdot \sum_{j=1}^\beta b_j\left(g_1^j \wedge \cdots \wedge g_q^j\right)\left(v_{J^c}\right) \\ & =\sum_{\# J=p}(-1)^{\sigma_J} A\left(v_J\right) B\left(v_{J^c}\right) \end{aligned}$$

which proves Eq. (7.10) and completes the proof of the proposition. ∎

With all of this preparation we are now in a position to complete the proof of Theorem 7.1.

Proof of Theorem 7.1. As we have see we may define $A \wedge B$ by either Eq. (7.4) or by Eq. (7.11). Equation (7.11) ensures $A \wedge B$ is well defined and is multi-linear while Eq. (7.4) ensures $A \wedge B \in \Lambda^m\left(V^*\right)$ and that Eq. (7.1) holds. This proves the existence assertion of the theorem. The uniqueness of $M_{p, q}(A, B)=A \wedge B$ follows by the necessity of defining $A \wedge B$ by Eq. (7.4). ∎

Corollary 7.20. *Suppose that $\left\{e_j\right\}_{j=1}^n$ is a basis of V and $\left\{\varepsilon_j\right\}_{j=1}^n$ is its dual basis of V^*. Then for $A \in \Lambda^p\left(V^*\right)$ and $B \in \Lambda^q\left(V^*\right)$ we have*

$$A \wedge B=\frac{1}{p! \cdot q!} \sum_{j_1, \ldots, j_m=1}^n A\left(e_{j_1}, \ldots, e_{j_p}\right) B\left(e_{j_{p+1}}, \ldots, e_{j_m}\right) \varepsilon_{j_1} \wedge \ldots \cdots \wedge \varepsilon_{j_m}. \tag{7.12}$$

Proof. By Theorem 6.30 we may write,

$$\begin{aligned} A & =\frac{1}{p!} \sum_{j_1, \ldots, j_p=1}^N A\left(e_{j_1}, \ldots, e_{j_p}\right) \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p} \text { and } \\ B & =\frac{1}{q!} \sum_{j_{p+1}, \ldots, j_m=1}^n B\left(e_{j_{p+1}}, \ldots, e_{j_m}\right) \varepsilon_{j_{p+1}} \wedge \ldots \cdots \wedge \varepsilon_{j_m} \end{aligned}$$

and therefore Eq. (7.12) holds by computing $A \wedge B$ as in Eq. (7.4). ∎