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Math 150B Differential Geometry

February 27, 2020 *File:150BNotes.tex*

Contents

Part IV Appendices

Part

Homework Problems

Math 150B Homework Problems: Winter 2020

Problems are our text book, "Differential Forms," by Guillemin and Haine or from the lecture notes as indicated. The problems from the lecture notes are restated here, however there may be broken references. If this is the case, please find the corresponding problem in the lecture notes for the proper references and for more context of the problem.

0.0 Homework 0, Due Wednesday, January 8, 2020 (Not to be collected)

- Lecture note Exercises: 3.1, 3.2, 3.3, 3.4, and 3.5.

0.1 Homework 1. Due Thursday, January 16, 2020

- Lecture note Exercises: 3.6, 5.1, 5.2, 5.5
- Book Exercises: 1.2.vi.

0.2 Homework 2. Due Thursday, January 23, 2020

- Lecture note Exercises: 5.3, 5.4, 6.1, 6.2, 6.3, 6.4, 6.5
- Book Exercises: 1.3.iii., 1.3.v., 1.3.vii, 1.4ix

0.3 Homework 3. Due Thursday, January 30, 2020

- Lecture note Exercises: 7.2, 7.3, 8.1, 8.2, 8.3, 8.4, 8.5
- Look at (but **don't hand in**) Exercises 7.4, 7.5 and the Book Exercises: 1.7.iv., 1.8vi.

0.4 Homework 4. Due Thursday, February 6, 2020

These problems are part of your midterm and are to be worked on by your-self. These are due at the start of the in-class portion of the midterm which is in class on **Thursday February 6, 2020.**

- Lecture note Exercises: 6.6, 6.7, 7.1, 8.6, 8.7

0.5 Homework 5. Due Thursday, February 13, 2020

- Lecture note Exercises: 8.8, 8.9, 8.10, 8.11, 8.12, 8.13
- Book Exercises: 2.3.ii., 2.3.iii., 2.4.i

0.6 Homework 6. Due Thursday, February 20, 2020

- Book Exercises: 2.1vii, 2.1viii, 2.4.ii, 2.4iii, 2.4iv. 2.6i, 2.6ii, 2.6iii (Refer to exercise 2.1.vii not 2.2viii), 3.2.i, 3.2viii
- Have a look at Reyer Sjamaar's notes: Manifolds and Differential Forms – especially see Chapter 6 starting on page 75 for the notions of a manifold, tangent spaces, and lots of pictures!

0.7 Homework 7. Now Due Friday, February 28, 2020 at 7:00PM

- Hand in Lecture note Exercises: 9.1 (now corrected), 10.1, 10.2, 10.3, 10.4, 10.6, 10.7, 10.8, 10.17, 10.18
- Look at but do not turn in Lecture note Exercise: 9.2

Part I

Background Material

1

Introduction

This class is devoted to understanding, proving, and exploring the multi-dimensional and "manifold" analogues of the classic one dimensional fundamental theorem of calculus and change of variables theorem. These theorems take on the following form;

$$\int_M d\omega = \int_{\partial M} \omega \quad \longleftrightarrow \quad \int_a^b g'(x)\, dx = g(x)|_a^b \text{ and} \tag{1.1}$$

$$\int_M f^*\omega = \deg(f) \cdot \int_N \omega \quad \longleftrightarrow \quad \int_a^b g(f(x)) f'(x)\, dx = \int_{f(a)}^{f(b)} g(y)\, dy. \tag{1.2}$$

In meeting our goals we will need to understand all the ingredients in the above formula including;

1. M is a **manifold.**
2. ∂M is the **boundary of** M.
3. ω is a **differential form** an $d\omega$ is its **differential.**
4. $f^*\omega$ is the **pull back** of ω by a "smooth map" $f : M \to N$.
5. $\deg(f) \in \mathbb{Z}$ is the **degree** of f.
6. There is also a hidden notion of **orientation** needed to make sense of the above integrals.

Remark 1.1. We will see that Eq. (1.1) encodes (all wrapped into one neat formula) the key integration formulas from 20E: Green's theorem, Divergence theorem, and Stoke's theorem.

Permutations Basics

The following proposition should be verified by the reader.

Proposition 2.1 (Permutation Groups). *Let Λ be a set and*

$$\Sigma(\Lambda) := \{\sigma : \Lambda \to \Lambda |\ \sigma \text{ is bijective}\}.$$

If we equip G with the binary operation of function composition, then G is a group. The identity element in G is the identity function, ε, and the inverse, σ^{-1}, to $\sigma \in G$ is the inverse function to σ.

Definition 2.2 (Finite permutation groups). *For $n \in \mathbb{Z}_+$, let $[n] := \{1, 2, \dots, n\}$, and $\Sigma_n := \Sigma([n])$ be the group described in Proposition 2.1. We will identify elements, $\sigma \in \Sigma_n$, with the following $2 \times n$ array,*

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}.$$

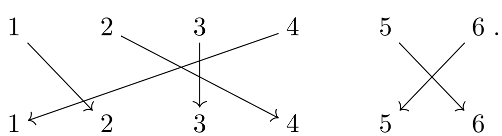
(Notice that $|\Sigma_n| = n!$ since there are n choices for $\sigma(1)$, $n-1$ for $\sigma(2)$, $n-2$ for $\sigma(3)$, ..., 1 for $\sigma(n)$.)

For examples, suppose that $n = 6$ and let

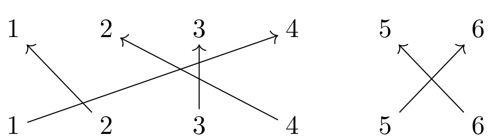
$$\varepsilon = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,2\,3\,4\,5\,6 \end{pmatrix} \text{ – the identity, and}$$

$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,4\,3\,1\,6\,5 \end{pmatrix}.$$

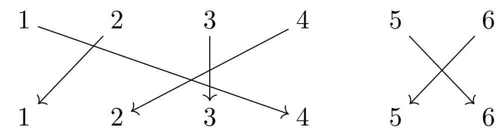
We identify σ with the following picture,



The inverse to σ is gotten pictorially by reversing all of the arrows above to find,



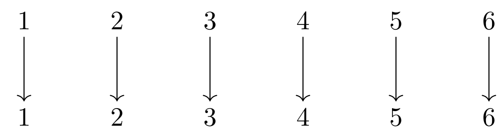
or equivalently,



and hence,

$$\sigma^{-1} = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 4\,1\,3\,2\,6\,5 \end{pmatrix}.$$

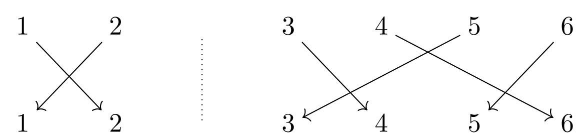
Of course the identity in this graphical picture is simply given by



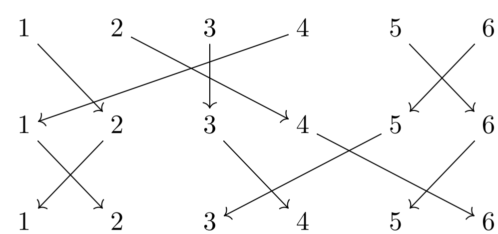
Now let $\beta \in S_6$ be given by

$$\beta = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,4\,6\,3\,5 \end{pmatrix},$$

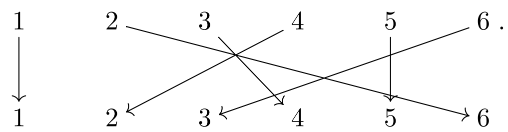
or in pictures;



We can now compose the two permutations $\beta \circ \sigma$ graphically to find,



which after erasing the intermediate arrows gives,



In terms of our array notation we have,

$$\beta \circ \sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,4\,6\,3\,5 \end{pmatrix} \circ \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,4\,3\,1\,6\,5 \end{pmatrix}$$
$$= \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,6\,4\,2\,5\,3 \end{pmatrix}.$$

Remark 2.3 (Optional). It is interesting to observe that β splits into a product of two permutations,

$$\beta = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,3\,4\,5\,6 \end{pmatrix} \circ \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,2\,4\,6\,3\,5 \end{pmatrix}$$
$$= \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,2\,4\,6\,3\,5 \end{pmatrix} \circ \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,3\,4\,5\,6 \end{pmatrix},$$

corresponding to the non-crossing parts in the graphical picture for β. Each of these permutations is called a "cycle."

Definition 2.4 (Transpositions). *A permutation, $\sigma \in \Sigma_k$, is a* ***transposition*** *if*

$$\#\{l \in [k] : \sigma(l) \neq l\} = 2.$$

We further say that σ is an ***adjacent transposition*** *if*

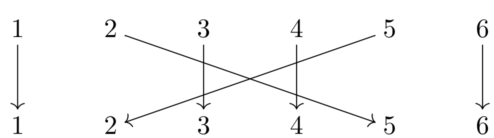
$$\{l \in [k] : \sigma(l) \neq l\} = \{i, i+1\}$$

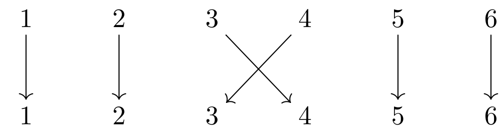
for some $1 \leq i < k$.

Example 2.5. If

$$\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,5\,3\,4\,2\,6 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 1\,2\,4\,3\,5\,6 \end{pmatrix}$$

then σ is a transposition and τ is an adjacent transposition. Here are the pictorial representation of σ and τ;





In terms of these pictures it is easy to recognize transpositions and adjacent transpositions.

Integration Theory Outline

In this course we are going to be considering integrals over open subsets of $\mathbb{R}^d$ and more generally over "manifolds." As the prerequisites for this class do not include real analysis, I will begin by summarizing a reasonable working knowledge of integration theory over $\mathbb{R}^d$. We will thus be neglecting some technical details involving measures and σ − algebras. The knowledgeable reader should be able to fill in the missing hypothesis while the less knowledgeable readers should not be too harmed by the omissions to follow.

Definition 3.1. *The* ***indicator function of*** *a subset, $A \subset \mathbb{R}^d$, is defined by*

$$1_A(x) := \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

Remark 3.2 (Optional). Every function, $f : \mathbb{R}^d \to \mathbb{R}$, may be approximated by a linear combination of indicator functions as follows. If $\varepsilon > 0$ is given we let

$$f_\varepsilon := \sum_{n \in \mathbb{N}} n\varepsilon \cdot 1_{\{n\varepsilon \le f < (n+1)\varepsilon\}}, \tag{3.1}$$

where $\{n\varepsilon \le f < (n+1)\varepsilon\}$ is shorthand for the set,

$$\left\{x \in \mathbb{R}^d : n\varepsilon \le f(x) < (n+1)\varepsilon\right\}.$$

We now summarize "modern" Lebesgue integration theory over $\mathbb{R}^d$.

1. For each d, there is a uniquely determined **volume measure,** m_d on all[1] subsets of $\mathbb{R}^d$ (subsets of $\mathbb{R}^d$) with the following properties;
   a) $m_d(A) \in [0, \infty]$ for all $A \subset \mathbb{R}^d$ with $m_d(\emptyset) = 0$.
   b) $m_d(A \cup B) = m_d(A) + m_d(B)$ is $A \cap B = \emptyset$. More generally, if $A_n \subset \mathbb{R}^d$ for all n with $A_n \cap A_m = \emptyset$ for $m \ne n$ we have
   $$m_d\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty m_d(A_n).$$
   c) $m_d(x + A) = m_d(A)$ for all $A \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, where
   $$x + A := \left\{x + y \in \mathbb{R}^d : y \in A\right\}.$$
   d) $m_d\left([0, 1]^d\right) = 1$.
   [The reader is supposed to view $m_d(A)$ as the d-dimensional volume of a subset, $A \subset \mathbb{R}^d$.]
2. Associated to this volume measure is an integral which takes (not all) functions, $f : \mathbb{R}^d \to \mathbb{R}$, and assigns to them a number denoted by
   $$\int_{\mathbb{R}^d} f dm_d = \int_{\mathbb{R}^d} f(x)\, dm_d(x) \in \mathbb{R}.$$
   This integral has the following properties;
   a) When $d = 1$ and f is continuous function with compact support, $\int_{\mathbb{R}} f dm_1$ is the ordinary integral you studied in your first few calculus courses.
   b) The integral is defined for "all" $f \ge 0$ and in this case
   $$\int_{\mathbb{R}^d} f dm_d \in [0, \infty] \text{ and } \int_{\mathbb{R}^d} 1_A dm_d = m_d(A) \text{ for all } A \subset \mathbb{R}^d.$$
   c) The integral is **"positive" linear**, i.e. if $f, g \ge 0$ and $c \in [0, \infty)$, then
   $$\int_{\mathbb{R}^d} (f + cg)\, dm_d = \int_{\mathbb{R}^d} f dm_d + c \int_{\mathbb{R}^d} g dm_d. \tag{3.2}$$
   d) The integral is **monotonic, i.e.** if $0 \le f \le g$, then
   $$\int_{\mathbb{R}^d} f dm_d \le \int_{\mathbb{R}^d} g dm_d. \tag{3.3}$$
   e) Let $L^1(m_d)$ denote those functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} |f|\, dm_d < \infty$. Then for $f \in L^1(m_d)$ we define
   $$\int_{\mathbb{R}^d} f dm_d := \int_{\mathbb{R}^d} f_+ dm_d - \int_{\mathbb{R}^d} f_- dm_d$$
   where
   $$f_\pm(x) = \max(\pm f(x), 0) \text{ and so that } f(x) = f_+(x) - f_-(x).$$

[1] This is a lie! Nevertheless, for our purposes it will be reasonably safe to ignore this lie.

f) The integral, $L^1(m_d) \ni f \to \int_{\mathbb{R}^d} f dm_d$ is linear, i.e. Eq. (3.2) holds for all $f, g \in L^1(m_d)$ and $c \in \mathbb{R}$.

g) If $f, g \in L^1(m_d)$ and $f \le g$ then Eq. (3.3) still holds.

3. The integral enjoys the following continuity properties.

a) **MCT**: the **monotone convergence theorem** holds; if $0 \le f_n \uparrow f$ then

$$\uparrow \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n dm_d = \int_{\mathbb{R}^d} f dm_d \text{ (with } \infty \text{ allowed as a possible value).}$$

Example 1: If $\{A_n\}_{n=1}^\infty$ is a sequence of subsets of $\mathbb{R}^d$ such that $A_n \uparrow A$ (i.e. $A_n \subset A_{n+1}$ for all n and $A = \cup_{n=1}^\infty A_n$), then

$$m_d(A_n) = \int_{\mathbb{R}^d} 1_{A_n} dm_d \uparrow \int_{\mathbb{R}^d} 1_A dm_d = m_d(A) \text{ as } n \to \infty$$

Example 2: If $g_n : \mathbb{R}^d \to [0, \infty]$ for $n \in \mathbb{N}$ then

$$\int_{\mathbb{R}^d} \sum_{n=1}^\infty g_n = \int_{\mathbb{R}^d} \lim_{N\to\infty} \sum_{n=1}^N g_n = \lim_{N\to\infty} \int_{\mathbb{R}^d} \sum_{n=1}^N g_n$$
$$= \lim_{N\to\infty} \sum_{n=1}^N \int_{\mathbb{R}^d} g_n = \sum_{n=1}^\infty \int_{\mathbb{R}^d} g_n.$$

b) **DCT**: the **dominated convergence theorem** holds**,** if $f_n : \mathbb{R}^d \to \mathbb{R}$ are functions **dominating** by a function $G \in L^1(m_d)$ is the sense that $|f_n(x)| \le G(x)$ for all $x \in \mathbb{R}^d$. Then assuming that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. $x \in \mathbb{R}^d$, we may conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n dm_d = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n dm_d = \int_{\mathbb{R}^d} f dm_d.$$

Example: If $\{g_n\}_{n=1}^\infty$ is a sequence of real valued random variables such that

$$\int_{\mathbb{R}^d} \sum_{n=1}^\infty |g_n| = \sum_{n=1}^\infty \int_{\mathbb{R}^d} |g_n| < \infty,$$

then; 1) $G := \sum_{n=1}^\infty |g_n| < \infty$ a.e. and hence $\sum_{n=1}^\infty g_n = \lim_{N\to\infty} \sum_{n=1}^N g_n$ exist a.e., 2) $\left|\sum_{n=1}^N g_n\right| \le G$ and $\int_{\mathbb{R}^d} G < \infty$, and so 3) by DCT,

$$\int_{\mathbb{R}^d} \sum_{n=1}^\infty g_n = \int_{\mathbb{R}^d} \lim_{N\to\infty} \sum_{n=1}^N g_n = \lim_{N\to\infty} \int_{\mathbb{R}^d} \sum_{n=1}^N g_n$$
$$= \lim_{N\to\infty} \sum_{n=1}^N \int_{\mathbb{R}^d} g_n = \sum_{n=1}^\infty \int_{\mathbb{R}^d} g_n.$$

c) **Fatou's Lemma (*Optional):** if $0 \le f_n \le \infty$, then

$$\int_{\mathbb{R}^d} \left[\liminf_{n\to\infty} f_n\right] \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} f_n dm_d.$$

This may be proved as an application of MCT.

4. **Tonelli's theorem;** if $f : \mathbb{R}^d \to [0, \infty]$, then for any $i \in [d]$,

$$\int_{\mathbb{R}^d} f dm_d = \int_{\mathbb{R}^{d-1}} \bar{f} dm_{d-1} \text{ where}$$
$$\bar{f}(x_1, \dots, \hat{x}_i, \dots x_d) := \int_{\mathbb{R}} f(x_1, \dots, x_i, \dots x_d) dx_i.$$

5. **Fubini's theorem;** if $f \in L^1(m_d)$ then the previous formula still hold.

6. For our purposes, by repeated use of use of items 4. and 5. we may compute $\int_{\mathbb{R}^d} f dm_d$ in terms of iterated integrals in any order we prefer. In more detail if $\sigma \in \Sigma_d$ is any permutation of $[d]$, then

$$\int_{\mathbb{R}^d} f dm_d = \int_{\mathbb{R}} dx_{\sigma(1)} \cdots \int_{\mathbb{R}} dx_{\sigma(d)} f(x_1, \dots, x_d)$$

provided either that $f \ge 0$ or

$$\int_{\mathbb{R}} dx_{\sigma(1)} \cdots \int_{\mathbb{R}} dx_{\sigma(d)} |f(x_1, \dots, x_d)| = \int_{\mathbb{R}^d} |f| dm_d < \infty.$$

This fact coupled with item 2a. will basically allow us to understand most integrals appearing in this course.

Notation 3.3 *For $A \subset \mathbb{R}^d$, we let*

$$\int_A f dm_d := \int_{\mathbb{R}^d} 1_A f \, dm_d.$$

Also when $d = 1$ and $-\infty \le s < t \le \infty$, we write

$$\int_s^t f dm_1 = \int_{(s,t)} f dm_1 = \int_{\mathbb{R}} 1_{(s,t)} f dm_1$$

and (as usual in Riemann integration theory)

$$\int_t^s f dm_1 := -\int_s^t f dm_1.$$

Example 3.4. Here is a MCT example,

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2} dt = \int_{-\infty}^{\infty} \lim_{n\to\infty} 1_{[-n,n]}(t) \frac{1}{1+t^2} dt$$
$$\overset{MCT}{=} \lim_{n\to\infty} \int_{-\infty}^{\infty} 1_{[-n,n]}(t) \frac{1}{1+t^2} dt = \lim_{n\to\infty} \int_{-n}^{n} \frac{1}{1+t^2} dt$$
$$= \lim_{n\to\infty} \left[\tan^{-1}(n) - \tan^{-1}(-n)\right] = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

Example 3.5. Similarly for any $x > 0$,

$$\int_0^{\infty} e^{-tx} dt = \int_{-\infty}^{\infty} \lim_{n\to\infty} 1_{[0,n]}(t) e^{-tx} dt \overset{MCT}{=} \lim_{n\to\infty} \int_{-\infty}^{\infty} 1_{[0,n]}(t) e^{-tx} dt$$
$$= \lim_{n\to\infty} \int_0^n e^{-tx} dt = \lim_{n\to\infty} \frac{-1}{x} e^{-tx}|_{t=0}^n = \frac{1}{x}. \tag{3.4}$$

Example 3.6. Here is a DCT example,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \sin\left(\frac{t}{n}\right) dt = \int_{-\infty}^{\infty} \lim_{n\to\infty} \frac{1}{1+t^2} \sin\left(\frac{t}{n}\right) dt = \int_{\mathbb{R}} 0 dm = 0$$

since

$$\lim_{n\to\infty} \frac{1}{1+t^2} \sin\left(\frac{t}{n}\right) = 0 \text{ for all } t \in \mathbb{R}$$

and

$$\left|\frac{1}{1+t^2} \sin\left(\frac{t}{n}\right)\right| \le \frac{1}{1+t^2} \text{ with } \int_{\mathbb{R}} \frac{1}{1+t^2} dt < \infty.$$

Example 3.7. In this example we will show

$$\lim_{M\to\infty} \int_0^M \frac{\sin x}{x} dx = \pi/2 \tag{3.5}$$

Let us first note that $\left|\frac{\sin x}{x}\right| \le 1$ for all x and hence by DCT,

$$\int_0^M \frac{\sin x}{x} dx = \lim_{\varepsilon\downarrow 0} \int_\varepsilon^M \frac{\sin x}{x} dx.$$

Moreover making use of Eq. (3.4), if $0 < \varepsilon < M < \infty$, then by Fubini's theorem, DCT, and FTC (Fundamental Theorem of Calculus) that

$$\int_\varepsilon^M \frac{\sin x}{x} dx = \int_\varepsilon^M \left[\lim_{N\to\infty} \int_0^N e^{-tx} \sin x \, dt\right] dx \quad \text{(DCT)}$$
$$= \lim_{N\to\infty} \int_\varepsilon^M dx \int_0^N dt e^{-tx} \sin x \quad \text{(DCT)}$$
$$= \lim_{N\to\infty} \int_0^N dt \int_\varepsilon^M dx e^{-tx} \sin x \quad \text{(Fubini)}$$
$$= \lim_{N\to\infty} \int_0^N dt \left[\frac{1}{1+t^2}(-\cos x - t\sin x) e^{-tx}\right]_{x=\varepsilon}^{x=M} \quad \text{(FTC)}$$
$$= \int_0^\infty dt \left[\frac{1}{1+t^2}(-\cos x - t\sin x) e^{-tx}\right]_{x=\varepsilon}^{x=M}. \quad \text{(DCT)}$$

Since

$$\left[\frac{1}{1+t^2}(-\cos x - t\sin x) e^{-tx}\right]_{x=\varepsilon}^{x=M} \to \frac{1}{1+t^2} \text{ as } M\uparrow\infty \text{ and } \varepsilon\downarrow 0,$$

we may again apply DCT with $G(t) = \frac{1}{1+t^2}$ being the dominating function in order to show

$$\int_0^M \frac{\sin x}{x} dx = \lim_{\varepsilon\downarrow 0} \int_\varepsilon^M \frac{\sin x}{x} dx = \lim_{\varepsilon\downarrow 0} \int_0^\infty dt \left[\frac{1}{1+t^2}(-\cos x - t\sin x) e^{-tx}\right]_{x=\varepsilon}^{x=M}$$
$$\overset{DCT}{=} \int_0^\infty dt \left[\frac{1}{1+t^2}(-\cos x - t\sin x) e^{-tx}\right]_{x=0}^{x=M}$$
$$\underset{M\to\infty}{\overset{DCT}{\longrightarrow}} \int_0^\infty \frac{1}{1+t^2} dt = \frac{\pi}{2}.$$

Theorem 3.8 (Linear Change of Variables Theorem). *If $T \in GL(d,\mathbb{R}) = GL(\mathbb{R}^d)$ – the space of $d\times d$ invertible matrices, then the change of variables formula,*

$$\int_{\mathbb{R}^d} f dm_d = |\det T| \int_{\mathbb{R}^d} f\circ T \, dm_d, \tag{3.6}$$

holds for all Riemann integrable functions $f : \mathbb{R}^d \to \mathbb{R}$.

Proof. From Exercise 3.6 below, we know that Eq. (3.6) is valid whenever T is an elementary matrix. From the elementary theory of row reduction in linear algebra, every matrix $T \in GL(\mathbb{R}^d)$ may be expressed as a finite product of the "elementary matrices", i.e. $T = T_1 \circ T_2 \circ \cdots \circ T_n$ where the T_i are elementary matrices. From these assertions we may conclude that

$$\int_{\mathbb{R}^d} f\circ T \, dm_d = \int_{\mathbb{R}^d} f\circ T_1 \circ T_2 \circ \cdots \circ T_n \, dm_d = \frac{1}{|\det T_n|} \int_{\mathbb{R}^d} f\circ T_1\circ T_2 \circ\cdots\circ T_{n-1} \, dm_d.$$

Repeating this procedure $n-1$ more times (i.e. by induction), we find,

$$\int_{\mathbb{R}^d} f\circ T\ dm_d = \frac{1}{|\det T_n|\dots|\det T_1|}\int_{\mathbb{R}^d} f\ dm_d.$$

Finally we use,

$$|\det T_n|\dots|\det T_1| = |\det T_n \dots \det T_1| = |\det(T_1T_2\dots T_n)| = |\det T|$$

in order to complete the proof. ■

3.1 Exercises

Exercise 3.1. Find the value of the following integral;

$$I := \int_1^9 dy \int_{\sqrt{y}}^3 dx\ xe^y.$$

Hint: use Tonelli's theorem to change the order of integrations.

Exercise 3.2. Write the following iterated integral

$$I := \int_0^1 dx \int_{y=x^{2/3}}^1 dy\ xe^{y^4}.$$

as a multiple integral and use this to change the order of integrations and then compute I.

For the next three exercises let

$$B(0,r) := \left\{x\in\mathbb{R}^d : \|x\| = \sqrt{\sum_{i=1}^d x_i^2} < r\right\}$$

be the d – dimensional ball of radius r and let

$$V_d(r) := m_d(B(0,r)) = \int_{\mathbb{R}^d} 1_{B(0,r)} dm_d$$

be its volume. For example,

$$V_1(r) = m_1((-r,r)) = \int_{-r}^r dx = 2r.$$

Exercise 3.3. Suppose that $d=2$, show $m_2(B(0,r)) = \pi r^2$.

Exercise 3.4. Suppose that $d=3$, show $m_3(B(0,r)) = \frac{4\pi}{3}r^3$.

Exercise 3.5. Let $V_d(r) := m_d(B(0,r))$. Show for $d\geq 1$ that

$$V_{d+1}(r) = \int_{-r}^r dz\cdot V_d\left(\sqrt{r^2-z^2}\right) = r\int_{-\pi/2}^{\pi/2} V_d(r\cos\theta)\cos\theta d\theta.$$

Remark 3.9. Using Exercise 3.5 we may deduce again that

$$V_1(r) = m_1((-r,r)) = 2r,$$
$$V_2(r) = r\int_{-\pi/2}^{\pi/2} 2r\cos\theta\cos\theta d\theta = \pi r^2,$$
$$V_3(r) = \int_{-r}^r dz\cdot V_2\left(\sqrt{r^2-z^2}\right) = \int_{-r}^r dz\cdot\pi\left(r^2-z^2\right) = \frac{4\pi}{3}r^3.$$

In principle we may now compute the volume of balls in all dimensions inductively this way.

Exercise 3.6 (Change of variables for elementary matrices). Let $f:\mathbb{R}^d\to\mathbb{R}$ be a continuous function with compact support. Show by direct calculation that;

$$|\det T|\int_{\mathbb{R}^d} f(T(x))\,dx = \int_{\mathbb{R}^d} f(y)\,dy \tag{3.7}$$

for each of the following linear transformations;

1. Suppose that $i<k$ and
$$T(x_1,x_2\dots,x_d) = (x_1,\dots,x_{i-1},x_k,x_{i+1}\dots,x_{k-1},x_i,x_{k+1},\dots x_d),$$
i.e. T swaps the i and k coordinates of x. [In matrix notation T is the identity matrix with the i and k column interchanged.]
2. $T(x_1,\dots x_k,\dots,x_d) = (x_1,\dots,cx_k,\dots x_d)$ where $c\in\mathbb{R}\setminus\{0\}$. [In matrix notation, $T = [e_1|\dots|e_{k-1}|ce_k|e_{k+1}|\dots|e_d]$.]
3. $T(x_1,x_2\dots,x_d) = (x_1,\dots,\overset{\text{i'th spot}}{x_i+cx_k},\dots x_k,\dots x_d)$ where $c\in\mathbb{R}$. [In matrix notation $T = [e_1|\dots|e_i|\dots|e_k+ce_i|e_{k+1}|\dots|e_d]$.

Hint: you should use Fubini's theorem along with the one dimensional change of variables theorem.

[To be more concrete here are examples of each of the T appearing above in the special case $d=4$,

1. If $i=2$ and $k=3$ then $T = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$.

2. If $k = 3$ then $T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

3. If $i = 2$ and $k = 4$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + cx_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

while if $i = 4$ and $k = 2$,

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + cx_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & 1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

3.2 *Appendix: Another approach to the linear change of variables theorem

Let $\langle x, y\rangle$ or $x \cdot y$ denote the standard dot product on $\mathbb{R}^d$, i.e.

$$\langle x, y\rangle = x \cdot y = \sum_{j=1}^{d} x_j y_j.$$

Recall that if A is a $d \times d$ real matrix then the transpose matrix, A^{tr}, may be characterized as the unique real $d \times d$ matrix such that

$$\langle Ax, y\rangle = \left\langle x, A^{\text{tr}} y\right\rangle \text{ for all } x, y \in \mathbb{R}^d.$$

Definition 3.10. *A $d \times d$ real matrix, S, is orthogonal iff $S^{\text{tr}}S = I$ or equivalently stated $S^{\text{tr}} = S^{-1}$.*

Here are a few basic facts about orthogonal matrices.

1. A $d \times d$ real matrix, S, is orthogonal iff $\langle Sx, Sy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^d$.
2. If $\{u_j\}_{j=1}^d$ is any orthonormal basis for $\mathbb{R}^d$ and S is the $d \times d$ matrix determined by $Se_j = u_j$ for $1 \le j \le d$, then S is orthogonal.[2] Here is a proof for your convenience; if $x, y \in \mathbb{R}^d$, then

$$\begin{aligned}\left\langle x, S^{\text{tr}} y\right\rangle &= \langle Sx, y\rangle = \sum_{j=1}^{d} \langle x, e_j\rangle \langle Se_j, y\rangle = \sum_{j=1}^{d} \langle x, e_j\rangle \langle u_j, y\rangle \\ &= \sum_{j=1}^{d} \left\langle x, S^{-1}u_j\right\rangle \langle u_j, y\rangle = \left\langle x, S^{-1}y\right\rangle\end{aligned}$$

   from which it follows that $S^{\text{tr}} = S^{-1}$.
3. If S is orthogonal, then $1 = \det I = \det(S^{\text{tr}}S) = \det S^{\text{tr}} \cdot \det S = (\det S)^2$ and hence $\det S = \pm 1$.

[2] This is a standard result from linear algebra often stated as a matrix, S, is orthogonal iff the columns of S form an orthonormal basis.

The following lemma is a special case the well known **singular value decomposition** or SVD for short..

Lemma 3.11 (SVD). *If T is a real $d \times d$ matrix, then there exists $D = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ with $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_d \ge 0$ and two orthogonal matrices R and S such that $T = RDS$. Further observe that $|\det T| = \det D = \lambda_1 \dots \lambda_d$.*

Proof. Since $T^{\text{tr}}T$ is symmetric, by the spectral theorem there exists an orthonormal basis $\{u_j\}_{j=1}^d$ of $\mathbb{R}^d$ and $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_d \ge 0$ such that $T^{\text{tr}}Tu_j = \lambda_j^2 u_j$ for all j. In particular we have

$$\langle Tu_j, Tu_k\rangle = \left\langle T^{\text{tr}}Tu_j, u_k\right\rangle = \lambda_j^2 \delta_{jk} \ \forall\ 1 \le j, k \le d.$$

Case where $\det T \ne 0$. In this case $\lambda_1 \dots \lambda_d = \det T^{\text{tr}}T = (\det T)^2 > 0$ and so $\lambda_d > 0$. It then follows that $\left\{v_j := \frac{1}{\lambda_j} Tu_j\right\}_{j=1}^d$ is an orthonormal basis for $\mathbb{R}^d$. Let us further let $D = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ (i.e. $De_j = \lambda_j e_j$ for $1 \le j \le d$) and R and S be the orthogonal matrices defined by

$$Re_j = v_j \text{ and } S^{\text{tr}}e_j = S^{-1}e_j = u_j \text{ for all } 1 \le j \le d.$$

Combining these definitions with the identity, $Tu_j = \lambda_j v_j$, implies

$$TS^{-1}e_j = \lambda_j Re_j = R\lambda_j e_j = RDe_j \text{ for all } 1 \le j \le d,$$

i.e. $TS^{-1} = RD$ or equivalently $T = RDS$.

Case where $\det T = 0$. In this case there exists $1 \le k < d$ such that $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_k > 0 = \lambda_{k+1} = \dots = \lambda_d$. The only modification needed for the above proof is to define $v_j := \frac{1}{\lambda_j} Tu_j$ for $j \le k$ and then extend choose $v_{k+1}, \dots, v_d \in \mathbb{R}^d$ so that $\{v_j\}_{j=1}^d$ is an orthonormal basis for $\mathbb{R}^d$. We still have $Tu_j = \lambda_j v_j$ for all j and so the proof in the first case goes through without change. ■

In the next theorem we will make use the characterization of m_d that it is the unique measure on $(\mathbb{R}^d)$ which is translation invariant assigns unit measure to $[0,1]^d$.

Theorem 3.12. *If T is a real $d \times d$ matrix, then $m_d \circ T = |\det T|\, m_d$.*

Proof. Recall that we know $m_d T = \delta(T)\, m_d$ for some $\delta(T) \in (0, \infty)$ and so we must show $\delta(T) = |\det T|$. We first consider two special cases.

1. If $T = R$ is orthogonal and B is the unit ball in $\mathbb{R}^d$,[3] then $\delta(R)\, m_d(B) = m_d(RB) = m_d(B)$ from which it follows $\delta(R) = 1 = |\det R|$.
2. If $T = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ with $\lambda_i \geq 0$, then $D[0,1]^d = [0, \lambda_1] \times \cdots \times [0, \lambda_d]$ so that

$$\delta(D) = \delta(D)\, m_d\left([0,1]^d\right) = m_d\left(D[0,1]^d\right) = \lambda_1 \ldots \lambda_d = \det D.$$

3. For the general case we use singular value decomposition (Lemma 3.11) to write $T = RDS$ and then find

$$\delta(T) = \delta(R)\,\delta(D)\,\delta(S) = 1 \cdot \det D \cdot 1 = |\det T|.$$

■

[3] $B = \left\{x \in \mathbb{R}^d : \|x\| < 1\right\}$.

Part II

Multi-Linear Algebra

4

Properties of Volumes

The goal of this short chapter is to show how computing volumes naturally gives rise to the idea of the key objects of this book, namely differential forms, i.e. alternating tensors. The point is that these objects are intimately related to computing areas and volumes.

Let $Q^n := \{x \in \mathbb{R}^n : 0 \le t_j \le 1 \ \forall\ j\} = [0,1]^n$ be the unit cube in $\mathbb{R}^n$ which we I think all agree should have volume equal to 1. For n-vectors, $a_1, \dots, a_n \in \mathbb{R}^n$, let

$$P(a_1, \dots, a_n) = [a_1 | \dots | a_n]\, Q = \left\{ \sum_{j=1}^n t_j a_j : 0 \le t_j \le 1 \ \forall\ j \right\}$$

be the parallelepiped spanned by $(a_1, \dots, a_n)$ and let

$$\delta(a_1, \dots, a_n) = \text{“signed”}\ \ \mathrm{Vol}\,(P(v_1, \dots, v_n)).$$

be the **signed volume** of the parallelepiped. To find the properties of this volume, let us fix $\{a_i\}_{i=1}^{n-1}$ and consider the function, $F(a_n) = \delta(a_1, \dots, a_n)$. This is easily computed using the formula of a slant cylinder, see Figure 4.1, as

$$F(a_n) = \delta(a_1, \dots, a_n) = \pm\,(\text{Area of base })\cdot \mathbf{n} \cdot a_n \tag{4.1}$$

where $\mathbf{n}$ is a unit vector orthogonal to $\{a_1, \dots, a_{n-1}\}$.

Example 4.1. When $n = 2$, let us first verify Eq. (4.1) in this case by considering

$$\delta(ae_1, b) = \int_0^{b_2} [\text{slice width}]_h\, dh = \int_0^{b_2} a dh = a\,(b \cdot e_2).$$

The sign in Eq. (4.1) is positive if $(a_1, \dots, a_{n-1}, \mathbf{n})$ is “positively oriented,” think of the right hand rule in dimensions 2 and 3. This show $a_n \to \delta(a_1, \dots, a_{n-1}, a_n)$ is a linear function. A similar argument shows

$$a_j \to \delta(a_1, \dots, a_j, \dots, a_n)$$

is linear as well. That is δ is a “**multi-linear function**” of its arguments. We further have that $\delta(a_1, \dots, a_n) = 0$ if $a_i = a_j$ for any $i \ne j$ as the parallelepiped generated by $(a_1, \dots, a_n)$ is degenerate and zero volume. We summarize these two properties by saying δ is an **alternating multi-linear n-function** on $\mathbb{R}^n$. Lastly as $P(e_1, \dots e_n) = Q$ we further have that

Fig. 4.1. The volume of a slant cylinder is it’s height, $\mathbf{n} \cdot a_n$.

$$\delta(e_1, \dots, e_n) = 1. \tag{4.2}$$

Fact 4.2 *We are going to show there is precisely one alternating multi-linear n-function, δ, on $\mathbb{R}^n$ such that Eq. (4.2) holds. This function is in fact the function you know and the determinant.*

Example 4.3 ($n = 1$ Det). When $n = 1$ we must have $\delta([a]) = \pm a$, we choose a by convention.

Example 4.4 ($n = 2$ Det). When $n = 2$, we find

$$\begin{aligned}
\delta(a, b) &= \delta(a_1 e_1 + a_2 e_2, b) = a_1 \delta(e_1, b) + a_2 \delta(e_2, b) \\
&= a_1 \delta(e_1, b_1 e_1 + b_2 e_2) + a_2 \delta(e_2, b_1 e_1 + b_2 e_2) \\
&= a_1 b_2 \delta(e_1, e_2) + a_2 b_1 \delta(e_2, e_1) = a_1 b_2 - a_2 b_1 \\
&= \det[a|b].
\end{aligned}$$

We now proceed to develop the theory of alternating multilinear functions in general.

Multi-linear Functions (Tensors)

For the rest of these notes, V will denote a real vector space. Typically we will assume that $n = \dim V < \infty$.

Example 5.1. $V = \mathbb{R}^n$, subspaces of $\mathbb{R}^n$, polynomials of degree $< n$. The most general overarching vector space is typically

$$V = \mathcal{F}(X, \mathbb{R}) = \{\text{all functions from } X \text{ to } \mathbb{R}\}.$$

An interesting subspace is the space of **finitely supported functions,**

$$\mathcal{F}_f(X, \mathbb{R}) = \{f \in \mathcal{F}(X, \mathbb{R}) : \#(\{f \neq 0\}) < \infty\},$$

where

$$\{f \neq 0\} = \{x \in X : f(x) \neq 0\}.$$

5.1 Basis and Dual Basis

Definition 5.2. *Let V^* denote the **dual space** of V, i.e. the vector space of all linear functions, $\ell : V \to \mathbb{R}$.*

Example 5.3. Here are some examples;

1. If $V = \mathbb{R}^n$, then $\ell(v) = w \cdot v = w^{\mathrm{tr}} v$ for $w \in V$ is in V^*.
2. V = polynomials of deg $< n$ is a vector space and $\ell_0(p) = p(0)$ or $\ell(p) = \int_{-1}^{1} p(x)\, dx$ given $\ell \in V^*$.
3. For $\{a_j\}_{j=1}^p \subset \mathbb{R}$ and $\{x_j\}_{j=1}^p \subset X$, let $\ell(f) = \sum_{j=1}^p a_j f(x_j)$, then $\ell \in \mathcal{F}(X, \mathbb{R})^*$.

Notation 5.4 *Let $\beta := \{e_j\}_{j=1}^n$ be a basis for V and $\beta^* := \{\varepsilon_j\}_{j=1}^n$ be its **dual basis**, i.e.*

$$\varepsilon_j\left(\sum_{i=1}^n a_i e_i\right) := a_j \text{ for all } j.$$

The book denotes ε_j as e_j^*. In case, $V = \mathbb{R}^n$ and $\{e_j\}_{j=1}^n$ is the standard basis, we will later write dx_j for $\varepsilon_j = e_j^*$.

Example 5.5. If $V = \mathbb{R}^n$ and $\beta = \{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$, then $\varepsilon_i(v) = e_i \cdot v = e_i^{\mathrm{tr}} v$ for $1 \leq i \leq n$ is the dual basis to β.

Example 5.6. If V denotes polynomials of degree $< n$, with basis $e_j(x) = x^j$ for $0 \leq j < n$, then $\varepsilon_j(p) := \frac{1}{j!} p^{(j)}(0)$ is the associated dual basis.

Example 5.7. For $x \in X$, let $\delta_x \in \mathcal{F}_f(X, \mathbb{R})$ be defined by

$$\delta_x(y) = 1_{\{x\}}(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases}.$$

One may easily show that $\{\delta_x\}_{x \in X}$ is a basis for $\mathcal{F}_f(X, \mathbb{R})$ and for $f \in \mathcal{F}_f(X, \mathbb{R})$,

$$f = \sum_{x : f(x) \neq 0} f(x)\, \delta_x.$$

The dual basis ideas are complicated in this case when X is an infinite set as Vaki mentioned in section. We will not consider such "infinite dimensional" problems in these notes.

Proposition 5.8. *Continuing the notation above, then*

$$v = \sum_{j=1}^n \varepsilon_j(v)\, e_j \quad \text{for all } v \in V, \text{ and} \tag{5.1}$$

$$\ell = \sum_{j=1}^n \ell(e_j)\, \varepsilon_j \quad \text{for all } \ell \in V^*. \tag{5.2}$$

Moreover, β^*, is indeed a basis for V^*.

Proof. Because $\{e_j\}$ is a basis, we know that $v = \sum_{j=1}^n a_j e_j$. Applying ε_k to this formula shows

$$\varepsilon_k(v) = \sum_{j=1}^n a_j \varepsilon_k(e_j) = a_k$$

and hence Eq. (5.1) holds. Now apply ℓ to Eq. (5.1) to find,

$$\ell(v)=\sum_{j=1}^n \varepsilon_j(v)\,\ell(e_j)=\sum_{j=1}^n \ell(e_j)\,\varepsilon_j(v)=\left(\sum_{j=1}^n \ell(e_j)\,\varepsilon_j\right)(v)$$

which proves Eq. (5.2). From Eq. (5.2) we know that $\{\varepsilon_j\}_{j=1}^n$ spans V^*. Moreover if

$$\mathbf{0}=\sum_{j=1}^n a_j\varepsilon_j \implies 0=\mathbf{0}(e_k)=\sum_{j=1}^n a_j\varepsilon_j(e_k)=a_k$$

which shows $\{\varepsilon_j\}_{j=1}^n$ is linearly independent. ■

Exercise 5.1. Let $V=\mathbb{R}^n$ and $\beta=\{u_j\}_{j=1}^n$ be a basis for $\mathbb{R}^n$. Recall that every $\ell\in(\mathbb{R}^n)^*$ is of the form $\ell_a(x)=a\cdot x$ for some $a\in\mathbb{R}^n$. Thus the dual basis, β^*, to β can be written as $\left\{u_j^*=\ell_{a_j}\right\}_{j=1}^n$ for some $\{a_j\}_{j=1}^n\subset\mathbb{R}^n$. In this problem you are asked to show how to find the $\{a_j\}_{j=1}^n$ by the following steps.

1. Show that for $j\in[n]$, a_j must solve the following k-linear equations;
$$\delta_{j,k}=\ell_{a_j}(u_k)=a_j\cdot u_k=u_k^{\mathrm{tr}}a_j \text{ for } k\in[n]. \tag{5.3}$$
2. Let $U:=[u_1|\dots|u_n]$ (i.e. the columns of U are the vectors from β). Show that the equations in (5.3) may be written in matrix form as, $U^{\mathrm{tr}}a_j=e_j$, where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.
3. Conclude that $a_j=[U^{\mathrm{tr}}]^{-1}e_j$ or equivalently;
$$[a_1|\dots|a_n]=\left[U^{\mathrm{tr}}\right]^{-1}$$

Exercise 5.2. Let $V=\mathbb{R}^2$ and $\beta=\{u_1,u_2\}$, where

$$u_1=\begin{bmatrix}1\\3\end{bmatrix} \text{ and } u_2=\begin{bmatrix}-1\\1\end{bmatrix}.$$

Find $a_1,a_2\in\mathbb{R}^2$ explicitly so that explicitly the dual basis $\beta^*:=\{u_1^*=\ell_{a_1},u_2^*=\ell_{a_2}\}$ is the dual basis to β. Please explicitly verify your answer is correct by showing $u_j^*(u_k)=\delta_{jk}$.

Exercise 5.3. Let $V=\mathbb{R}^n$, $\{a_j\}_{j=1}^k\subset V$, and $\ell_j(x)=a_j\cdot x$ for $x\in\mathbb{R}^n$ and $j\in[k]$. Show $\{\ell_j\}_{j=1}^k\subset V^*$ is a linearly independent set if and only if $\{a_j\}_{j=1}^k\subset V$ is a linearly independent set.

Exercise 5.4. Let $V=\mathbb{R}^n$, $\{a_j\}_{j=1}^k\subset V$, and $\ell_j(x)=a_j\cdot x$ for $x\in\mathbb{R}^n$ and $j\in[k]$. If $\{\ell_j\}_{j=1}^k\subset V^*$ is a linearly independent set, show there exists $\{u_j\}_{j=1}^k\subset V$ so that $\ell_i(u_j)=\delta_{ij}$ for $i,j\in[k]$. Here is a possible outline.

1. Using Exercise 5.3 and citing a basic fact from Linear algebra, you may choose $\{a_j\}_{j=k+1}^n\subset V$ so that $\{a_j\}_{j=1}^n$ is a basis for V.
2. Argue that it suffices to find $u_j\in V$ so that
$$a_i\cdot u_j=\delta_{ij} \text{ for all } i,j\in[n]. \tag{5.4}$$
3. Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$ and $A:=[a_1|\dots|a_n]$ be the $n\times n$ matrix with columns given by that $\{a_j\}_{j=1}^n$. Show that the Eqs. (5.4) may be written as
$$A^{\mathrm{tr}}u_j=e_j \text{ for } j\in[n]. \tag{5.5}$$
4. Cite basic facts from linear algebra to explain why $A:=[a_1|\dots|a_n]$ and A^{tr} are both invertible $n\times n$ matrices.
5. Argue that Eq. (5.5) has a unique solution, $u_j\in\mathbb{R}^n$, for each j.

5.2 Multi-linear Forms

Definition 5.9. *A function $T:V^k\to\mathbb{R}$ is **multi-linear** (k-linear to be precise) if for each $1\le i\le k$, the map*

$$V\ni v_i\to T(v_1,\dots,v_i,\dots v_k)\in\mathbb{R}$$

*is linear. We denote the space of k-linear maps by $\mathcal{L}^k(V)$ and element of this space is a k-**tensor on (in)** V.*

Lemma 5.10. *Note that $\mathcal{L}^k(V)$ is a **vector subspace** of all functions from $V^k\to\mathbb{R}$.*

Example 5.11. If $\ell_1,\dots,\ell_k\in V^*$, we let $\ell_1\otimes\dots\otimes\ell_k\in\mathcal{L}^k(V)$ be defined

$$(\ell_1\otimes\dots\otimes\ell_k)(v_1,\dots,v_k)=\prod_{j=1}^k \ell_j(v_j)$$

for all $(v_1,\dots,v_k)\in V^k$.

Exercise 5.5. In this problem, let

$$v=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} \text{ and } w=\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix}.$$

Which of the following functions formulas for T define a 2-tensors on $\mathbb{R}^3$. Please justify your answers.

1. $T(v,w) = v_1 w_3 + v_1 w_2 + v_2 w_1 + 7v_1 w_1$.
2. $T(v,w) = v_1 + 7v_1 + v_2$.
3. $T(v,w) = v_1^2 w_3 + v_2 w_1$,
4. $T(v,w) = \sin(v_1 w_3 + v_1 w_2 + v_2 w_1 + 7v_1 w_1)$.

Theorem 5.12. *If $\{e_j\}_{j=1}^n$ is a basis for V, then $\{\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} : j_i \in [n]\}$ is a basis for $\mathcal{L}^k(V)$ and moreover if $T \in \mathcal{L}^k(V)$, then*

$$T = \sum_{j_1,\dots,j_k \in [n]} T(e_{j_1},\dots,e_{j_k}) \cdot \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} \tag{5.6}$$

and this decomposition is unique. [One might identify 2-tensors with matrices via $T \to A_{ij} := T(e_i, e_j)$.]

Proof. Given $v_1,\dots,v_k \in V$, we know that

$$v_i = \sum_{j_i=1}^n \varepsilon_{j_i}(v_i)\, e_{j_i}$$

and hence

$$\begin{aligned} T(v_1,\dots,v_k) &= T\left(\sum_{j_1=1}^n \varepsilon_{j_1}(v_1)\, e_{j_1}, \dots, \sum_{j_k=1}^n \varepsilon_{j_k}(v_k)\, e_{j_k}\right) \\ &= \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n T(\varepsilon_{j_1}(v_1)\, e_{j_1}, \dots, \varepsilon_{j_k}(v_k)\, e_{j_k}) \\ &= \sum_{j_1,\dots,j_k \in [n]} T(e_{j_1},\dots,e_{j_k})\, \varepsilon_{j_1}(v_1) \dots \varepsilon_{j_k}(v_k) \\ &= \sum_{j_1,\dots,j_k \in [n]} T(e_{j_1},\dots,e_{j_k})\, \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}(v_1,\dots,v_k). \end{aligned}$$

This verifies that Eq. (5.6) holds and also that

$$\{\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} : j_i \in [n]\} \text{ spans } \mathcal{L}^k(V).$$

For linearly independence, if $\{a_{j_1,\dots,j_k}\} \subset \mathbb{R}$ are such that

$$\mathbf{0} = \sum_{j_1,\dots,j_k \in [n]} a_{j_1,\dots,j_k} \cdot \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k},$$

then evaluating this expression at $(e_{i_1},\dots e_{i_k})$ shows

$$\begin{aligned} 0 = \mathbf{0}(e_{i_1},\dots e_{i_k}) &= \sum_{j_1,\dots,j_k \in [n]} a_{j_1,\dots,j_k} \cdot \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}(e_{i_1},\dots e_{i_k}) \\ &= \sum_{j_1,\dots,j_k \in [n]} a_{j_1,\dots,j_k} \cdot \varepsilon_{j_1}(e_{i_1}) \dots \varepsilon_{j_k}(e_{i_k}) \\ &= \sum_{j_1,\dots,j_k \in [n]} a_{j_1,\dots,j_k} \cdot \delta_{j_1,i_1} \dots \delta_{j_k,i_k} = a_{i_1\dots,i_k} \end{aligned}$$

which shows $a_{i_1\dots,i_k} = 0$ for all indices and completes the proof. ■

Corollary 5.13. $\dim \mathcal{L}^k(V) = n^k$.

Definition 5.14. *If $S \in \mathcal{L}^p(V)$ and $T \in \mathcal{L}^q(V)$, then we define $S \otimes T \in \mathcal{L}^{p+q}(V)$ by,*

$$S \otimes T(v_1,\dots,v_p,w_1,\dots,w_q) = S(v_1,\dots,v_p)\, T(w_1,\dots,w_q).$$

Definition 5.15. *If $A : V \to W$ is a linear transformation, and $T \in \mathcal{L}^k(W)$, then we define the **pull back** $A^*T \in \mathcal{L}^k(V)$ by*

$$(A^*T)(v_1,\dots,v_k) = A(Tv_1,\dots,Tv_k).$$

$$\begin{aligned} V \times \cdots \times V &\longrightarrow W \times \cdots \times W \longrightarrow \mathbb{R} \\ (v_1,\dots,v_k) &\longrightarrow (Av_1,\dots,Av_k) \longrightarrow T(Av_1,\dots,Av_k). \end{aligned}$$

It is called pull back since $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ maps the opposite direction of A.

Remark 5.16. As shown in the book the tensor product satisfies

$$\begin{aligned} (R \otimes S) \otimes T &= R \otimes (S \otimes T), \\ T \otimes (S_1 + S_2) &= T \otimes S_1 + T \otimes S_2, \\ (S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T, \\ &\vdots \end{aligned}$$

Remark 5.17. The definition of $T_1 \otimes T_2$ and the associated "tensor algebra." [Typically the tensor symbol, $\otimes$, in mathematics is used to denote the product of two functions which have distinct arguments. Thus if $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are two functions on the sets X and Y respectively, then $f \otimes g : X \times Y \to \mathbb{R}$ is defined by

$$(f \otimes g)(x,y) = f(x)\, g(y).$$

In contrast, if $Y = X$ we may also define the more familiar product, $f \cdot g : X \to \mathbb{R}$, by

$$(f \cdot g)(x) = f(x)\, g(x).$$

Incidentally, the relationship between these two products is

$$(f \cdot g)(x) = (f \otimes g)(x,x).$$

Lemma 5.18. *The product, $\otimes$, defined in the previous remark is associative and distributive over addition. We also have for $\lambda \in \mathbb{R}$, that*

$$(\lambda f) \otimes g = f \otimes (\lambda g) = \lambda \cdot f \otimes g. \tag{5.7}$$

That is $\otimes$ satisfies the rules we expect of a "product," i.e. plays nicely with the vector space operations.

Proof. If $h : Z \to \mathbb{R}$ is another function, then

$$\begin{aligned}((f \otimes g) \otimes h)(x, y, z) &= (f \otimes g)(x, y) \cdot h(z) = (f(x) g(y)) h(z) \\ &= f(x)(g(y) h(z)) = (f \otimes (g \otimes h))(x, y, z).\end{aligned}$$

This shows in general that $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, i.e. $\otimes$ is associative.

Similarly if $Z = Y$, then

$$\begin{aligned}(f \otimes (g + h))(x, y) &= f(x) \cdot (g + h)(y) = f(x) \cdot (g(y) + h(y)) \\ &= f(x) \cdot g(y) + f(x) \cdot h(y) \\ &= (f \otimes g)(x, y) + (f \otimes h)(x, y) \\ &= (f \otimes g + f \otimes h)(x, y)\end{aligned}$$

from which we conclude that

$$f \otimes (g + h) = f \otimes g + f \otimes h$$

Similarly one shows $(f + h) \otimes g = f \otimes g + h \otimes g$ when $Z = X$. These are the distributive rules. The easy proof of Eq. (5.7) is left to the reader. ■

Alternating Multi-linear Functions

Definition 6.1. *$T \in \mathcal{L}^k(V)$ is said to be* ***alternating*** *if $T(v_1, \dots, v_k) = -T(w_1, \dots, w_k)$ whenever $(w_1, \dots, w_k)$ is the list $(v_1, \dots, v_k)$ with any two entries interchanged. We denote the subspace[1] of alternating functions by $\mathcal{A}^k(V)$ or by $\Lambda^k(V^*)$ with the convention that $\mathcal{A}^0(V) = \Lambda^0(V^*) = \mathbb{R}$. An element, $T \in \mathcal{A}^k(V) = \Lambda^k(V^*)$ will be called a k-**form.***

Remark 6.2. If $f(v, w)$ is a multi-linear function such that $f(v, v) = 0$ then for all $v, w \in V$, then

$$\begin{aligned} 0 &= f(v + w, v + w) = f(v, v) + f(w, w) + f(v, w) + f(w, v) \\ &= f(w, v) + f(v, w) \implies f(v, w) = -f(w, v). \end{aligned}$$

Conversely, if $f(v, w) = -f(w, v)$ for all v, and w, then $f(v, v) = -f(v, v)$ which shows $f(v, v) = 0$.

Lemma 6.3. *If $T \in \mathcal{L}^k(V)$, then the following are equivalent;*

1. *T is alternating, i.e. $T \in \Lambda^k(V^*)$.*
2. *$T(v_1, \dots, v_k) = 0$ whenever any two distinct entries are equal.*
3. *$T(v_1, \dots, v_k) = 0$ whenever any two consecutive entries are equal.*

Proof. 1. $\implies$ 2. If $v_i = v_j$ for some $i < j$ and $T \in \Lambda^k(V^*)$, then by interchanging the i and j entries we learn that $T(v_1, \dots, v_k) = -T(v_1, \dots, v_k)$ which implies $T(v_1, \dots, v_k) = 0$.

2. $\implies$ 3. This is obvious.

3. $\implies$ 1. Applying Remark 6.2 with

$$f(v, w) := T(v_1, \dots, v_{j-1}, v, w, v_{j+2}, \dots, v_k)$$

shows that $T(v_1, \dots, v_k) = -T(w_1, \dots, w_k)$ if $(w_1, \dots, w_k)$ is the list $(v_1, \dots, v_k)$ with the j and $j + 1$ entries interchanged. If $(w_1, \dots, w_k)$ is the list $(v_1, \dots, v_k)$ with the $i < j$ entries interchanged, then $(w_1, \dots, w_k)$ can be transformed back to $(v_1, \dots, v_k)$ by an odd number of nearest neighbor interchanges and therefore it follows by what we just proved that

$$T(v_1, \dots, v_k) = -T(w_1, \dots, w_k).$$

For example, to transform

$$(v_1, v_5, v_3, v_4, v_2, v_6) \text{ back to } (v_1, v_2, v_3, v_4, v_5, v_6),$$

we transpose v_5 with its nearest neighbor to the right 2 times to arrive at the list $(v_1, v_3, v_4, v_5, v_2, v_6)$. We then we transpose v_2 with its nearest neighbor to the left 3 times to arrive (after a sum total of 5 adjacent transpositions) back to the list $(v_1, v_2, v_3, v_4, v_5, v_6)$. For the general $i < j$ the number of adjacent transposition needed needed is $2(j - i) - 1$ which is always odd. ■

Exercise 6.1. If $T \in \Lambda^k(V^*)$, show $T(v_1, \dots, v_k) = 0$ whenever $\{v_i\}_{i=1}^k \subset V$ are linearly dependent.

A simple consequence of this exercise is the following basic lemma.

Lemma 6.4. *If $T \in \Lambda^k(V^*)$ with $k > \dim V$, then $T \equiv 0$, i.e. $\Lambda^k(V^*) = \{0\}$ for all $k > \dim V$.*

At this point we have not given any non-zero examples of alternating forms. The next definition and proposition gives a mechanism for constructing many (in fact a full basis of) alternating forms.

Definition 6.5. *For $\ell \in V^*$ and $\varphi \in \Lambda^k(V^*)$, let $L_\ell \varphi$ be the multi-linear $k+1$ – form on V defined by*

$$(L_\ell \varphi)(v_0, \dots, v_k) = \sum_{i=0}^{k} (-1)^i \, \ell(v_i) \, \varphi(v_0, \dots, \hat{v}_i, \dots, v_k).$$

for all $(v_0, \dots, v_k) \in V^{k+1}$.

Proposition 6.6. *If $\ell \in V^*$ and $\varphi \in \Lambda^k(V^*)$, then $(L_\ell \varphi) \in \Lambda^{k+1}(V^*)$.*

Proof. We must show $L_\ell \varphi$ is alternating. According to Lemma 6.3, it suffices to show $(L_\ell \varphi)(v_0, \dots, v_k) = 0$ whenever $v_j = v_{j+1}$ for some $0 \le j < k$. So suppose that $v_j = v_{j+1}$, then since φ is alternating

[1] The alternating conditions are linear equations that $T \in \mathcal{L}^k(V)$ must satisfy and hence $\mathcal{A}^k(V)$ is a subspace of $\mathcal{L}^k(V)$.

$$\begin{aligned}(L_\ell\varphi)(v_0,\dots,v_k) &= \sum_{i=0}^{k}(-1)^i\,\ell(v_i)\,\varphi(v_0,\dots,\hat{v}_i,\dots,v_k)\\ &= \sum_{i=j}^{j+1}(-1)^i\,\ell(v_i)\,\varphi(v_0,\dots,\hat{v}_i,\dots,v_k)\\ &= \left[(-1)^j+(-1)^{j+1}\right]\ell(v_j)\,\varphi(v_0,\dots,\hat{v}_j,\dots,v_k)=0.\end{aligned}$$

■

Proposition 6.7. *Let $\{e_i\}_{i=1}^n$ be a basis for V and $\{\varepsilon_i\}_{i=1}^n$ be its dual basis for V^*. Then*

$$\varphi_j := L_{\varepsilon_j}L_{\varepsilon_{j+1}}\dots L_{\varepsilon_{n-1}}\varepsilon_n \in \Lambda^{n-j+1}(V^*)\setminus\{0\}$$

for all $j\in[n]$ and in particular, $\dim\Lambda^k(V^*)\geq 1$ *for all $0\leq k\leq n$. [We will see in Theorem 6.33 below that* $\dim\Lambda^k(V^*)=\binom{n}{k}$ *for all $0\leq k\leq n$.]*

Proof. We will show that φ_j is not zero by showing that

$$\varphi_j(e_j,\dots,e_n)=1 \text{ for all } j\in[n].$$

This is easily proved by (reverse induction) on j. Indeed, for $j=n$ we have $\varphi_n(e_n)=\varepsilon_n(e_n)=1$ and for $1\leq j<n$ we have $\varphi_j := L_{\varepsilon_j}\varphi_{j+1}$ so that

$$\begin{aligned}\varphi_j(e_j,\dots,e_n) &= \sum_{k=j}^{n}(-1)^{k-j}\,\varepsilon_j(e_k)\,\varphi_{j+1}(e_j,\dots,\widehat{e_k},\dots,e_n)\\ &= \varphi_{j+1}(\widehat{e_j},e_{j+1},\dots,e_n)=\varphi_{j+1}(e_{j+1},\dots,e_n)=1\end{aligned}$$

wherein we used the induction hypothesis for the last equality. This completes the proof for $j\in[n]$. Finally for $k=0$, we have $\Lambda^0(V^*)=\mathbb{R}$ by convention and hence $\dim\Lambda^0(V^*)=1$. ■

Notation 6.8 *Fix a basis $\{e_i\}_{i=1}^n$ of V with dual basis, $\{\varepsilon_i\}_{i=1}^n\subset V^*$, and then let*

$$\varphi=\varphi_1=L_{\varepsilon_1}L_{\varepsilon_2}\dots L_{\varepsilon_{n-1}}\varepsilon_n. \quad (6.1)$$

Proposition 6.9. *When $V=\mathbb{R}^n$ and $\{e_j\}_{j=1}^n$ is the standard basis for V, then*

$$\varphi(a_1,\dots,a_n)=\det[a_1|\dots|a_n]\ \ \forall\ \{a_i\}_{i=1}^n\subset\mathbb{R}^n. \quad (6.2)$$

Proof. Let us note that if

$$\begin{aligned}&\varphi(a_1,\dots,ca_i,\dots,a_n)=c\varphi(a_1,\dots,a_n)\ \text{ and}\\ &\varphi(a_1,\dots,a_i,\dots,a_j+ca_i,\dots,a_n)\\ &\quad=\varphi(a_1,\dots,a_i,\dots,a_j,\dots,a_n)+c\varphi(a_1,\dots,a_i,\dots,a_i,\dots,a_n)\\ &\quad=\varphi(a_1,\dots,a_n)+c\cdot 0=\varphi(a_1,\dots,a_n).\end{aligned}$$

Thus both φ and det behave the same way under column operations and agree with $a_i=e_i$ which already shows Eq. (6.2) holds when $\{a_i\}_{i=1}^n$ are linearly independent. As both sides of Eq. (6.2) are zero when $\{a_i\}_{i=1}^n$ are linearly dependent, the proof is complete. ■

Definition 6.10 (Signature of σ). *For $\sigma\in\Sigma_n$, let*

$$(-1)^\sigma := \varphi(e_{\sigma1},\dots,e_{\sigma n}),$$

where φ is as in Notation 6.8. We call $(-1)^\sigma$ the ***sign of the permutation***, *σ.*

Lemma 6.11. *If $\sigma\in\Sigma_n$, then $(-1)^\sigma$ may be computed as $(-1)^N$ where N is the number of transpositions*[2] *needed to bring $(\sigma1,\dots,\sigma n)$ back to $(1,2,\dots,n)$ and so $(-1)^\sigma$ does not depend on the choices made in defining $(-1)^\sigma$. Moreover, if $\{v_j\}_{j=1}^n\subset V$, then*

$$\varphi(v_{\sigma1},\dots,v_{\sigma n})=(-1)^\sigma\varphi(v_1,\dots,v_d)\ \ \forall\ \sigma\in\Sigma_n.$$

Proof. Straightforward and left to the reader. ■

Corollary 6.12. *If $\sigma\in\Sigma_n$ is a transposition, then $(-1)^\sigma=-1$.*

Proof. This has already been proved in the course of proving Lemma 6.3. ■

Lemma 6.13. *If $\sigma,\tau\in\Sigma_n$, then $(-1)^{\sigma\tau}=(-1)^\sigma(-1)^\tau$ and in particular it follows that $(-1)^{\sigma^{-1}}=(-1)^\sigma$.*

Proof. Let $v_j:=e_{\sigma j}$ for each j, then

$$\begin{aligned}(-1)^{\sigma\tau} &:= \varphi(e_{\sigma\tau1},\dots,e_{\sigma\tau n})=\varphi(v_{\tau1},\dots,v_{\tau n})\\ &= (-1)^\tau\varphi(v_1,\dots,v_d)=(-1)^\tau\varphi(e_{\sigma1},\dots,e_{\sigma d})\\ &= (-1)^\tau(-1)^\sigma.\end{aligned}$$

■

Lemma 6.14. *A multi-linear map, $T\in\mathcal{L}^k(V)$, is alternating (i.e. $T\in\mathcal{A}^k(V)=\Lambda^k(V^*)$) iff*

$$T(v_{\sigma1},\dots,v_{\sigma k})=(-1)^\sigma T(v_1,\dots,v_k)\ \text{ for all } \sigma\in\Sigma_k.$$

[2] N is not unique but $(-1)^N=(-1)^\sigma$ is unique.

Proof. $(-1)^\sigma = (-1)^N$ where N is the number of transpositions need to transform σ to the identity permutation. For each of these transpositions produce an interchange of entries of the T function and hence introduce a (-1) factor. Thus in total,

$$T(v_{\sigma 1}, \dots, v_{\sigma k}) = (-1)^N T(v_1, \dots, v_k) = (-1)^\sigma T(v_1, \dots, v_k).$$

The converse direction follows from the simple fact that the sign of a transposition is -1. ■

Notation 6.15 (Pull Backs) *Let V and W be finite dimensional vector spaces. To each linear transformation, $T : V \to W$, there is linear transformation, $T^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ defined by*

$$(T^*\varphi)(v_1, \dots, v_k) := \varphi(Tv_1, \dots, Tv_k)$$

for all $\varphi \in \Lambda^k(W^*)$ and $(v_1, \dots, v_k) \in V^k$. [We leave to the reader the easy proof that $T^*\varphi$ is indeed in $\Lambda^k(V^*)$.]

Exercise 6.2. Let $V, W,$ and Z be three finite dimensional vector spaces and suppose that $V \overset{T}{\to} W \overset{S}{\to} Z$ are linear transformations. Noting that $V \overset{ST}{\to} Z$, show $(ST)^* = T^* S^*$.

6.1 Structure of $\Lambda^n(V^*)$ and Determinants

In what follows we will continue to use the notation introduced in Notation 6.8.

Proposition 6.16 (Structure of $\Lambda^n(V^*)$). *If $\psi \in \Lambda^n(V^*)$, then $\psi = \psi(e_1, \dots, e_n)\varphi$ and in particular, $\dim \Lambda^n(V^*) = 1$. Moreover for any $\{v_j\}_{j=1}^n \subset V$,*

$$\varphi(v_1, \dots, v_n) = \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1}(v_1) \dots \varepsilon_{\sigma n}(v_n)$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_1(v_{\sigma 1}) \dots \varepsilon_n(v_{\sigma n}).$$

The first equality may be rewritten as

$$\varphi = \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1} \otimes \dots \otimes \varepsilon_{\sigma n}.$$

Proof. Let $\{v_j\}_{j=1}^n \subset V$ and recall that

$$v_j = \sum_{k_j=1}^n \varepsilon_{k_j}(v_j) e_{k_j}.$$

Using the fact that ψ is multi-linear and alternating we find,

$$\psi(v_1, \dots, v_n) = \sum_{k_1, \dots, k_n = 1}^n \left[\prod_{j=1}^n \varepsilon_{k_j}(v_j)\right] \psi(e_{k_1}, \dots, e_{k_n})$$
$$= \sum_{\sigma \in \Sigma_n} \left[\prod_{j=1}^n \varepsilon_{\sigma j}(v_j)\right] \psi(e_{\sigma_1}, \dots, e_{\sigma n})$$
$$= \sum_{\sigma \in \Sigma_n} \left[\prod_{j=1}^n \varepsilon_{\sigma j}(v_j)\right] (-1)^\sigma \psi(e_1, \dots, e_n)$$

while the same computation shows

$$\varphi(v_1, \dots, v_n) = \sum_{\sigma \in \Sigma_n} \left[\prod_{j=1}^n \varepsilon_{\sigma j}(v_j)\right] (-1)^\sigma \varphi(e_1, \dots, e_n)$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1}(v_1) \dots \varepsilon_{\sigma n}(v_n).$$

Lastly let us note that

$$\prod_{j=1}^n \varepsilon_{\sigma j}(v_j) = \prod_{j=1}^n \varepsilon_{\sigma\sigma^{-1} j}(v_{\sigma^{-1} j}) = \prod_{j=1}^n \varepsilon_j(v_{\sigma^{-1} j})$$

so that

$$\varphi(v_1, \dots, v_n) = \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n \varepsilon_j(v_{\sigma^{-1} j})(-1)^\sigma$$
$$= \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n \varepsilon_j(v_{\sigma^{-1} j})(-1)^{\sigma^{-1}}$$
$$= \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n \varepsilon_j(v_{\sigma j})(-1)^\sigma$$

wherein we have used $\Sigma_n \ni \sigma \to \sigma^{-1} \in \Sigma_n$ is a bijection for the last equality. ■

Exercise 6.3. If $\psi \in \Lambda^n(V^*) \setminus \{0\}$, show $\psi(v_1, \dots, v_n) \neq 0$ whenever $\{v_i\}_{i=1}^n \subset V$ are linearly independent. [Coupled with Exercise 6.1, it follows that $\psi(v_1, \dots, v_n) \neq 0$ iff $\{v_i\}_{i=1}^n \subset V$ are linearly independent.]

Definition 6.17. *Suppose that $T : V \to V$ is a linear map between a finite dimensional vector space, then we define $\det T \in \mathbb{R}$ by the relationship, $T^*\psi = \det T \cdot \psi$ where ψ is any non-zero element in $\Lambda^n(V^*)$. [The reader should verify that $\det T$ is independent of the choice of $\psi \in \Lambda^n(V^*) \setminus \{0\}$.]*

The next lemma gives a slight variant of the definition of the determinant.

Lemma 6.18. *If $\psi \in \Lambda^n(V^*) \setminus \{0\}$, $\{e_j\}_{j=1}^n$ is a basis for V, and $T : V \to V$ is a linear transformation, then*

$$\det T = \frac{\psi(Te_1, \dots, Te_n)}{\psi(e_1, \dots, e_n)}. \tag{6.3}$$

Proof. Evaluation the identity, $\det T \cdot \psi = T^*\psi$, at $(e_1, \dots, e_n)$ shows

$$\det T \cdot \psi(e_1, \dots, e_n) = (T^*\psi)(e_1, \dots, e_n) = \psi(Te_1, \dots, Te_n)$$

from which the lemma directly follows. ∎

Corollary 6.19. *Let T be as in Definition 6.17 and suppose $\{e_j\}_{j=1}^n$ is a basis for V and $\{\varepsilon_j\}_{j=1}^n$ is its dual basis, then*

$$\det T = \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_1(Te_{\sigma 1}) \dots \varepsilon_n(Te_{\sigma n})$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1}(Te_1) \dots \varepsilon_{\sigma n}(Te_n).$$

Proof. We take $\varphi \in \Lambda^n(V^*)$ so that $\varphi(e_1, \dots, e_n) = 1$. Since $T^*\varphi \in \Lambda^n(V^*)$ we have seen that $T^*\varphi = \lambda\varphi$ where

$$\lambda = (T^*\varphi)(e_1, \dots, e_n) = \varphi(Te_1, \dots, Te_n)$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1}(Te_1) \dots \varepsilon_{\sigma n}(Te_n)$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_1(Te_{\sigma 1}) \dots \varepsilon_n(Te_{\sigma n}).$$

∎

Corollary 6.20. *Suppose that $S, T : V \to V$ are linear maps between a finite dimensional vector space, V, then*

$$\det(ST) = \det(S) \cdot \det(T).$$

Proof. On one hand

$$(ST)^* \varphi = \det(ST)\varphi.$$

On the other using Exercise 6.2 we have

$$(ST)^* \varphi = T^*(S^*\varphi) = T^*(\det S \cdot \varphi) = \det S \cdot T^*(\varphi) = \det S \cdot \det T \cdot \varphi.$$

Comparing the last two equations completes the proof. ∎

6.2 Determinants of Matrices

In this section we will restrict our attention to linear transformations on $V = \mathbb{R}^n$ which we identify with $n \times n$ matrices. Also, for the purposes of this section let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$. Finally recall that the i^{th} column of A is $v_i = Ae_i$ and so we may express A as

$$A = [v_1 | \dots | v_n] = [Ae_1 | \dots | Ae_n].$$

Proposition 6.21. *The function, $A \to \det(A)$ is the unique alternating multi-linear function of the columns of A such that $\det(I) = \det[e_1 | \dots | e_n] = 1$.*

Proof. Let $\psi \in \mathcal{A}^n(\mathbb{R}^n) \setminus \{0\}$. Then by Lemma 6.18,

$$\det A = \frac{\psi(Ae_1, \dots, Ae_n)}{\psi(e_1, \dots, e_n)}$$

which shows that $\det A$ is and alternating multi-linear function of the columns of A. We have already seen in Proposition 6.16 that there is only one such function.

∎

Theorem 6.22. *If A is a $n \times n$ matrix which we view as a linear transformation on $\mathbb{R}^n$, then;*

1. $\det A = \sum_{\sigma \in \Sigma_n} (-1)^\sigma a_{\sigma 1,1} \dots a_{\sigma n,n}$,
2. $\det A = \sum_{\sigma \in \Sigma_n} (-1)^\sigma a_{1,\sigma 1} \dots a_{n,\sigma n}$, *and*
3. $\det A = \det A^{\text{tr}}$.
4. *The map $A \to \det A$ is the unique alternating multilinear function of the rows of A such that $\det I = 1$.*

Proof. We take $\{e_i\}_{i=1}^n$ to be the standard basis for $\mathbb{R}^n$ and $\{\varepsilon_i\}_{i=1}^n$ be its dual basis. Then by Corollary 6.19,

$$\det A = \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_1(Ae_{\sigma 1}) \dots \varepsilon_n(Ae_{\sigma n})$$
$$= \sum_{\sigma \in \Sigma_n} (-1)^\sigma \varepsilon_{\sigma 1}(Ae_1) \dots \varepsilon_{\sigma n}(Ae_n)$$

which completes the proof of item 1. and 2. since $\varepsilon_i(Ae_j) = a_{i,j}$. For item 3 we use item 1. with A replaced by A^{tr} to find,

$$\det A^{\text{tr}} = \sum_{\sigma \in \Sigma_n} (-1)^\sigma (A^{\text{tr}})_{\sigma 1,1} \dots (A^{\text{tr}})_{\sigma n,n} = \sum_{\sigma \in \Sigma_n} (-1)^\sigma a_{1,\sigma 1} \dots a_{n,\sigma n}.$$

This completes the proof item 3. since the latter expression is equality to $\det A$ by item 2. Finally item 4. follows from item 3. and Proposition 6.21. ∎

Proposition 6.23. *Suppose that $n = n_1 + n_2$ with $n_i \in \mathbb{N}$ and T is a $n \times n$ matrix which has the block form,*

$$T = \begin{bmatrix} A & B \\ 0_{n_2 \times n_1} & C \end{bmatrix},$$

where A is a $n_1 \times n_1$ – matrix, C is a $n_2 \times n_2$ – matrix and B is a $n_1 \times n_2$ – matrix. Then

$$\det T = \det A \cdot \det C.$$

Proof. Fix B and C and consider $\delta(A) := \det \begin{bmatrix} A & B \\ 0_{n_2 \times n_1} & C \end{bmatrix}$. Then $\delta \in \Lambda^{n_1}(\mathbb{R}^{n_1})$ and hence

$$\delta(A) = \delta(I) \cdot \det(A) = \det(A) \cdot \det \begin{bmatrix} I & B \\ 0_{n_2 \times n_1} & C \end{bmatrix}.$$

By doing standard column operations it follows that

$$\det \begin{bmatrix} I & B \\ 0_{n_2 \times n_1} & C \end{bmatrix} = \det \begin{bmatrix} I & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & C \end{bmatrix} =: \tilde{\delta}(C).$$

Working as we did with δ we conclude that $\tilde{\delta}(C) = \det[C] \cdot \tilde{\delta}(I) = \det C$. Putting this all together completes the proof. ■

Next we want to prove the standard cofactor expansion of $\det A$.

Notation 6.24 *If A is a $n \times n$ matrix and $1 \le i, j \le n$, let $A(i,j)$ denotes A with its i^{th} row and j^{th} – column being deleted.*

Proposition 6.25 (Co-factor Expansion). *If A is a $n \times n$ matrix and $1 \le j \le n$, then*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det[A(i,j)] \tag{6.4}$$

and similarly if $1 \le i \le n$, then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det[A(i,j)]. \tag{6.5}$$

We refer to Eq. (6.4) as the ***cofactor expansion along the j^{th} – column*** *and Eq. (6.5) as the* ***cofactor expansion along the i^{th} – row.***

Proof. Equation (6.5) follows from Eq. (6.4) with that aid of item 3. of Theorem 6.22. To prove Eq. (6.4), let $A = [v_1|\dots|v_n]$ and for $b \in \mathbb{R}^n$ let $b^{(i)} := b - b_i e_i$ and then write $v_j = \sum_{i=1}^n a_{ij} e_i$. We then find,

$$\begin{aligned}
\det A &= \sum_{i=1}^{n} a_{ij} \det[v_1|\dots|v_{j-1}|e_i|v_{j+1}|\dots|v_n] \\
&= \sum_{i=1}^{n} a_{ij} \det\left[v_1^{(i)}|\dots|v_{j-1}^{(i)}|e_i|v_{j+1}^{(i)}|\dots|v_n^{(i)}\right] \\
&= \sum_{i=1}^{n} a_{ij} (-1)^{j-1} \det\left[e_i|v_1^{(i)}|\dots|v_{j-1}^{(i)}|v_{j+1}^{(i)}|\dots|v_n^{(i)}\right] \\
&= \sum_{i=1}^{n} a_{ij} (-1)^{j-1} (-1)^{i-1} \det\begin{bmatrix} 1 & 0 \\ 0 & A(i,j) \end{bmatrix} \\
&= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det[A(i,j)]
\end{aligned}$$

wherein we have used the determinant changes sign any time one interchanges two columns or two rows. ■

Example 6.26. Let us illustrate the above proof in the 3×3 case by expanding along the second column. To shorten the notation we we write $\det A = |A|$;

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{12} \begin{vmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 1 & a_{33} \end{vmatrix}$$

where

$$\begin{vmatrix} a_{11} & 1 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\det A(1,2),$$

$$\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 1 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & a_{13} \\ 0 & 1 & 0 \\ a_{31} & 0 & a_{33} \end{vmatrix} = - \begin{vmatrix} 0 & 1 & 0 \\ a_{11} & 0 & a_{13} \\ a_{31} & 0 & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{13} \\ 0 & a_{31} & a_{33} \end{vmatrix} = \det[A(2,2)],$$

and

$$\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 1 & a_{33} \end{vmatrix} = - \begin{vmatrix} 0 & a_{11} & a_{13} \\ 0 & a_{21} & a_{23} \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a_{11} & a_{13} \\ 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{13} \\ 0 & a_{21} & a_{23} \end{vmatrix} = -\det[A(3,1)].$$

6.3 The structure of $\Lambda^k(V^*)$

Definition 6.27. *Let $m \in \mathbb{N}$ and $\{\ell_j\}_{j=1}^m \subset V^*$, we define $\ell_1 \wedge \dots \wedge \ell_m \in \mathcal{A}^m(V)$ by*

$$\left(\ell_1\wedge\dots\wedge\ell_m\right)\left(v_1,\dots,v_m\right)=\det\begin{bmatrix}\ell_1(v_1) & \dots & \ell_1(v_m)\\ \ell_2(v_1) & \dots & \ell_2(v_m)\\ \vdots & \vdots & \vdots\\ \ell_m(v_1) & \dots & \ell_m(v_m)\end{bmatrix} \tag{6.6}$$

$$=\sum_{\sigma\in\Sigma_m}(-1)^\sigma\prod_{i=1}^m\ell_i\left(v_{\sigma i}\right)=\sum_{\sigma\in\Sigma_m}(-1)^\sigma\,\ell_1\left(v_{\sigma1}\right)\dots\ell_m\left(v_{\sigma m}\right). \tag{6.7}$$

or alternatively using $\det A^{\mathrm{tr}}=\det A$,

$$\left(\ell_1\wedge\dots\wedge\ell_m\right)\left(v_1,\dots,v_m\right)=\det\begin{bmatrix}\ell_1(v_1) & \ell_2(v_1) & \dots & \ell_m(v_1)\\ \ell_1(v_2) & \ell_2(v_2) & \dots & \ell_m(v_2)\\ \vdots & \vdots & \vdots & \vdots\\ \ell_1(v_m) & \ell_2(v_m) & \dots & \ell_m(v_m)\end{bmatrix} \tag{6.8}$$

$$=\sum_{\sigma\in\Sigma_m}(-1)^\sigma\prod_{i=1}^m\ell_{\sigma i}\left(v_i\right)=\sum_{\sigma\in\Sigma_m}(-1)^\sigma\,\ell_{\sigma1}\left(v_1\right)\dots\ell_{\sigma m}\left(v_m\right). \tag{6.9}$$

which may be written as,

$$\ell_1\wedge\dots\wedge\ell_m=\sum_{\sigma\in\Sigma_m}(-1)^\sigma\,\ell_{\sigma1}\otimes\dots\otimes\ell_{\sigma m}. \tag{6.10}$$

Remark 6.28. It is perhaps easier to remember these equations as

$$\begin{aligned}&\left(\ell_1\wedge\dots\wedge\ell_m\right)\left(v_1,\dots,v_m\right)\\ &=\det\begin{bmatrix}\ell_1\left(v_1,\dots,v_m\right)\\ \ell_2\left(v_1,\dots,v_m\right)\\ \vdots\\ \ell_m\left(v_1,\dots,v_m\right)\end{bmatrix}\text{ and}\\ &=\det\begin{bmatrix}\ell_1\begin{pmatrix}v_1\\ v_2\\ \vdots\\ v_m\end{pmatrix} & \ell_2\begin{pmatrix}v_1\\ v_2\\ \vdots\\ v_m\end{pmatrix} & \dots & \ell_m\begin{pmatrix}v_1\\ v_2\\ \vdots\\ v_m\end{pmatrix}\end{bmatrix}\end{aligned}$$

where

$$\ell\left(v_1,\dots,v_m\right):=\begin{bmatrix}\ell(v_1) & \dots & \ell(v_m)\end{bmatrix}\text{ and}$$
$$\ell\begin{pmatrix}v_1\\ v_2\\ \vdots\\ v_m\end{pmatrix}:=\begin{bmatrix}\ell(v_1)\\ \ell(v_2)\\ \vdots\\ \ell(v_m)\end{bmatrix}.$$

Exercise 6.4. Let $\{e_i\}_{i=1}^4$ be the standard basis for $\mathbb{R}^4$ and $\{\varepsilon_i=e_i^*\}_{i=1}^4$ be the associated dual basis (i.e. $\varepsilon_i(v)=v_i$ for all $v\in\mathbb{R}^4$.) Compute;

1. $\varepsilon_3\wedge\varepsilon_2\wedge\varepsilon_4\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\3\\2\end{bmatrix}\right)$,
2. $\varepsilon_3\wedge\varepsilon_2\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right)$,
3. $\varepsilon_1\wedge\varepsilon_2\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right)$,
4. $(\varepsilon_1+\varepsilon_3)\wedge\varepsilon_2\left(\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}\right)$, and
5. $\varepsilon_4\wedge\varepsilon_3\wedge\varepsilon_2\wedge\varepsilon_1\left(e_1,e_2,e_3,e_4\right)$.

The next problem is a special case of Theorem 6.30 below.

Exercise 6.5. Show, using basic knowledge of determinants, that for $\ell_0,\ell_1,\ell_2,\ell_3\in V^*$, that

$$\left(\ell_0+\ell_1\right)\wedge\ell_2\wedge\ell_3=\ell_0\wedge\ell_2\wedge\ell_3+\ell_1\wedge\ell_2\wedge\ell_3.$$

Remark 6.29. Note that

$$\ell_{\sigma1}\wedge\dots\wedge\ell_{\sigma m}=(-1)^\sigma\,\ell_1\wedge\dots\wedge\ell_m$$

for all $\sigma\in\Sigma_m$ and in particular if $m=p+q$ with $p,q\in\mathbb{N}$, then

$$\ell_{p+1}\wedge\dots\wedge\ell_m\wedge\ell_1\wedge\dots\wedge\ell_p=(-1)^{pq}\,\ell_1\wedge\dots\wedge\ell_p\wedge\ell_{p+1}\wedge\dots\wedge\ell_m.$$

Theorem 6.30. *For any fixed* $\ell_2,\dots,\ell_k\in V^*$, *the map,*

$$V^*\ni\ell_1\to\ell_1\wedge\dots\wedge\ell_k\in\Lambda^k\left(V^*\right)$$

is linear.

Proof. From Eq. (6.7) we find,

$$\begin{aligned}
&\left(\left(\ell_1 + c\tilde{\ell}_1\right) \wedge \cdots \wedge \ell_k\right)(v_1, \dots, v_k) \\
&= \sum_{\sigma \in \Sigma_k} (-1)^\sigma \left(\ell_1 + c\tilde{\ell}_1\right)(v_{\sigma 1}) \dots \ell_k(v_{\sigma k}) \\
&= \sum_{\sigma \in \Sigma_k} (-1)^\sigma \ell_1(v_{\sigma 1}) \dots \ell_k(v_{\sigma k}) + c \sum_{\sigma \in \Sigma_k} (-1)^\sigma \tilde{\ell}_1(v_{\sigma 1}) \dots \ell_k(v_{\sigma k}) \\
&= \ell_1 \wedge \cdots \wedge \ell_k(v_1, \dots, v_k) + c \cdot \tilde{\ell}_1 \wedge \cdots \wedge \ell_k(v_1, \dots, v_k) \\
&= \left(\ell_1 \wedge \cdots \wedge \ell_k + c \cdot \tilde{\ell}_1 \wedge \cdots \wedge \ell_k\right)(v_1, \dots, v_k).
\end{aligned}$$

As this holds for all $(v_1, \dots, v_k)$, it follows that

$$\left(\ell_1 + c\tilde{\ell}_1\right) \wedge \cdots \wedge \ell_k = \ell_1 \wedge \cdots \wedge \ell_k + c \cdot \tilde{\ell}_1 \wedge \cdots \wedge \ell_k$$

which is the desired linearity. ■

Remark 6.31. If W is another finite dimensional vector space and $T : W \to V$ is a linear transformation, then $T^*(\ell_1 \wedge \cdots \wedge \ell_m) = (T^*\ell_1) \wedge \cdots \wedge (T^*\ell_m)$. To see this is the case, let $w_i \in W$ for $i \in [m]$, then

$$\begin{aligned}
&T^*(\ell_1 \wedge \cdots \wedge \ell_m)(w_1, \dots, w_m) \\
&\quad = (\ell_1 \wedge \cdots \wedge \ell_m)(Tw_1, \dots, Tw_m) \\
&\quad = \sum_{\sigma \in \Sigma_m} (-1)^\sigma \prod_{i=1}^m \ell_i(Tw_{\sigma i}) = \sum_{\sigma \in \Sigma_m} (-1)^\sigma \prod_{i=1}^m (T^*\ell_i)(w_{\sigma i}) \\
&\quad = (T^*\ell_1) \wedge \cdots \wedge (T^*\ell_m)(w_1, \dots, w_m)
\end{aligned}$$

Example 6.32. Let $T \in \Lambda^2\left(\left[\mathbb{R}^3\right]^*\right)$ and $v, w \in \mathbb{R}^3$. Then

$$\begin{aligned}
T(v, w) =& T(v_1 e_1 + v_2 e_2 + v_3 e_3, w_1 e_1 + w_2 e_2 + w_3 e_3) \\
=& T(e_1, e_2)(v_1 w_2 - w_1 v_2) + T(e_1, e_3)(v_1 w_3 - w_1 v_3) \\
&+ T(e_2, e_3)(v_2 w_3 - w_2 v_3) \\
=& \left[T(e_1, e_2)\varepsilon_1 \wedge \varepsilon_2 + T(e_1, e_3)\varepsilon_1 \wedge \varepsilon_3 + T(e_2, e_3)\varepsilon_2 \wedge \varepsilon_3\right](v, w)
\end{aligned}$$

from this it follows that

$$T = T(e_1, e_2)\varepsilon_1 \wedge \varepsilon_2 + T(e_1, e_3)\varepsilon_1 \wedge \varepsilon_3 + T(e_2, e_3)\varepsilon_2 \wedge \varepsilon_3.$$

Further note that if

$$a_{12}\varepsilon_1 \wedge \varepsilon_2 + a_{13}\varepsilon_1 \wedge \varepsilon_3 + a_{23}\varepsilon_2 \wedge \varepsilon_3 = 0$$

then evaluating this expression at (e_i, e_j) for $1 \le i < j \le 3$ allows us to conclude that $a_{ij} = 0$ for $1 \le i \le j \le 3$. Therefore $\{\varepsilon_i \wedge \varepsilon_j : 1 \le i < j \le 3\}$ is a basis for $\Lambda^2\left(\left[\mathbb{R}^3\right]^*\right)$. This example is generalized in the next theorem.

Theorem 6.33. *Let $\{e_i\}_{i=1}^N$ be a basis for V and $\{\varepsilon_i\}_{i=1}^N$ be its' dual basis and for*

$$J = \{1 \le a_1 < a_2 < \cdots < a_p \le N\} \subset [N],$$

let with $\#(J) = p$,

$$e_J := \left(e_{a_1}, \dots, e_{a_p}\right), \text{ and } \varepsilon_J := \varepsilon_{a_1} \wedge \cdots \wedge \varepsilon_{a_p}. \tag{6.11}$$

Then;

1. *$\beta_p := \{\varepsilon_J : J \subset [N] \text{ with } \#(J) = p\}$ is a basis for $\Lambda^p(V^*)$ and so $\dim(\Lambda^p(V^*)) = \binom{N}{p}$, and*
2. *any $A \in \Lambda^p(V^*)$ admits the following expansions,*

$$A = \frac{1}{p!} \sum_{j_1, \dots, j_p = 1}^N A\left(e_{j_1}, \dots, e_{j_p}\right) \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p} \tag{6.12}$$

$$= \sum_{J \subset [N]} A(e_J)\, \varepsilon_J. \tag{6.13}$$

Proof. We begin by proving Eqs. (6.12) and (6.13). To this end let $v_1, \dots, v_p \in V$ and then compute using the multi-linear and alternating properties of A that

$$\begin{aligned}
A(v_1, \dots, v_p) &= \sum_{j_1, \dots, j_p = 1}^N \varepsilon_{j_1}(v_1) \dots \varepsilon_{j_p}(v_p) A\left(e_{j_1}, \dots, e_{j_p}\right) \qquad (6.14) \\
&= \sum_{j_1, \dots, j_p = 1}^N \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \varepsilon_{j_{\sigma 1}}(v_1) \dots \varepsilon_{j_{\sigma p}}(v_p) A\left(e_{j_{\sigma 1}}, \dots, e_{j_{\sigma p}}\right) \\
&= \sum_{j_1, \dots, j_p = 1}^N \frac{1}{p!} \sum_{\sigma \in \Sigma_p} (-1)^\sigma \varepsilon_{j_1}\left(v_{\sigma^{-1}1}\right) \dots \varepsilon_{j_p}\left(v_{\sigma^{-1}p}\right) A\left(e_{j_1}, \dots, e_{j_p}\right) \\
&= \frac{1}{p!} \sum_{j_1, \dots, j_p = 1}^N A\left(e_{j_1}, \dots, e_{j_p}\right) \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p}(v_1, \dots, v_p),
\end{aligned}$$

which is Eq. (6.12). Alternatively we may write Eq. (6.14) as

$$
\begin{aligned}
A\left(v_1,\dots,v_p\right) &= \sum_{j_1,\dots,j_p=1}^{N} 1_{\#(\{j_1,\dots,j_p\})=p}\varepsilon_{j_1}\left(v_1\right)\dots\varepsilon_{j_p}\left(v_p\right)A\left(e_{j_1},\dots,e_{j_p}\right)\\
&= \sum_{1\le a_1<a_2<\dots<a_p\le N}\sum_{\sigma\in\Sigma_n}\varepsilon_{a_{\sigma 1}}\left(v_1\right)\dots\varepsilon_{a_{\sigma p}}\left(v_p\right)A\left(e_{a_{\sigma 1}},\dots,e_{a_{\sigma p}}\right)\\
&= \sum_{1\le a_1<a_2<\dots<a_p\le N} A\left(e_{a_1},\dots,e_{a_p}\right)\sum_{\sigma\in\Sigma_n}\left(-1\right)^{\sigma}\varepsilon_{a_{\sigma 1}}\left(v_1\right)\dots\varepsilon_{a_{\sigma p}}\left(v_p\right)\\
&= \sum_{1\le a_1<a_2<\dots<a_p\le N} A\left(e_{a_1},\dots,e_{a_p}\right)\varepsilon_{a_1}\wedge\dots\wedge\varepsilon_{a_p}\left(v_1,\dots,v_p\right)\\
&= \sum_{J\subset[N]} A\left(e_J\right)\varepsilon_J\left(v_1,\dots,v_p\right).
\end{aligned}
$$

which verifies Eq. (6.13) and hence item 2. is proved.

To prove item 1., since (by Eq. (6.13) we know that β_p spans $\Lambda^p\left(V^*\right)$, it suffices to show β_p is linearly independent. The key point is that for

$$
\begin{aligned}
J &= \{1\le a_1<a_2<\dots<a_p\le N\} \text{ and}\\
K &= \{1\le b_1<b_2<\dots<b_p\le N\}
\end{aligned}
$$

we have

$$
\varepsilon_J\left(e_K\right) = \det\begin{bmatrix}\varepsilon_{a_1}\left(e_{b_1}\right) & \dots & \varepsilon_{a_1}\left(e_{b_p}\right)\\ \varepsilon_{a_2}\left(e_{b_1}\right) & \dots & \varepsilon_{a_2}\left(e_{b_p}\right)\\ \vdots & \vdots & \vdots\\ \varepsilon_{a_p}\left(e_{b_1}\right) & \dots & \varepsilon_{a_p}\left(e_{b_p}\right)\end{bmatrix} = \delta_{J,K}.
$$

Thus if $\sum_{J\subset[N]} a_J\varepsilon_J = 0$, then

$$
0 = 0\left(e_K\right) = \sum_{J\subset[N]} a_J\varepsilon_J\left(e_K\right) = \sum_{J\subset[N]} a_J\delta_{J,K} = a_K
$$

which shows that $a_K = 0$ for all K as above. ■

Exercise 6.6. Suppose $\{\ell_j\}_{j=1}^k \subset [\mathbb{R}^n]^*$.

1. Explaining why $\ell_1\wedge\dots\wedge\ell_k = 0$ if $\ell_i = \ell_j$ for some $i\ne j$.
2. Show $\ell_1\wedge\dots\wedge\ell_k = 0$ if $\{\ell_j\}_{j=1}^k$ are linear **dependent.** [You may assume that $\ell_1 = \sum_{j=2}^k a_j\ell_j$ for some $a_j\in\mathbb{R}$.]

Exercise 6.7. If $\{\ell_j\}_{j=1}^k \subset [\mathbb{R}^n]^*$ are linearly **independent**, show

$$
\ell_1\wedge\dots\wedge\ell_k \ne 0.
$$

Hint: make use of Exercise 5.4.

7

Exterior/Wedge and Interior Products

The main goal of this chapter is to define a good notion of how to multiply two alternating multi-linear forms. The multiplication will be referred to as the "wedge product." Here is the result we wish to prove whose proof will be delayed until Section 7.4.

Theorem 7.1. *Let V be a finite dimensional vector space, $n = \dim(V)$, $p, q \in [n]$, and let $m = p + q$. Then there is a unique bilinear map,*

$$M_{p,q} : \Lambda^p(V^*) \times \Lambda^q(V^*) \to \Lambda^m(V^*),$$

such that for any $\{f_i\}_{i=1}^p \subset V^*$ and $\{g_j\}_{j=1}^q \subset V^*$, we have,

$$M_{p,q}(f_1 \wedge \cdots \wedge f_p, g_1 \wedge \cdots \wedge g_q) = f_1 \wedge \cdots \wedge f_p \wedge g_1 \wedge \cdots \wedge g_q. \tag{7.1}$$

The notation, $M_{p,q}$, in the previous theorem is a bit bulky and so we introduce the following (also temporary) notation.

Notation 7.2 (Preliminary) *For $A \in \Lambda^p(V^*)$ and $B \in \Lambda^q(V^*)$, let us simply denote $M_{p,q}(A, B)$ by $A \cdot B$.*[1]

Remark 7.3. If $m = p + q > n$, then $\Lambda^m(V^*) = \{0\}$ and hence $A \cdot B = 0$.

7.1 Consequences of Theorem 7.1

Before going to the proof of Theorem 7.1 (see Section 7.4) let us work out some of its consequences. By Theorem 6.33 it is always possible to write $A \in \Lambda^p(V^*)$ in the form

$$A = \sum_{i=1}^{\alpha} a_i f_1^i \wedge \cdots \wedge f_p^i \tag{7.2}$$

for some $\alpha \in \mathbb{N}$, $\{a_i\}_{i=1}^N \subset \mathbb{R}$, and $\left\{f_j^i : j \in [p] \text{ and } i \in [\alpha]\right\} \subset V^*$. Similarly we may write $B \in \Lambda^q(V^*)$ in the form,

$$B = \sum_{j=1}^{\beta} b_j g_1^j \wedge \cdots \wedge g_q^j \tag{7.3}$$

for some $\beta \in \mathbb{N}$, $\{b_j\}_{j=1}^N \subset \mathbb{R}$, and $\left\{g_j^j : j \in [q] \text{ and } j \in [\beta]\right\} \subset V^*$. Thus by Theorem 7.1 we must have

$$\begin{aligned} A \cdot B = M_{p,q}(A, B) &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j M_{p,q}\left(f_1^i \wedge \cdots \wedge f_p^i, g_1^j \wedge \cdots \wedge g_q^j\right) \\ &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j. \end{aligned} \tag{7.4}$$

Proposition 7.4 (Associativity). *If $A \in \Lambda^p(V^*)$, $B \in \Lambda^q(V^*)$, and $C \in \Lambda^r(V^*)$ for some $r \in [n]$, then*

$$(A \cdot B) \cdot C = A \cdot (B \cdot C). \tag{7.5}$$

Proof. Let us express C as

$$C = \sum_{k=1}^{\gamma} c_k h_1^k \wedge \cdots \wedge h_r^k.$$

Then working as above we find with the aid of Eq. (7.4) that

$$(A \cdot B) \cdot C = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} a_i b_j c_k f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j \wedge h_1^k \wedge \cdots \wedge h_r^k.$$

A completely analogous computation then shows that $A \cdot (B \cdot C)$ is also given by the right side of the previously displayed equation and so Eq. (7.5) is proved. ∎

Remark 7.5. Since our multiplication rule is associative it now makes sense to simply write $A \cdot B \cdot C$ rather than $(A \cdot B) \cdot C$ or $A \cdot (B \cdot C)$. More generally if $A_j \in \Lambda^{p_j}(V^*)$ we may now simply write $A_1 \cdot \cdots \cdot A_k$. For example by the above associativity we may easily show,

[1] We will see shortly that it is reasonable and more suggestive to write $A \wedge B$ rather than $A \cdot B$. We will make this change after it is justified, see Notation 7.7 below.

$$A \cdot (B \cdot (C \cdot D)) = (A \cdot B) \cdot (C \cdot D) = ((A \cdot B) \cdot C) \cdot D$$

and so it makes sense to simply write $A \cdot B \cdot C \cdot D$ for any one of these expressions.

Corollary 7.6. *If* $\{\ell_j\}_{j=1}^p \subset V^*$, *then*

$$\ell_1 \cdot \dots \cdot \ell_p = \ell_1 \wedge \dots \wedge \ell_p.$$

Proof. For clarity of the argument let us suppose that $p = 5$ in which case we have

$$\begin{aligned} \ell_1 \cdot \ell_2 \cdot \ell_3 \cdot \ell_4 \cdot \ell_5 &= \ell_1 \cdot (\ell_2 \cdot (\ell_3 \cdot (\ell_4 \cdot \ell_5))) \\ &= \ell_1 \cdot (\ell_2 \cdot (\ell_3 \cdot (\ell_4 \wedge \ell_5))) \\ &= \ell_1 \cdot (\ell_2 \cdot (\ell_3 \wedge \ell_4 \wedge \ell_5)) \\ &= \ell_1 \cdot (\ell_2 \wedge \ell_3 \wedge \ell_4 \wedge \ell_5) \\ &= \ell_1 \wedge \ell_2 \wedge \ell_3 \wedge \ell_4 \wedge \ell_5. \end{aligned}$$

■

Because of Corollary 7.6 there is no longer any danger in denoting $A \cdot B = M_{p,q}(A, B)$ by $A \wedge B$. Moreover, this notation suggestively leads one to the correct multiplication formulas.

Notation 7.7 (Wedge=Exterior Product) *For* $A \in \Lambda^p(V^*)$ *and* $B \in \Lambda^q(V^*)$, *we will from now on denote* $M_{p,q}(A, B)$ *by* $A \wedge B$.

Although the wedge product is associative, one must be careful to observe that the wedge product is not commutative, i.e. groupings do not matter but order may matter.

Lemma 7.8 (Non-commutativity). *For* $A \in \Lambda^p(V^*)$ *and* $B \in \Lambda^q(V^*)$ *we have*

$$A \wedge B = (-1)^{pq} B \wedge A.$$

Proof. See Remark 6.29. ■

Example 7.9. Suppose that $\{\varepsilon_j\}_{j=1}^5$ is the standard dual basis on $\mathbb{R}^5$ and

$$\alpha = 2\varepsilon_1 - 3\varepsilon_3, \ \beta = \varepsilon_2 \wedge \varepsilon_4 + (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5.$$

Find and simplify formulas for $\alpha \wedge \alpha$, $\alpha \wedge \beta$ and $\beta \wedge \beta$.

1. $\alpha \wedge \alpha = 0$ since $\alpha \wedge \alpha = -\alpha \wedge \alpha$.
2.
$$\begin{aligned} \alpha \wedge \beta &= (2\varepsilon_1 - 3\varepsilon_3) \wedge \varepsilon_2 \wedge \varepsilon_4 + (2\varepsilon_1 - 3\varepsilon_3) \wedge (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5 \\ &= 2\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 + 3\varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \\ &+ 2\varepsilon_1 \wedge (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5 - 3\varepsilon_3 \wedge (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5 \\ &= 2\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 + 3\varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 + 2\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_5 + 3\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_5 \\ &= 2\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 + 3\varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 + 5\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_5. \end{aligned}$$
3. Finally,
$$\begin{aligned} \beta \wedge \beta &= [\varepsilon_2 \wedge \varepsilon_4 + (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5] \wedge [\varepsilon_2 \wedge \varepsilon_4 + (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5] \\ &= \varepsilon_2 \wedge \varepsilon_4 \wedge (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5 + (\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_5 \wedge \varepsilon_2 \wedge \varepsilon_4 \\ &= \varepsilon_2 \wedge \varepsilon_4 \wedge \varepsilon_1 \wedge \varepsilon_5 + \varepsilon_2 \wedge \varepsilon_4 \wedge \varepsilon_3 \wedge \varepsilon_5 \\ &+ \varepsilon_1 \wedge \varepsilon_5 \wedge \varepsilon_2 \wedge \varepsilon_4 + \varepsilon_3 \wedge \varepsilon_5 \wedge \varepsilon_2 \wedge \varepsilon_4 \\ &= \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 \wedge \varepsilon_1 \wedge \varepsilon_5 - \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \wedge \varepsilon_5 \\ &+ \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 \wedge \varepsilon_5 - \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4 \wedge \varepsilon_5. \end{aligned}$$

Theorem 7.10 (Pull-Backs and Wedges). *If* $A : V \to W$ *is a linear transformation,* $\omega \in \Lambda^k(W^*)$, *and* $\eta \in \Lambda^l(W^*)$, *then*

$$A^*(\omega \wedge \eta) = A^*\omega \wedge A^*\eta \tag{7.6}$$

and in particular if $\ell_1, \dots, \ell_k \in W^*$, *then*

$$A^*[\ell_1 \wedge \dots \wedge \ell_k] = A^*\ell_1 \wedge \dots \wedge A^*\ell_k. \tag{7.7}$$

Proof. Equation (7.6) follows directly from Eq. (7.18) used below in the proof of Theorem 7.1. Equation (7.7) then follows from Eq. (7.6) by induction on k. **However,** not wanting to use the proof of Theorem 7.1 in this proof we will give another proof which only use the material presented so far.

To prove Eq. (7.7), simply let $\{v_i\}_{i=1}^k \subset V$ and compute

$$\begin{aligned} A^*[\ell_1 \wedge \dots \wedge \ell_k](v_1, \dots, A_k) &= \ell_1 \wedge \dots \wedge \ell_k (Av_1, \dots, Av_k) \\ &= \det[\{\ell_i(Av_j)\}] = \det[\{A^*\ell_i(v_j)\}] \\ &= (A^*\ell_1 \wedge \dots \wedge A^*\ell_k)(v_1, \dots, v_k). \end{aligned}$$

As this is true for all $(v_1, \dots, v_k) \in V^k$, Eq. (7.7) follows.

Since both sides of Eq. (7.6) are bilinear functions of ω and η, it suffices to verify Eq. (7.6) in the special case where

$$\omega = \ell_1 \wedge \dots \wedge \ell_k \text{ and } \eta = f_1 \wedge \dots \wedge f_l$$

for some $\ell_1, \dots, \ell_k, f_1, \dots f_l \in W^*$. However this is now simply done using Eq. (7.7),

$$\begin{aligned}A^*(\omega\wedge\eta) &= A^*(\ell_1\wedge\dots\wedge\ell_k\wedge f_1\wedge\dots\wedge f_l)\\ &= A^*\ell_1\wedge\dots\wedge A^*\ell_k\wedge A^*f_1\wedge\dots\wedge A^*f_l\\ &= [A^*\ell_1\wedge\dots\wedge A^*\ell_k]\wedge[A^*f_1\wedge\dots\wedge A^*f_l]\\ &= A^*\omega\wedge A^*\eta.\end{aligned}$$

■

7.2 Interior product

There is yet one more product structure on $\Lambda^m(V^*)$ that we will used throughout these notes given in the following definition.

Definition 7.11 (Interior product). *For $v\in V$ and $T\in\Lambda^m(V^*)$, let $i_vT\in\Lambda^{m-1}(V^*)$ be defined by $i_vT=T(v,\cdots)$.*

Lemma 7.12. *If $\{\ell_i\}_{i=1}^m\subset V^*$, $T=\ell_1\wedge\dots\wedge\ell_m$, and $v\in V$, then*

$$i_v(\ell_1\wedge\dots\wedge\ell_m)=\sum_{j=1}^m(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m. \tag{7.8}$$

Proof. Expanding the determinant along its first column we find,

$$T(v_1,\dots,v_m)=\begin{vmatrix}\ell_1(v_1)&\dots&\ell_1(v_m)\\ \ell_2(v_1)&\dots&\ell_2(v_m)\\ \vdots&\vdots&\vdots\\ \ell_m(v_1)&\dots&\ell_m(v_m)\end{vmatrix}$$

$$=\sum_{j=1}^m(-1)^{j-1}\ell_j(v_1)\cdot\begin{vmatrix}\ell_1(v_2)&\dots&\ell_1(v_m)\\ \ell_2(v_2)&\dots&\ell_2(v_m)\\ \vdots&\vdots&\vdots\\ \ell_{j-1}(v_2)&\dots&\ell_{j-1}(v_m)\\ \ell_{j+1}(v_2)&\dots&\ell_{j+1}(v_m)\\ \vdots&\vdots&\vdots\\ \ell_m(v_2)&\dots&\ell_m(v_m)\end{vmatrix}$$

$$=\sum_{j=1}^m(-1)^{j-1}\ell_j(v_1)\left(\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m\right)(v_2,\dots,v_m)$$

from which Eq. (7.8) follows. ■

Example 7.13. Let us work through the above proof when $m=3$. Letting $T=\ell_1\wedge\ell_2\wedge\ell_3$ we have

$$\begin{aligned}T(v_1,v_2,v_3)&=\begin{vmatrix}\ell_1(v_1)&\ell_1(v_2)&\ell_1(v_3)\\ \ell_2(v_1)&\ell_2(v_2)&\ell_2(v_3)\\ \ell_3(v_1)&\ell_3(v_2)&\ell_3(v_3)\end{vmatrix}\\ &=\ell_1(v_1)\begin{vmatrix}\ell_2(v_2)&\ell_2(v_3)\\ \ell_3(v_2)&\ell_3(v_3)\end{vmatrix}-\ell_2(v_1)\begin{vmatrix}\ell_1(v_2)&\ell_1(v_3)\\ \ell_3(v_2)&\ell_3(v_3)\end{vmatrix}\\ &\quad+\ell_3(v_1)\begin{vmatrix}\ell_1(v_2)&\ell_1(v_3)\\ \ell_2(v_2)&\ell_2(v_3)\end{vmatrix}\\ &=(\ell_1(v_1)\ell_2\wedge\ell_3-\ell_2(v_1)\ell_1\wedge\ell_3+\ell_3(v_1)\ell_1\wedge\ell_2)(v_2,v_3)\end{aligned}$$

and so

$$i_{v_1}(\ell_1\wedge\ell_2\wedge\ell_3)=\ell_1(v_1)\ell_2\wedge\ell_3-\ell_2(v_1)\ell_1\wedge\ell_3+\ell_3(v_1)\ell_1\wedge\ell_2.$$

Exercise 7.1. Let $\{\varepsilon_j\}_{j=1}^3$ be the standard dual basis and $v=(1,2,3)^{\mathrm{tr}}\in\mathbb{R}^3$, find $a_1,a_2,a_3\in\mathbb{R}$ so that

$$i_v(\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3)=a_1\varepsilon_2\wedge\varepsilon_3+a_2\varepsilon_1\wedge\varepsilon_3+a_3\varepsilon_1\wedge\varepsilon_2.$$

Corollary 7.14. *For $A\in\Lambda^p(V^*)$ and $B\in\Lambda^q(V^*)$ and $v\in V$, we have*

$$i_v[A\wedge B]=(i_vA)\wedge B+(-1)^pA\wedge(i_vB).$$

Proof. It suffices to verify this identity on decomposable forms, $A=\ell_1\wedge\dots\wedge\ell_p$ and $B=\ell_{p+1}\wedge\dots\wedge\ell_m$ so that $A\wedge B=\ell_1\wedge\dots\wedge\ell_m$ and we have

$$\begin{aligned}&i_v(A\wedge B)\\ &=\sum_{j=1}^m(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m\\ &=\sum_{j=1}^p(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\ell_p\wedge\ell_{p+1}\wedge\dots\wedge\ell_m\\ &\qquad+\sum_{j=p+1}^m(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\ell_p\wedge\ell_{p+1}\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\wedge\ell_m\\ &=:T_1+T_2\end{aligned}$$

where

$$T_1=\left[\sum_{j=1}^p(-1)^{j-1}\ell_j(v)\cdot\ell_1\wedge\dots\wedge\widehat{\ell_j}\wedge\dots\ell_p\right]\wedge B=(i_vA)\wedge B$$

and

$$\begin{aligned}T_2 &= A \wedge \left[\sum_{j=p+1}^{m} (-1)^{j-1} \ell_j(v)\, \ell_{p+1} \wedge \cdots \wedge \widehat{\ell_j} \wedge \cdots \wedge \ell_m\right] \\ &= (-1)^p A \wedge \left[\sum_{j=p+1}^{m} (-1)^{j-(p+1)} \ell_j(v)\, \ell_{p+1} \wedge \cdots \wedge \widehat{\ell_j} \wedge \cdots \wedge \ell_m\right] \\ &= (-1)^p A \wedge (i_v B).\end{aligned}$$

■

Lemma 7.15. *If $v, w \in V$, then $i_v^2 = 0$ and $i_v i_w = -i_w i_v$.*

Proof. Let $T \in \Lambda^k(V^*)$, then

$$i_v i_w T = T(w, v, —) = T(v, w, —) = i_w i_v T.$$

■

Definition 7.16 (Cross product on $\mathbb{R}^3$). *For $a, b \in \mathbb{R}^3$, let $a \times b$ be the unique vector in $\mathbb{R}^3$ so that*

$$\det[c|a|b] = c \cdot (a \times b) \text{ for all } c \in \mathbb{R}^3.$$

Such a unique vector exists since we know that $c \to \det[c|a|b]$ is a linear functional on $\mathbb{R}^3$ for each $a, b \in \mathbb{R}^3$.

Lemma 7.17 (Cross product). *The cross product in Definition 7.16 agrees with the "usual definition,*

$$\begin{aligned}a \times b &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &=: \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},\end{aligned}$$

where $\mathbf{i} = e_1$, $\mathbf{j} = e_2$, and $\mathbf{k} = e_3$ is the standard basis for $\mathbb{R}^3$.

Proof. Suppose that $a \times b$ is defined by the formula in the lemma, then for all $c \in \mathbb{R}$,

$$\begin{aligned}(a \times b) \cdot c &= c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \det[c|a|b],\end{aligned}$$

wherein we have used the cofactor expansion along the top row for the second equality and the fact that $\det A = \det A^{\mathrm{tr}}$ for the last equality.

■

Remark 7.18 (Generalized Cross product). If $a_1, a_2 \ldots, a_{n-1} \in \mathbb{R}^n$, let $a_1 \times a_2 \times \cdots \times a_{n-1}$ denote the unique vector in $\mathbb{R}^n$ such that

$$\det[c|a_1|a_2|\ldots|a_{n-1}] = c \cdot a_1 \times a_2 \times \cdots \times a_{n-1} \ \forall\, c \in \mathbb{R}^n.$$

This "multi-product" is the $n > 3$ analogue of the cross product in $\mathbb{R}^3$. I don't anticipate using this generalized cross product.

7.3 Exercises

Exercise 7.2 (Cross I). For $a \in \mathbb{R}^3$, let $\ell_a(v) = a \cdot v = a^{\mathrm{tr}} v$, so that $\ell_a \in (\mathbb{R}^3)^*$. In particular we have $\varepsilon_i = \ell_{e_i}$ for $i \in [3]$ is the dual basis to the standard basis $\{e_i\}_{i=1}^3$. Show for $a, b \in \mathbb{R}^3$,

$$\ell_a \wedge \ell_b = i_{a \times b}[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3] \tag{7.9}$$

Hints: 1) write $\ell_a = \sum_{i=1}^3 a_i \varepsilon_i$ and 2) make use of Eq. (7.8)

Exercise 7.3 (Cross II). Use Exercise 7.2 to prove the standard vector calculus identity;

$$(a \times b) \cdot (x \times y) = (a \cdot x)(b \cdot y) - (b \cdot x)(a \cdot y)$$

which is valid for all $a, b, x, y \in \mathbb{R}^3$.Hint: evaluate Eq. (7.9) at (x, y) while using Lemma 7.17.

Exercise 7.4 (Surface Integrals). In this exercise, let $\omega \in \mathcal{A}_3(\mathbb{R}^3)$ be the standard volume form, $\omega(v_1, v_2, v_3) := \det[v_1|v_2|v_3]$, suppose D is an open subset of $\mathbb{R}^2$, and $\Sigma : D \to S \subset \mathbb{R}^3$ is a "parametrized surface," refer to Figure 7.1. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field on $\mathbb{R}^3$, then from your vector calculus class,

$$\iint_S F \cdot N dA = \varepsilon \cdot \iint_D F(\Sigma(u,v)) \cdot [\Sigma_u(u,v) \times \Sigma_v(u,v)]\, dudv \tag{7.10}$$

where $\varepsilon = 1$ ($\varepsilon = -1$) if $N(\Sigma(u,v))$ points in the same (opposite) direction as $\Sigma_u(u,v) \times \Sigma_v(u,v)$. We assume that ε is independent of $(u,v) \in D$.

Show the formula in Eq. (7.10) may be rewritten as

$$\iint_S F \cdot N dA = \varepsilon \iint_D \left(i_{F(\Sigma(u,v))}\omega\right)(\Sigma_u(u,v), \Sigma_v(u,v))\, dudv \tag{7.11}$$

where

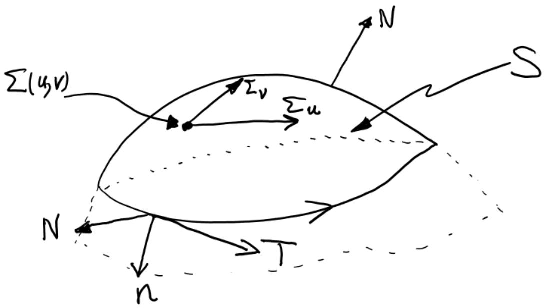


Fig. 7.1. In this figure N is a smoothly varying normal to S, n is a normal to the boundary of S, and T is a tangential vector to the boundary of S. Moreover, $D \ni (u,v) \to \Sigma(u,v) \in S$ is a parametrization of S where $D \subset \mathbb{R}^2$.

$$\varepsilon := \operatorname{sgn}(\omega(N \circ \Sigma, \Sigma_u, \Sigma_v)) = \begin{cases} 1 & \text{if } \omega(N \circ \Sigma, \Sigma_u, \Sigma_v) > 0 \\ -1 & \text{if } \omega(N \circ \Sigma, \Sigma_u, \Sigma_v) < 0. \end{cases}$$

Remarks: Once we introduce the proper notation, we will be able to write Eq. (7.11) more succinctly as

$$\iint_S F \cdot N dA = \iint_S i_F \omega := \varepsilon \iint_D \Sigma^* (i_F \omega).$$

Definition 7.19 (Curl). *If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field on $\mathbb{R}^3$, we define a new vector field called the* ***curl of*** *F by*

$$\nabla \times F = (\partial_2 F_3 - \partial_3 F_2) e_1 - (\partial_1 F_3 - \partial_3 F_1) e_2 + (\partial_1 F_2 - \partial_2 F_1) e_3 \tag{7.12}$$

where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$. This usually remembered by the following mnemonic formulas;

$$\begin{aligned} \nabla \times F &= \det \begin{bmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{bmatrix} \\ &= e_1 \det \begin{bmatrix} \partial_2 & \partial_3 \\ F_2 & F_3 \end{bmatrix} - e_2 \det \begin{bmatrix} \partial_1 & \partial_3 \\ F_1 & F_3 \end{bmatrix} + e_3 \det \begin{bmatrix} \partial_1 & \partial_2 \\ F_1 & F_2 \end{bmatrix}. \end{aligned}$$

Exercise 7.5 (Boundary Orientation). Referring to the set up in Exercise 7.4, the tangent vector T has been chosen by using the "right-hand" rule in order to determine the orientation on the boundary, ∂S, of S so that Stoke's theorem holds, i.e.

$$\iint_S [\nabla \times F] \cdot N dA = \int_{\partial S} F \cdot T ds. \tag{7.13}$$

Show by using the "right hand rule" that $T = c \cdot N \times n$ with $c > 0$ and then also show

$$c = \omega(N, n, T) = (i_n i_N \omega)(T).$$

Also note by Exercise 7.4, that Eq. (7.13) may be written as

$$\iint_S i_{\nabla \times F} \omega = \int_{\partial S} F \cdot T ds \tag{7.14}$$

Remark: We will introduce the "one form", $F \cdot dx$ and an "exterior derivative" operator, d, so that

$$d[F \cdot dx] = i_{\nabla \times F} \omega$$

and Eq. (7.14) may be written in the pleasant form,

$$\iint_S d[F \cdot dx] = \int_{\partial S} F \cdot dx.$$

7.4 *Proof of Theorem 7.1

[This section may safely be skipped if you are willing to believe the results as stated!]

If Theorem 7.1 is going to be true we must have $M_{p,q}(A,B) = A \cdot B = D$ where, as written in Eq. (7.4),

$$D = \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j f_1^i \wedge \cdots \wedge f_p^i \wedge g_1^j \wedge \cdots \wedge g_q^j. \tag{7.15}$$

The problem with this presumed definition is that the formula for D in Eq. (7.15) seems to depend on the expansions of A and B in Eqs. (7.2) and (7.3) rather than on only A and B. [The expansions for A and B in Eqs. (7.2) and (7.3) are highly non-unique!] In order to see that D is independent of the possible choices of expansions of A and B, we are going to show in Proposition 7.23 below that $D(v_1, \dots, v_m)$ (with D as in Eq. (7.15)) may be expressed by a formula which only involves A and B and **not** their expansions. Before getting to this proposition we need some more notation and a preliminary lemma.

Notation 7.20 *Let $m = p + q$ be as in Theorem 7.1 and let $\{v_i\}_{i=1}^m \subset V$ be fixed. For each $J \subset [m]$ with $\#J = p$ write*

$$\begin{aligned} J &= \{1 \le a_1 < a_2 < \cdots < a_p \le m\}, \\ J^c &= \{1 \le b_1 < b_2 < \cdots < b_q \le m\}, \\ v_J &:= (v_{a_1}, \dots, v_{a_p}), \text{ and } v_{J^c} := (v_{b_1}, \dots, v_{b_q}). \end{aligned}$$

Also for any $\alpha \in \Sigma_p$ and $\beta \in \Sigma_q$, let

$$\sigma_{J,\alpha,\beta} = \begin{pmatrix} 1 & \dots & p & p+1 & \dots & m \\ a_{\alpha 1} & \dots & a_{\alpha p} & b_{\beta 1} & \dots & b_{\beta q} \end{pmatrix}.$$

When α and β are the identity permutations in Σ_p and Σ_q respectively we will simply denote $\sigma_{J,\alpha,\beta}$ by σ_J, i.e.

$$\sigma_J = \begin{pmatrix} 1 & \dots & p & p+1 & \dots & m \\ a_1 & \dots & a_p & b_1 & \dots & b_q \end{pmatrix}.$$

The point of this notation is contained in the following lemma.

Lemma 7.21. *Assuming Notation 7.20,*

1. *the map,*
$$\mathcal{P}_{p,m} \times \Sigma_p \times \Sigma_q \ni (J, \alpha, \beta) \to \sigma_{J,\alpha,\beta} \in \Sigma_m,$$
is a bijection, and
2. $(-1)^{\sigma_{J,\alpha,\beta}} = (-1)^{\sigma_J} (-1)^{\alpha} (-1)^{\beta}$.

Proof. We leave proof of these assertions to the reader. ∎

Lemma 7.22 (Wedge Product I). *Let $n = \dim V$, $p, q \in [n]$, $m := p + q$, $\{f_i\}_{i=1}^p \subset V^*$, $\{g_j\}_{j=1}^q \subset V^*$, and $\{v_j\}_{j=1}^m \subset V$, then*

$$\begin{aligned}&(f_1 \wedge \dots \wedge f_p \wedge g_1 \wedge \dots \wedge g_q)(v_1, \dots, v_m) \\ &\quad = \sum_{\#J=p} (-1)^{\sigma_J} (f_1 \wedge \dots \wedge f_p)(v_J)(g_1 \wedge \dots \wedge g_q)(v_{J^c}).\end{aligned} \tag{7.16}$$

Proof. In order to simplify notation in the proof let, $\ell_i = f_i$ for $1 \le i \le p$ and $\ell_{j+p} = g_j$ for $1 \le j \le q$ so that

$$f_1 \wedge \dots \wedge f_p \wedge g_1 \wedge \dots \wedge g_q = \ell_1 \wedge \dots \wedge \ell_m.$$

Then by Definition 6.27 of $\ell_1 \wedge \dots \wedge \ell_m$ along with Lemma 7.21, we find,

$$\begin{aligned}&(\ell_1 \wedge \dots \wedge \ell_m)(v_1, \dots, v_m) \\ &= \det\left[\{\ell_i(v_j)\}_{i,j=1}^m\right] = \sum_{\sigma \in \Sigma_m} (-1)^{\sigma} \prod_{i=1}^m \ell_i(v_{\sigma i}). \\ &= \sum_J \sum_{\alpha \in \Sigma_p} \sum_{\beta \in \Sigma_q} (-1)^{\sigma_{J,\alpha,\beta}} \prod_{i=1}^m \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) \\ &= \sum_J (-1)^{\sigma_J} \sum_{\alpha \in \Sigma_p} \sum_{\beta \in \Sigma_q} (-1)^{\sigma_\alpha} \prod_{i=1}^p \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) (-1)^{\sigma_\beta} \prod_{i=p+1}^m \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right).\end{aligned}$$

Combining this with the following identity,

$$\begin{aligned}&\sum_{\alpha \in \Sigma_p} \sum_{\beta \in \Sigma_q} (-1)^{\sigma_\alpha} \prod_{i=1}^p \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) (-1)^{\sigma_\beta} \prod_{i=p+1}^m \ell_i\left(v_{\sigma_{J,\alpha,\beta} i}\right) \\ &= \sum_{\alpha \in \Sigma_p} (-1)^{\sigma_\alpha} \prod_{i=1}^p \ell_i(v_{a_{\alpha i}}) \sum_{\beta \in \Sigma_q} (-1)^{\sigma_\beta} \prod_{i=p+1}^m \ell_i(v_{b_{\beta i}}) \\ &= (\ell_1 \wedge \dots \wedge \ell_p)(v_J)(\ell_{p+1} \wedge \dots \wedge \ell_m)(v_{J^c}) \\ &= (f_1 \wedge \dots \wedge f_p)(v_J)(g_1 \wedge \dots \wedge g_q)(v_{J^c})\end{aligned}$$

completes the proof. ∎

Proposition 7.23 (Wedge Product II). *If $A \in \Lambda^p(V^*)$ and $B \in \Lambda^q(V^*)$ are written as in Eqs. (7.2–7.3) and $D \in \Lambda^m(V^*)$ is defined as in Eq. (7.15), then*

$$D(v_1, \dots, v_m) = \sum_{\#J=p} (-1)^{\sigma_J} A(v_J) B(v_{J^c}) \quad \forall \ \{v_j\}_{j=1}^m \subset V. \tag{7.17}$$

This shows defining $A \wedge B$ by Eq. (7.4)is well defined and in fact could have been defined intrinsically using the formula,

$$A \wedge B(v_1, \dots, v_m) = \sum_{\#J=p} (-1)^{\sigma_J} A(v_J) B(v_{J^c}). \tag{7.18}$$

Proof. By Lemma 7.22,

$$\begin{aligned}&f_1^i \wedge \dots \wedge f_p^i \wedge g_1^j \wedge \dots \wedge g_q^j(v_1, \dots, v_m) \\ &= \sum_{\#J=p} (-1)^{\sigma_J} \left(f_1^i \wedge \dots \wedge f_p^i\right)(v_J) \cdot \left(g_1^j \wedge \dots \wedge g_q^j\right)(v_{J^c})\end{aligned}$$

and therefore,

$$\begin{aligned}&D(v_1, \dots, v_m) \\ &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j \left(f_1^i \wedge \dots \wedge f_p^i \wedge g_1^j \wedge \dots \wedge g_q^j\right)(v_1, \dots, v_m) \\ &= \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i b_j \sum_{\#J=p} (-1)^{\sigma_J} \left(f_1^i \wedge \dots \wedge f_p^i\right)(v_J) \cdot \left(g_1^j \wedge \dots \wedge g_q^j\right)(v_{J^c}) \\ &= \sum_{\#J=p} (-1)^{\sigma_J} \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} a_i \left(f_1^i \wedge \dots \wedge f_p^i\right)(v_J) \cdot \sum_{j=1}^{\beta} b_j \left(g_1^j \wedge \dots \wedge g_q^j\right)(v_{J^c}) \\ &= \sum_{\#J=p} (-1)^{\sigma_J} A(v_J) B(v_{J^c})\end{aligned}$$

which proves Eq. (7.17) and completes the proof of the proposition. ■

With all of this preparation we are now in a position to complete the proof of Theorem 7.1.

Proof of Theorem 7.1. As we have see we may define $A \wedge B$ by either Eq. (7.4) or by Eq. (7.18). Equation (7.18) ensures $A \wedge B$ is well defined and is multi-linear while Eq. (7.4) ensures $A \wedge B \in \Lambda^m (V^*)$ and that Eq. (7.1) holds. This proves the existence assertion of the theorem. The uniqueness of $M_{p,q}(A, B) = A \wedge B$ follows by the necessity of defining $A \wedge B$ by Eq. (7.4). ■

Corollary 7.24. *Suppose that* $\{e_j\}_{j=1}^n$ *is a basis of* V *and* $\{\varepsilon_j\}_{j=1}^n$ *is its dual basis of* V^*. *Then for* $A \in \Lambda^p (V^*)$ *and* $B \in \Lambda^q (V^*)$ *we have*

$$A \wedge B = \frac{1}{p! \cdot q!} \sum_{j_1,\dots,j_m=1}^{n} A\left(e_{j_1}, \dots, e_{j_p}\right) B\left(e_{j_{p+1}}, \dots, e_{j_m}\right) \varepsilon_{j_1} \wedge \dots \dots \wedge \varepsilon_{j_m}. \tag{7.19}$$

Proof. By Theorem 6.33 we may write,

$$A = \frac{1}{p!} \sum_{j_1,\dots,j_p=1}^{N} A\left(e_{j_1}, \dots, e_{j_p}\right) \varepsilon_{j_1} \wedge \dots \wedge \varepsilon_{j_p} \text{ and}$$

$$B = \frac{1}{q!} \sum_{j_{p+1},\dots,j_m=1}^{n} B\left(e_{j_{p+1}}, \dots, e_{j_m}\right) \varepsilon_{j_{p+1}} \wedge \dots \dots \wedge \varepsilon_{j_m}$$

and therefore Eq. (7.19) holds by computing $A \wedge B$ as in Eq. (7.4). ■

Part III

Differential Forms on $U \subset_o \mathbb{R}^n$

Derivatives, Tangent Spaces, and Differential Forms

In this chapter we will develop calculus and the language of differential forms on open subsets of Euclidean space in such a way that our result will transfer to the more general manifold setting.

8.1 Derivatives and Chain Rules

Notation 8.1 (Open subset) *I use the symbol "$\subset_o$" to denote containment with the smaller set being open in the bigger. Thus writing $U \subset_o \mathbb{R}^n$ means U is an open subset of $\mathbb{R}^n$ which we always assume to be non-empty.*

Notation 8.2 *For $U \subset_o \mathbb{R}^n$, we write $f : U \to \mathbb{R}^m$ as short hand for saying that f is a function from U to $\mathbb{R}^m$, thus for each $x \in U$,*

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

where $f_i : U \to \mathbb{R}$ for each $i \in [m]$.

Definition 8.3 (Directional Derivatives). *Suppose $U \subset_o \mathbb{R}^n$ that $f : U \to \mathbb{R}^m$ is a function, so For $p \in U$ and $v \in \mathbb{R}^n$, let*

$$(\partial_v f)(p) := \frac{d}{dt}|_0 f(p + tv)$$

*be the **directional derivative**[1] **of** f **at** p **in the direction** v. By definition, the j^{th}**-partial derivative** of f at p, is*

$$\frac{\partial f}{\partial x_j}(p) = \left(\partial_{e_j} f\right)(p) = \frac{d}{dt}|_0 f(p + te_j).$$

where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$. We will also write $\partial_j f$ for $\frac{\partial f}{\partial x_j} = \partial_{e_j} f$.

[1] We use this terminology even though no assumption about v being a unit vector is being made.

Definition 8.4. *A function, $f : U \to \mathbb{R}$, is **smooth** if f has partial derivatives to all orders and all of these partial derivatives are continuous. We say $f : U \to \mathbb{R}^m$ is **smooth** if each of the functions, $f_i : U \to \mathbb{R}$, are smooth functions.*

Notation 8.5 *We let $C^\infty(U, \mathbb{R}^m)$ denote the smooth functions from U to $\mathbb{R}^m$. When $m = 1$ we will also write $C^\infty(U, \mathbb{R}) = \Omega^0(U)$ and refer to these as the **smooth** 0-forms on U. We also let $C^k(U, \mathbb{R}^m)$ denote those $f : U \to \mathbb{R}^m$ such that each coordinate function, f_i, has partial derivatives to order k an all of these partial derivatives are continuous.*

Let us recall a some version of the chain rule.

Theorem 8.6. *If $f \in C^1(U, \mathbb{R}^m)$, $p \in U$, and $v = (v_1, \dots, v_n)^{\mathrm{tr}} \in \mathbb{R}^n$, then*

$$(\partial_v f)(p) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p) v_j = f'(p) v$$

where $f'(p) = Df(p)$ is the $m \times n$ matrix defined by

$$\begin{aligned} f'(p) &= \left[\frac{\partial f}{\partial p_1}(p) | \frac{\partial f}{\partial p_2}(p) | \dots | \frac{\partial f}{\partial p_n}(p)\right] \\ &= \begin{bmatrix} \partial_1 f_1(p) & \partial_2 f_1(p) & \dots & \partial_n f_1(p) \\ \partial_1 f_2(p) & \partial_2 f_2(p) & \dots & \partial_n f_2(p) \\ \vdots & \vdots & \dots & \vdots \\ \partial_1 f_m(p) & \partial_2 f_m(p) & \cdots & \partial_n f_m(p) \end{bmatrix} \end{aligned}$$

We refer to $f'(p) = Df(p)$ as the differential of f at p.

More generally, *if, $\sigma : (-\varepsilon, \varepsilon) \to U$ is a curve in U such that $\dot{\sigma}(0) = \frac{d}{dt}|_0 \sigma(t) \in \mathbb{R}^n$ exists, then*

$$\begin{aligned} \frac{d}{dt}|_0 f(\sigma(t)) &= \left(\partial_{\dot{\sigma}(0)} f\right)(\sigma(0)) = f'(\sigma(0))\dot{\sigma}(0) \\ &= \sum_{j=1}^n \frac{\partial f}{\partial x_j}(\sigma(0)) \dot{\sigma}_j(0). \end{aligned} \tag{8.1}$$

Example 8.7. If

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ \sin(x_1) \\ x_2 e^{x_1} \end{pmatrix}$$

then

$$f'\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} x_2 & x_1 \\ \cos(x_1) & 0 \\ x_2 e^{x_1} & e^{x_1} \end{bmatrix}.$$

Example 8.8. Let

$$p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; v = \begin{bmatrix} \pi \\ 11 \end{bmatrix}, \text{ and}$$
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} ye^x \\ x^2 + y^2 \end{bmatrix}.$$

Then

$$f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} ye^x & e^x \\ 2x & 2y \end{bmatrix},$$
$$f'(p) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \text{ and}$$
$$(\partial_v f)(p) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \pi \\ 11 \end{bmatrix} = \begin{bmatrix} \pi + 11 \\ 22 \end{bmatrix}.$$

Exercise 8.1. Let

$$f\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \text{ for } \begin{pmatrix} r \\ \theta \end{pmatrix} \in \mathbb{R}^2.$$

Find;

$$f'\begin{pmatrix} r \\ \theta \end{pmatrix} \text{ and } \det\left[f'\begin{pmatrix} r \\ \theta \end{pmatrix}\right].$$

Exercise 8.2. Let

$$f\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} = \begin{bmatrix} r\sin\varphi \cdot \cos\theta \\ r\sin\varphi \cdot \sin\theta \\ r\cos\varphi \end{bmatrix} \text{ for } \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \in \mathbb{R}^3.$$

Find;

$$f'\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \text{ and } \det\left[f'\begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix}\right].$$

The following rewriting of the chain rule is often useful for computing directional derivatives.

Lemma 8.9 (Chain Rule II). *Let $0 \in U \subset_o \mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ be smooth function. Then*

$$\frac{d}{dt}|_0 f(t, t, \dots, t) = \sum_{j=1}^n \frac{d}{dt}|_0 f(te_j) = \sum_{j=1}^n \frac{d}{dt}|_0 f\left(0, \dots, 0, \overset{j \text{ position}}{t}, 0, \dots, 0\right).$$

Proof. Let $\sigma(t) = (t, t, \dots, t)^{\mathrm{tr}}$, then by the chain rule,

$$\begin{aligned} \frac{d}{dt}|_0 f(t, t, \dots, t) &= \frac{d}{dt}|_0 f(\sigma(t)) = f'(\sigma(0))\,\dot\sigma(0) \\ &= f'(0)\left[e_1 + \dots + e_n\right] \\ &= \sum_{j=1}^n \left(\partial_{e_j} f\right)(0) = \sum_{j=1}^n \frac{d}{dt}|_0 f(te_j). \end{aligned}$$

■

Exercise 8.3. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} = [a_1| \dots |a_n]$$

be an $n \times n$ matrix with i^{th}-column

$$a_i = \begin{bmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{ni} \end{bmatrix}.$$

Given another $n \times n$ matrix B with analogous notation, show

$$(\partial_B \det)(A) = \sum_{j=1}^n \det\left[a_1| \dots |a_{j-1}|b_j|a_{j+1}| \dots |b_n\right]. \tag{8.2}$$

For example if $n = 3$, this formula reads,

$$(\partial_B \det)(A) = \det[b_1|a_2|a_3] + \det[a_1|b_2|a_3] + \det[a_1|a_2|b_3].$$

Suggestions; by definition,

$$(\partial_B \det)(A) := \frac{d}{dt}|_0 \det(A+tB) = \frac{d}{dt}|_0 \det[a_1+tb_1|\dots|a_n+tb_n].$$

Now apply Lemma 8.9 with

$$f(x_1,\dots,x_n) = \det[a_1+x_1b_1|\dots|a_n+x_nb_n].$$

Exercise 8.4 (Exercise 8.3 continued). Continuing the notation and results from Exercise 8.3, show;

1. If $A = I$ is the $n \times n$ identity matrix in Eq. (8.2), then
$$(\partial_B \det)(I) = \operatorname{tr}(B) = \sum_{j=1}^n B_{j,j}.$$
2. If A is an $n \times n$ invertible matrix, shows
$$(\partial_B \det)(A) = \det(A)\cdot \operatorname{tr}\left(A^{-1}B\right).$$
Hint: Verify the identity,
$$\det(A+tB) = \det(A)\cdot\det\left(I+tA^{-1}B\right)$$
which you should then use along with first item of this exercise.

Corollary 8.10. *If A is an $n \times n$ matrix, then* $\det\left(e^A\right) = e^{\operatorname{tr}(A)}$.

Proof. Let $f(t) := \det\left(e^{tA}\right)$, then

$$\begin{aligned}
\dot f(t) &= \frac{d}{ds}|_0 f(t+s) = \frac{d}{ds}|_0 \det\left(e^{(t+s)A}\right)\\
&= \frac{d}{ds}|_0 \det\left(e^{tA}e^{sA}\right) = \det\left(e^{tA}\right)\frac{d}{ds}|_0 \det\left(e^{sA}\right)\\
&= f(t)\left(\partial_{\frac{d}{ds}|_0 e^{sA}}\det\right)\left(e^{0A}\right) = f(t)(\partial_A \det)(I)\\
&= f(t)\operatorname{tr}(A) \text{ with } f(0) = \det(I) = 1.
\end{aligned}$$

Solving this differential equation then shows,

$$\det\left(e^{tA}\right) = e^{t\cdot\operatorname{tr}(A)}.$$

■

8.2 Tangent Spaces and More Chain Rules

Definition 8.11 (Tangent space). *To each open set, $U \subset_o \mathbb{R}^n$, let*

$$TU := U \times \mathbb{R}^n = \{v_p = (p,v) : p \in U \text{ and } v \in \mathbb{R}^n\}.$$

For a given $p \in U$, we let

$$T_pU = \{v_p = (p,v) : v \in \mathbb{R}^n\}$$

and refer this as ***the tangent space to** U **at** p. Note that*

$$TU = \cup_{p\in U} T_pU.$$

For $v_p, w_p \in T_pU$ and $\lambda \in \mathbb{C}$ we define,

$$v_p + \lambda w_p := (v+\lambda w)_p$$

which makes T_pU into a vector space isomorphic to $\mathbb{R}^n$.

Notation 8.12 (Cotangent spaces) *For $p \in \mathbb{R}^n$, let $T_p^*U := [T_pU]^*$ be the dual space to T_pU.*

Definition 8.13. *If $f \in C^\infty(U,\mathbb{R}^m)$ and $v_p \in T_pU$ let*

$$df(v_p) := (\partial_v f)(p) = f'(p)\,v.$$

We call df the ***differential*** *of f and further write df_p for $df|_{T_pU} \in [T_pU]^*$.*

We will mostly (probably exclusively) use the df notation in the case where $m = 1$.

Example 8.14. Let $f(x_1,x_2) = x_1x_2^2$, then

$$f'\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{bmatrix}x_2^2 & 2x_1x_2\end{bmatrix} \implies f'\begin{pmatrix}p_1\\p_2\end{pmatrix} = \begin{bmatrix}p_2^2 & 2p_1p_2\end{bmatrix}.$$

Therefore,

$$df(v_p) = \begin{bmatrix}p_2^2 & 2p_1p_2\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = p_2^2v_1 + 2p_1p_2v_2.$$

Notation 8.15 (Coordinate functions) ***Note well:*** *from now on we will usually consider $x = (x_1,\dots,x_n)^{\mathrm{tr}}$ to be the identity function from $\mathbb{R}^n$ to $\mathbb{R}^n$ rather than a point in $\mathbb{R}^n$, i.e. if $p = (p_1,p_2,\dots,p_n)^{\mathrm{tr}}$ then $x_i(p) = p_i$. We still however write*

$$\frac{\partial f}{\partial x_i}(p) := \partial_i f(p) = (\partial_{e_i} f)(p)$$

Example 8.16. We have for each $i \in [n]$

$$dx_i(v_p) = (\partial_v x_i)(p) = \frac{d}{dt}|_0 x_i(p+tv) = \frac{d}{dt}|_0 (p_i + tv_i) = v_i.$$

Proposition 8.17. *If $f \in \Omega^0(U)$, then*

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

where the right side of this equation evaluated at v_p is by definition,

$$\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right)(v_p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \cdot dx_i(v_p)$$

Proof. By definition and the chain rule,

$$df(v_p) = (\partial_v f)(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) v_i = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) \cdot dx_i(v_p).$$

■

Exercise 8.5. Using Proposition 8.17, find df when

$$f(x_1, x_2, x_3) = x_1^2 \sin(e^{x_2}) + \cos(x_3).$$

Lemma 8.18 (Product Rule). *Suppose that $f, g \in C^\infty(U)$, then $d(fg) = fdg + gdf$ which in more detail means,*

$$d(fg)(v_p) = f(p)\, dg(v_p) + g(p)\, df(v_p) \text{ for all } v_p \in TU.$$

[You are asked to generalize this result in Exercise 8.6.]

Proof. This is the product rule. Here are two ways to prove this result.

1. The first method used the product rule for directional derivatives,

$$\begin{aligned} d(fg)(v_p) &= (\partial_v(fg))(p) = (\partial_v f \cdot g + f\partial_v g)(p) \\ &= g(p)\, df(v_p) + f(p)\, dg(v_p). \end{aligned}$$

2. For the second we use Proposition 8.17 and the product rule for partial derivatives to find,

$$\begin{aligned} d(fg) &= \sum_{j=1}^n \partial_j(fg)\, dx_j = \sum_{j=1}^n [\partial_j f \cdot g + f\partial_j g]\, dx_j \\ &= g\sum_{j=1}^n \partial_j f dx_j + f\sum_{j=1}^n \partial_j g dx_j = gdf + fdg. \end{aligned}$$

■

Example 8.19 (Example 8.14 revisited). Let $f(x_1, x_2) = x_1 x_2^2$ be as in Example 8.14, then using the product rule,

$$df = x_2^2 dx_1 + x_1 d\left[x_2^2\right] = x_2^2 dx_1 + 2x_1 x_2 dx_2.$$

Proposition 8.20. *If $f \in \Omega^0(U)$ and $\sigma(t)$ is curve in U so that $\dot\sigma(0)$ exists, then*

$$\frac{d}{dt}|_0 f(\sigma(t)) = df\left(\dot\sigma(0)_{\sigma(0)}\right).$$

Proof. By the chain rule and the definition of $df(v_p)$,

$$\frac{d}{dt}|_0 f(\sigma(t)) = f'(\sigma(0))\,\dot\sigma(0) = \left(\partial_{\dot\sigma(t)} f\right)(\sigma(0)) = df\left(\dot\sigma(0)_{\sigma(0)}\right).$$

■

Exercise 8.6. Let $g_1, g_2, \dots, g_n \in C^1(U, \mathbb{R})$, $f \in C^1(\mathbb{R}^n, \mathbb{R})$, and $u = f(g_1, \dots, g_n)$, i.e.

$$u(p) = f(g_1(p), \dots, g_n(p)) \text{ for all } p \in U.$$

Show

$$du = \sum_{j=1}^n (\partial_j f)(g_1, \dots, g_n)\, dg_j$$

which is to be interpreted to mean,

$$du(v_p) = \sum_{j=1}^n (\partial_j f)(g_1(p), \dots, g_n(p))\, dg_j(v_p) \text{ for all } v_p \in TU.$$

Hint: For $v_p \in TU$, let $\sigma(t) = (g_1(p+tv), \dots, g_n(p+tv))$ and then make use of the chain rule (see Eq. (8.1)) to compute $du(v_p)$.

Here is yet one more version of the chain rule. [This next version essentially encompasses all of the previous versions.]

Exercise 8.7 (Chain Rule for Maps). Suppose that $f : U \to V$ and $g : V \to W$ are C^1-functions where U, V, and W are open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$ respectively and let $g \circ f : U \to W$ be the composition map,

$$g \circ f : U \xrightarrow{f} V \xrightarrow{g} W.$$

Show

$$(g \circ f)'(p) = g'(f(p))\, f'(p) \text{ for all } p \in U. \tag{8.3}$$

Hint: Let $v \in \mathbb{R}^n$ and $\sigma(t) := f(p+tv)$ – a differentiable curve in V. Then use the chain rule in Theorem 8.6 twice in order to compute,

$$(g \circ f)'(p)\, v = \frac{d}{dt}|_0 g(f(p+tv)) = \frac{d}{dt}|_0 g(\sigma(t)).$$

We now want to define a derivative map which fully keeps track of the base points, unlike df which forgets the target base point.

Definition 8.21 ($f_* : TU \to TV$). *If $U \subset_o \mathbb{R}^n$ and $V \subset_o \mathbb{R}^m$ and $f : U \to V$ is a smooth function, i.e. $f : U \to \mathbb{R}^m$ is smooth with $f(U) \subset V$, then we define a map, $f_* : TU \to TV$ by*

$$f_* v_p := \left[(\partial_v f)(p)\right]_{f(p)} = \left[f'(p)\,v\right]_{f(p)} \text{ for all } (p,v) \in TU = U \times \mathbb{R}^n. \quad (8.4)$$

We further let f_{*p} denote the restriction of f_* to T_pU in which case $f_{*p} : T_pU \to T_{f(p)}V$ which is seen to be linear by the formula in Eq. (8.4).

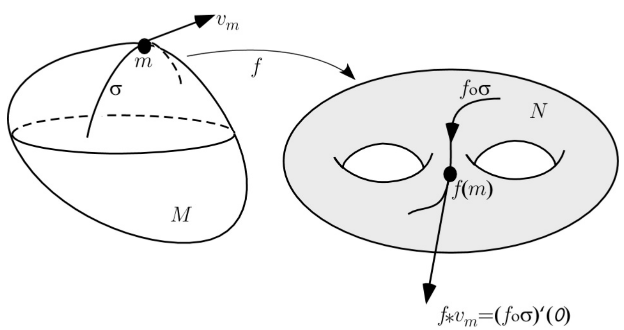


Fig. 8.1. Describing the differential in geometric context.

Proposition 8.22 (Chain rule again). *Let f and g be as in Exercises 8.7. Here are are last two reformulations of the chain rule.*

1. *If $\sigma(t)$ is a curve in U such that $\dot{\sigma}(0) = v$ and $\sigma(0) = p$, then*

$$f_* v_p = f_*\left(\dot{\sigma}(0)_{\sigma(0)}\right) = \left[\frac{d}{dt}|_0 f(\sigma(t))\right]_{f(\sigma(0))}.$$

2. *The chain rule in Eq. (8.3) may be written in the following pleasing form,*

$$(g \circ f)_* = g_* f_*.$$

Proof. We take each item in turn.

1. Let $v_p \in TU$. By the chain rule,

$$\frac{d}{dt}|_0 f(\sigma(t)) = f'(\sigma(0))\,\dot{\sigma}(0) = f'(p)\,v$$

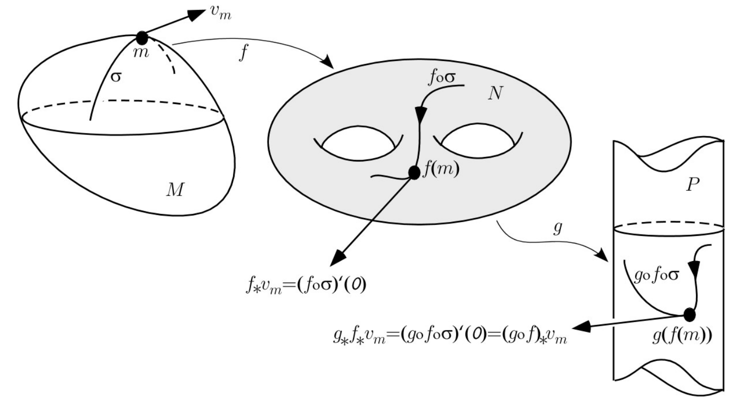


Fig. 8.2. The chain rule in pictures.

and therefore,

$$\left[\frac{d}{dt}|_0 f(\sigma(t))\right]_{f(\sigma(0))} = \left[f'(p)\,v\right]_p =: f_* v_p.$$

2. By the chain rule in Exercise 8.7,

$$(g \circ f)_* v_p = \left[(g \circ f)'(p)\,v\right]_{(g\circ f)(p)} = \left[g'(f(p))\,f'(p)\,v\right]_{g(f(p))}.$$

On the other hand,

$$g_* f_* v_p = g_*\left(\left[f'(p)\,v\right]_{f(p)}\right) = \left[g'(f(p))\,f'(p)\,v\right]_{g(f(p))}$$

and hence $(g \circ f)_* v_p = g_* f_* v_p$ for all $v_p \in TU$, i.e. $(g \circ f)_* = g_* f_*$.

■

8.3 Differential Forms

Standing notation: throughout this section, let $\{e_i\}_{i=1}^n$ be the standard basis on $\mathbb{R}^n$, $\{\varepsilon_i\}_{i=1}^n$ be its dual basis, $\{x_i\}_{i=1}^n$ be the standard coordinate functions on $\mathbb{R}^n$ (so $x_i(v) = \varepsilon_i(v) = v_i$ for all $v = (v_1, \dots, v_n)^{\mathrm{tr}} \in \mathbb{R}^n$) and U be an open subset of $\mathbb{R}^n$.

Definition 8.23 (Differential k-form). *A 0-**form** on U is just a function, $f : U \to \mathbb{R}$ while (for $k \in \mathbb{N}$) a **differential** k-**form** (ω) on U is an assignment;*

$$U \ni p \to \omega_p \in \Lambda^k\left([T_p\mathbb{R}^n]^*\right) \text{ for all } p \in U.$$

The form ω is said to be C^r if for every fixed $v_1, \dots, v_k \in \mathbb{R}^n$ the function,

$$U \ni p \to \omega_p \left([v_1]_p, \dots, [v_k]_p \right) \in \mathbb{R}$$

is a C^r function.

In order to simplify notation, I will usually just write

$$\omega \left([v_1]_p, \dots, [v_k]_p \right) \text{ for } \omega_p \left([v_1]_p, \dots, [v_k]_p \right).$$

Notation 8.24 *For an open subset, $U \subset \mathbb{R}^n$ and $k \in [n]$, we let $\Omega^k (U)$ denote the collection of C^∞ (smooth) k-forms on U.*

Example 8.25. If $\{f_i\}_{i=0}^k$ are smooth functions on U, then $\omega = f_0 df_1 \wedge \cdots \wedge df_k$ defined by

$$\begin{aligned} \omega \left([v_1]_p, \dots, [v_k]_p \right) &= f_0 (p)_0 \, df_1 \wedge \cdots \wedge df_k \left([v_1]_p, \dots, [v_k]_p \right) \\ &= f_0 (p) \det \left[\left\{ df_i \left([v_j]_p \right) \right\}_{i,j=1}^k \right] \\ &= f_0 (p) \det \left[\left\{ (\partial_{v_j} f_i) (p) \right\}_{i,j=1}^k \right] \end{aligned}$$

If $f_1 = x_{l_1}, \dots, f_k = x_{l_k}$ for some $1 \le l_1 < l_2 < \cdots < l_k \le n$, then

$$\begin{aligned} \omega \left([v_1]_p, \dots, [v_k]_p \right) &= f_0 (p) \det \left[\{ \varepsilon_{l_i} (v_j) \}_{i,j=1}^k \right] \\ &= f_0 (p) \, \varepsilon_{l_1} \wedge \cdots \wedge \varepsilon_{l_k} (v_1, \dots, v_k) \end{aligned}$$

Lemma 8.26. *There is a one to one correspondence between k-forms (ω) on U and functions $\tilde{\omega} : U \to \Lambda^k ([\mathbb{R}^n]^*)$. The correspondence is determined by;*

$$\tilde{\omega} (p) (v_1, \dots, v_k) = \omega_p \left([v_1]_p, \dots, [v_k]_p \right) \text{ for all } p \in U \text{ and } \{v_i\}_{i=1}^k \subset \mathbb{R}^n.$$

Under this correspondence, ω is a C^r k-form iff $\tilde{\omega} : U \to \Lambda^k ([\mathbb{R}^n]^*)$ is a C^r-function.

Definition 8.27 (Multiplication Rules). *If $\alpha \in \Omega^k (U)$ and $\beta \in \Omega^l (U)$, we define $\alpha \wedge \beta \in \Omega^{k+l} (U)$ by requiring*

$$[\alpha \wedge \beta]_p = \alpha_p \wedge \beta_p \in \Lambda^{k+l} \left([T_p \mathbb{R}^n]^* \right) \text{ for all } p \in U.$$

If $\alpha = f \in \Omega^0 (U)$, then the above formula is to be interpreted as

$$[f \beta]_p = f (p) \beta_b \in \Lambda^l \left([T_p \mathbb{R}^n]^* \right) \text{ for all } p \in U.$$

Remark 8.28. Using the identification in Lemma 8.26, these multiplication rules are equivalent to requiring

$$\widetilde{\alpha \wedge \beta} (p) = \tilde{\alpha} (p) \wedge \tilde{\beta} (b) \text{ for all } p \in U.$$

Notation 8.29 *For*

$$J = \{1 \le j_1 < j_2 < \cdots < j_k \le n\} \subset [n], \tag{8.5}$$

let

$$\begin{aligned} dx_J &:= dx_{j_1} \wedge \cdots \wedge dx_{j_k} \in \Omega^k (\mathbb{R}^n) \text{ and} \\ \varepsilon_J &= \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k} \in \Lambda^k \left([\mathbb{R}^n]^* \right) \end{aligned}$$

Proposition 8.30. *If ω is a k-form on U, there exist unique functions $\omega_J : U \to \mathbb{R}$ such that*

$$\omega = \sum_{J \subset [n] : |J| = k} \omega_J dx_J, \tag{8.6}$$

and all possible functions $\omega_J : U \to \mathbb{R}$ may occur. Moreover, if $J \subset [n]$ as in Eq. (8.5), then ω_J is related to ω by

$$\omega_J (p) := \omega \left([e_{j_1}]_p, \dots, [e_{j_k}]_p \right) = \tilde{\omega} (p) (e_{j_1}, \dots, e_{j_k}) \text{ for all } p \in U.$$

Corollary 8.31. *If ω is given as in Eq. (8.6) then*

$$\tilde{\omega} (p) = \sum_{J \subset [n] : |J| = k} \omega_J (p) \, \varepsilon_J$$

and ω is smooth iff the functions ω_J are smooth for each $J \subset [n]$ with $|J| = k$.

Example 8.32. If $\omega \in \Omega^2 (U)$, then

$$\omega = \sum_{1 \le i < j \le n} \omega_{ij} dx_i \wedge dx_j \text{ and } \tilde{\omega} = \sum_{1 \le i < j \le n} \omega_{ij} \varepsilon_i \wedge \varepsilon_j$$

for some functions $\omega_{ij} \in \Omega^0 (U)$.

The following lemma is a direct consequence of our development of the multi-linear algebra in the previous part.

Lemma 8.33. *If $\{\alpha_i\}_{i=1}^k \subset \Omega^1 (U)$, then $\alpha_1 \wedge \cdots \wedge \alpha_k \in \Omega^k (U)$ and moreover if $\left\{ v_p^i \right\}_{i=1}^k \subset T_p U$, then*

$$\alpha_1 \wedge \cdots \wedge \alpha_k \left(v_p^1, \dots, v_p^k \right) = \det \left[\left\{ \alpha_i \left(v_p^j \right) \right\}_{i,j=1}^k \right]$$

where

$$\left[\left\{\alpha_i\left(v_p^j\right)\right\}_{i,j=1}^k\right] = \begin{bmatrix} \alpha_1(v_1) & \alpha_1(v_2) & \dots & \alpha_1(v_k) \\ \alpha_2(v_1) & \alpha_2(v_2) & \dots & \alpha_2(v_k) \\ \vdots & \vdots & \dots & \vdots \\ \alpha_k(v_1) & \alpha_k(v_2) & \dots & \alpha_k(v_k) \end{bmatrix}$$

or its transpose if you prefer.

Remark 8.34 (Book Exercise 2.3.iii). Here is some help on Exercise 8.34 in the book which asks you to show the following. Suppose U is an open subset of $\mathbb{R}^n$ and $f_j \in C^\infty(U)$ for each $j \in [n]$. Let $F(p) = (f_1(p), \dots, f_n(p))^{\mathrm{tr}}$, show

$$df_1 \wedge \dots \wedge df_n = \det F' \cdot dx_1 \wedge \dots \wedge dx_n.$$

Well by Proposition 8.30, we know that

$$df_1 \wedge \dots \wedge df_n = \omega \cdot dx_1 \wedge \dots \wedge dx_n$$

where

$$\begin{aligned} \omega_p &= df_1 \wedge \dots \wedge df_n\left([e_1]_p, \dots, [e_n]_p\right) \\ &= \det \begin{bmatrix} df_1\left([e_1]_p\right) & df_1\left([e_2]_p\right) & \dots & df_1\left([e_n]_p\right) \\ df_2\left([e_1]_p\right) & df_2\left([e_2]_p\right) & \dots & df_2\left([e_n]_p\right) \\ \vdots & \vdots & \ddots & \vdots \\ df_n\left([e_1]_p\right) & df_n\left([e_2]_p\right) & \dots & df_n\left([e_n]_p\right) \end{bmatrix} \\ &= \det\left[F'(p)\right]. \end{aligned}$$

Example 8.35. If

$$\begin{aligned} \alpha &= f_0 df_1 \wedge \dots \wedge df_k \in \Omega^k(U) \text{ and} \\ \beta &= g_0 dg_1 \wedge \dots \wedge dg_l \in \Omega^l(U) \end{aligned}$$

for some functions $\{f_j\}_{j=0}^k \cup \{g_i\}_{i=0}^l \subset \Omega^0(U)$, then

$$\alpha \wedge \beta = f_0 g_0 df_1 \wedge \dots \wedge df_k \wedge dg_1 \wedge \dots \wedge dg_l \in \Omega^{k+l}(U).$$

Exercise 8.8. Suppose that $\{x_j\}_{j=1}^4$ are the standard coordinates on $\mathbb{R}^4$, $p = (1, -1, 2, 3)^{\mathrm{tr}} \in \mathbb{R}^4$, $v^1 = (1, 2, 3, 4)^{\mathrm{tr}}$, $v^2 = (0, 1, -1, 1)^{\mathrm{tr}}$, $v^3 = (1, 0, 3, 2)$,

$$\alpha = x_4(dx_1 + dx_2), \ \beta = x_1 x_2(dx_3 + dx_4), \text{ and } \omega = \left(x_1^2 + x_3^2\right) dx_3 \wedge dx_2 \wedge dx_4.$$

Compute the following quantities;

1. $\alpha\left(v_p^1\right)$,
2. $\alpha \wedge \alpha\left(v_p^1, v_p^2\right)$,
3. $\alpha \wedge \beta\left(v_p^1, v_p^2\right)$,
4. $\omega\left(v_p^1, v_p^2, v_p^3\right)$.

Exercise 8.9. Let $\{x_i\}_{i=1}^6$ be the standard coordinates on $\mathbb{R}^6$ and let

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 \in \Omega^2\left(\mathbb{R}^6\right).$$

Show

$$\omega \wedge \omega \wedge \omega = c dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6,$$

for some $c \in \mathbb{R}$ which you should find.

8.4 Vector-Fields and Interior Products

Definition 8.36. *A* ***vector field*** *on $U \subset_o \mathbb{R}^n$, is an assignment to each $p \in U$ to and element $F(p) \in T_pU$. Necessarily, this means there exists a unique function, $f = (f_1, \dots, f_n)^{\mathrm{tr}} : U \to \mathbb{R}^n$, such that $F(p) = [f(p)]_p$ for all $p \in U$. We say F is smooth if $f \in C^\infty(U, \mathbb{R}^n)$. To simplify notation, we will often simply identify f with F.*

Definition 8.37 (Interior Product). *For $\omega \in \Omega^k(U)$ and $v_p \in T_pM$, let*

$$i_{v_p}\omega_p := \omega_p(v_p, \dots)$$

be the ***interior product*** *of v_p with $\omega_p \in \Lambda^k\left(T_p^*U\right)$ as in Definition 7.11. If F is a vector field as in Definition 8.36 we let $i_F\omega \in \Omega^{k-1}(U)$ be defined by*

$$[i_F\omega]_p = i_{F(p)}\omega_p = i_{f(p)_p}\omega_p.$$

[We will abuse notation and often just (improperly) write $i_f\omega$ for $i_F\omega$.]

Example 8.38. If $\omega = g_0 dg_1 \wedge \dots \wedge dg_k$, then from Lemma 7.12,

$$\begin{aligned} i_F\omega &= g_0 \sum_{j=1}^k (-1)^{j-1} dg_j(F_j)\, dg_1 \wedge \dots \widehat{dg_j} \wedge \dots \wedge dg_k \\ &= g_0 \sum_{j=1}^k (-1)^{j-1} (\partial_f g_j)\, dg_1 \wedge \dots \widehat{dg_j} \wedge \dots \wedge dg_k. \end{aligned}$$

If $g_0 = 1$ and $g_j = x_j$ for $1 \le j \le k$, then $dx_j(F) = f_j$ and the above formula becomes,

$$i_F(dx_1 \wedge \dots \wedge dx_k) = \sum_{j=1}^k (-1)^{j-1} f_j dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_k. \tag{8.7}$$

8.5 Pull Backs

Definition 8.39 (Pull-Back). *Suppose that $V \subset_o \mathbb{R}^m$ and $U \subset_o \mathbb{R}^n$ and $\varphi : V \to U$ is a smooth function. Then for $\omega \in \Omega^p(U)$ we define $\varphi^*\omega \in \Omega^p(V)$ by*

$$(\varphi^*\omega)(v_1, \dots, v_p) = \omega(\varphi_* v_1, \dots, \varphi_* v_p).$$

Lemma 8.40. *If $f \in \Omega^0(V)$ and $\omega = df \in \Omega^1(V)$, then*

$$\varphi^* df = d[\varphi^* f] = d(f \circ \varphi). \tag{8.8}$$

Proof. For $v_p \in T_p V$, let $\sigma(t) = \varphi(p + tv)$ and use the chain rule to find,

$$\frac{d}{dt}|_0 f(\varphi(p + tv)) = \frac{d}{dt}|_0 f(\sigma(t)) = df\left(\dot{\sigma}(0)_{\sigma(0)}\right) = df(\varphi_* v_p).$$

Therefore,

$$\begin{aligned}(\varphi^* df)(v_p) &= df(\varphi_* v_p) = \frac{d}{dt}|_0 f(\varphi(p + tv)) \\ &= \frac{d}{dt}|_0 [f \circ \varphi(p + tv)] = d(f \circ \varphi)(v_p) = d[\varphi^* f](v_p).\end{aligned}$$

■

Proposition 8.41. *If $\omega \in \Omega^k(U)$, $\eta \in \Omega^l(U)$, and φ and ψ are maps such that $\psi \circ \varphi$ makes sense, then*

$$\varphi^* \psi^* \omega = (\psi \circ \varphi)^* \omega \tag{8.9}$$

and

$$\varphi^*(\omega \wedge \eta) = \varphi^* \omega \wedge \varphi^* \eta. \tag{8.10}$$

Proof. The first identity follows from Exercise 6.2 and the second from Theorem 7.10.

■

Corollary 8.42. *Suppose that $V \subset_o \mathbb{R}^m$, $U \subset_o \mathbb{R}^n$, $\varphi : V \to U$ is a smooth function, $g_j \in C^\infty(U)$ for $0 \le j \le k$. Then*

$$\varphi^*[g_0 dg_1 \wedge \dots \wedge dg_k] = g_0 \circ \varphi \cdot d[g_1 \circ \varphi] \wedge \dots \wedge d[g_k \circ \varphi]. \tag{8.11}$$

Proof. Let $\alpha = g_0 dg_1 \wedge \dots \wedge dg_k$.
First proof. Using Eq. (8.10) it follows that

$$\varphi^* \alpha = \varphi^*(g_0 dg_1 \wedge \dots \wedge dg_k) = \varphi^* g_0 [\varphi^* dg_1 \wedge \dots \wedge \varphi^* dg_k].$$

This result along with Lemma 8.40 completes the proof of Eq. (8.11).

Second proof. Let $\{v^i\}_{i=1}^k \subset \mathbb{R}^m$ and $q \in V$, then

$$\begin{aligned}(\varphi^* \alpha)\left(v_q^1, \dots, v_q^k\right) &= \alpha_{\varphi(q)}\left(\varphi_* v_q^1, \dots, \varphi_* v_q^k\right) \\ &= g_0(\varphi(q)) \, dg_1 \wedge \dots \wedge dg_k \left(\varphi_* v_q^1, \dots, \varphi_* v_q^k\right) \\ &= g_0(\varphi(q)) \cdot \det\left[\left\{dg_i\left(\varphi_* v_q^j\right)\right\}_{i,j=1}^k\right].\end{aligned}$$

But finally we have by Lemma 8.40, that

$$dg_i\left(\varphi_* v_q^j\right) = (\varphi^* dg_i)\left(v_q^j\right) = (d[g_i \circ \varphi])\left(v_q^j\right)$$

and so

$$\det\left[\left\{dg_i\left(\varphi_* v_q^j\right)\right\}_{i,j=1}^k\right] = (d[g_1 \circ \varphi] \wedge \dots \wedge d[g_k \circ \varphi])\left(v_q^1, \dots, v_q^k\right)$$

and hence

$$(\varphi^* \alpha)\left(v_q^1, \dots, v_q^k\right) = (g_0 \circ \varphi)(q) \cdot (d[g_1 \circ \varphi] \wedge \dots \wedge d[g_k \circ \varphi])\left(v_q^1, \dots, v_q^k\right)$$

which again proves Eq. (8.11). ■

Example 8.43. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $f(x_1, x_2, x_3) = \left(x_1^2 e^{x_2}, x_1 x_3\right)$ and $\omega = x dy$ and $\alpha = \cos(xy) \, dx \wedge dy$ as forms on $\mathbb{R}^2$ where (x, y) are the standard coordinates on $\mathbb{R}^2$. Here are the solutions;

$$f^* \omega = x \circ f \cdot d[y \circ f] = x_1^2 e^{x_2} \cdot d[x_1 x_3] = x_1^2 e^{x_2} \cdot (x_3 dx_1 + x_1 dx_3)$$

and

$$\begin{aligned}f^* \alpha &= \cos\left(x_1^2 e^{x_2} x_1 x_3\right) \left[d\left(x_1^2 e^{x_2}\right)\right] \wedge [d(x_1 x_3)] \\ &= \cos\left(x_1^3 x_3 e^{x_2}\right) e^{x_2} [2x_1 dx_1 + dx_2] \wedge (x_3 dx_1 + x_1 dx_3) \\ &= \cos\left(x_1^3 x_3 e^{x_2}\right) e^{x_2} \left[2x_1^2 dx_1 \wedge dx_3 - x_3 dx_1 \wedge dx_2 + x_1 dx_2 \wedge dx_3\right].\end{aligned}$$

Basically in this case we need only let "$x = x_1^2 e^{x_2}$" and $y = x_1 x_3$ and then follows our nose in computing $\omega = x dy$ and $\alpha = \cos(xy) \, dx \wedge dy$.

Example 8.44. Let $\omega = u dv$ where $u(x, y) = \sin(x + y)$ and $v(x, y) = e^{xy}$ and suppose again that $f(x_1, x_2, x_3) = \left(x_1^2 e^{x_2}, x_1 x_3\right)$. Again the rule is to let $x = x_1^2 e^{x_2}$ and $y = x_1 x_3$ and then compute

$$\begin{aligned}f^* \omega &= \sin\left(x_1^2 e^{x_2} + x_1 x_3\right) \cdot d \exp\left(x_1^2 e^{x_2} \cdot x_1 x_3\right) \\ &= \sin\left(x_1^2 e^{x_2} + x_1 x_3\right) \cdot d \exp\left(x_1^3 x_3 e^{x_2}\right) \\ &= \sin\left(x_1^2 e^{x_2} + x_1 x_3\right) \cdot \exp\left(x_1^3 x_3 e^{x_2}\right) d\left(x_1^3 x_3 e^{x_2}\right) \\ &= \sin\left(x_1^2 e^{x_2} + x_1 x_3\right) \cdot \exp\left(x_1^3 x_3 e^{x_2}\right) \left(3x_1^2 x_3 e^{x_2} dx_1 + x_1^3 e^{x_2} dx_3 + x_1^3 x_3 e^{x_2} dx_2\right) \\ &= \sin\left(x_1^2 e^{x_2} + x_1 x_3\right) \cdot \exp\left(x_1^3 x_3 e^{x_2}\right) x_1^2 e^{x_2} \left(3x dx_1 + x_1 x_3 dx_2 + x_1 e^{x_2} dx_3\right).\end{aligned}$$

8.6 Exterior Differentiation

Now that we have defined forms it is natural to try to differentiate these forms. We have already differentiated 0-forms, f, to get a 1-form df. So it is natural to generalize this definition as follows.

Definition 8.45 (Exterior Differentiation). *If $\omega = \sum_J \omega_J dx_J \in \Omega^k(U)$, we define*

$$d\omega := \sum_J d\omega_J \wedge dx_J \in \Omega^{k+1}(U) \tag{8.12}$$

or equivalently,

$$d\omega = \sum_{i=1}^n \sum_J (\partial_i \omega_J)\, dx_i \wedge dx_J.$$

It turns out in order to compute $d\omega$ you only need to use the Properties of d explained in the next proposition. You may wish to skip the proof of this proposition until after seeing examples of computing $d\omega$ and doing the related exercises.

Proposition 8.46 (Properties of d). *The exterior derivative d satisfies the following properties;*

1. *$df(v_p) = (\partial_v f)(p)$ for $f \in \Omega^0(U)$.*
2. *$d : \Omega^p(U) \to \Omega^{p+1}(U)$ is a linear map for all $0 \le p < n$.*
3. *d satisfies the product rule*
$$d[\omega \wedge \eta] = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$$
for all $\omega \in \Omega^p(U)$ and $\eta \in \Omega^q(U)$.
4. *$d^2\omega = 0$ for all $\omega \in \Omega^p(U)$.*
Suggestion: *rather than reading the proof on your first pass, instead jump to Lemma 8.47 and continue reading from there. Come back to the proof after you have some experience with computing with d.*

Proof. In terms of the identification of $\omega \in \Omega^p(U)$ with $\tilde{\omega} \in C^\infty\left(U, \Lambda^k\left([\mathbb{R}^n]^*\right)\right)$ in Lemma 8.26 we have

$$\widetilde{d\omega} = \sum_{i=1}^n \sum_J (\partial_i \omega_J)\, \varepsilon_i \wedge \varepsilon_J = \sum_{i=1}^n \varepsilon_i \wedge \sum_J (\partial_i \omega_J)\, \varepsilon_J$$

which may be written as

$$\widetilde{d\omega} = \hat{d}\tilde{\omega} := \sum_{i=1}^n \varepsilon_i \wedge \partial_i \tilde{\omega}. \tag{8.13}$$

This last equation describes $d\omega$ without first expanding ω as a linear combination of the $\{dx_J\}$. This turns out to be quite convenient for deducing the basic properties of the exterior derivative stated in this proposition. To simplify notation in this proof we will not distinguish between ω and $\tilde{\omega}$ and d and $\hat{d}$ and we will exclusively (in this proof) view forms as function from U to $\Lambda^k\left([\mathbb{R}^n]^*\right)$. We now go to the proof proper.

The first item immediate from the linearity of the derivative operator. The second item is consequence of the product rule for differentiation;

$$\begin{aligned} d[\omega \wedge \eta] &= \sum_{j=1}^n \varepsilon_j \wedge \frac{\partial}{\partial x_j}[\omega \wedge \eta] = \sum_{j=1}^n \varepsilon_j \wedge \left[\frac{\partial \omega}{\partial x_j} \wedge \eta + \omega \wedge \frac{\partial \eta}{\partial x_j}\right] \\ &= \sum_{j=1}^n \varepsilon_j \wedge \frac{\partial \omega}{\partial x_j} \wedge \eta + \sum_{j=1}^n \varepsilon_j \wedge \omega \wedge \frac{\partial \eta}{\partial x_j} \\ &= d\omega \wedge \eta + (-1)^p \omega \wedge \left(\sum_{j=1}^n \varepsilon_j \wedge \frac{\partial \eta}{\partial x_j}\right) \\ &= d\omega \wedge \eta + (-1)^p \omega \wedge d\eta. \end{aligned}$$

Lastly,

$$\begin{aligned} d^2\omega &= \sum_{i=1}^n \varepsilon_i \wedge \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n \varepsilon_j \wedge \frac{\partial \omega}{\partial x_j}\right) \\ &= \sum_{i,j=1}^n \varepsilon_i \wedge \varepsilon_j \wedge \frac{\partial^2 \omega}{\partial x_i \partial x_j} \\ &= \frac{1}{2}\sum_{i,j=1}^n \left[\varepsilon_i \wedge \varepsilon_j \wedge \frac{\partial^2 \omega}{\partial x_i \partial x_j} + \varepsilon_j \wedge \varepsilon_i \wedge \frac{\partial^2 \omega}{\partial x_j \partial x_i}\right] \\ &= \frac{1}{2}\sum_{i,j=1}^n \left[\varepsilon_i \wedge \varepsilon_j \wedge \frac{\partial^2 \omega}{\partial x_i \partial x_j} - \varepsilon_i \wedge \varepsilon_j \wedge \frac{\partial^2 \omega}{\partial x_i \partial x_j}\right] = 0, \end{aligned}$$

wherein we have used the fact that mixed partial derivatives of C^2-functions (vector-valued or not) are equal. ■

Lemma 8.47. *If $\{g_j\}_{j=0}^p \subset \Omega^0(U)$, then*

$$d[g_0 \cdot g_1 \wedge \dots \wedge dg_p] = dg_0 \wedge dg_1 \wedge \dots \wedge dg_p. \tag{8.14}$$

This formula along with the knowing df for $f \in \Omega^0(U)$ completely determines d on $\Omega^p(U)$.

Proof. The proof is by induction on p. Rather than do the general induction argument, let me explain the case $p=3$ in detail so that $\omega = g_0 dg_1 \wedge dg_2 \wedge dg_3$. Then using **only** the properties developed in Proposition 8.46,

$$d\omega = dg_0 \wedge [dg_1 \wedge dg_2 \wedge dg_3] + g_0 d\,[dg_1 \wedge dg_2 \wedge dg_3]$$

where

$$\begin{aligned} d\,[dg_1 \wedge dg_2 \wedge dg_3] &= d\,[dg_1 \wedge (dg_2 \wedge dg_3)] \\ &= d^2 g_1 \wedge (dg_2 \wedge dg_3) - dg_1 \wedge d\,(dg_2 \wedge dg_3) \\ &= 0 - dg_1 \wedge \left[d^2 g_2 \wedge dg_3 - dg_2 \wedge d^2 g_3\right] = 0. \end{aligned}$$

Thus we have shown

$$d\omega = dg_0 \wedge dg_1 \wedge dg_2 \wedge dg_3$$

as desired. ■

The next corollary shows that the properties in Proposition 8.46 actually uniquely determines the exterior derivative, d.

Corollary 8.48. *If $d : \Omega^*(U) \to \Omega^{*+1}(U)$ is any linear operator satisfying the four properties in Proposition 8.46, then d is in fact given as in Definition 8.45.*

Proof. Let $\omega = \sum_J \omega_J dx_J \in \Omega^k(U)$ where the sum is over $J \subset [n]$ with $|J| = k$. By Lemma 8.47, which was proved using only the properties in Proposition 8.46, we know that

$$\begin{aligned} d\,[\omega_J dx_J] &= d\,[\omega_J dx_{j_1} \wedge \cdots \wedge dx_{j_k}] = d\omega_J \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k} \\ &= d\omega_J \wedge dx_J. \end{aligned}$$

Thus using the assumed linearity of d, it follows that

$$d\omega = \sum_J d\omega_J \wedge dx_J$$

in agreement with the definition in Eq. (8.12). ■

Example 8.49. In this example, let x, y, z be the standard coordinates on $\mathbb{R}^3$ (actually any smooth function on $\mathbb{R}^3$ or $\mathbb{R}^k$ for that matter would work). If

$$\alpha = xdy - ydx + zdz,$$

then

$$d\alpha = dx \wedge dy - dy \wedge dx + dz \wedge dz = 2dx \wedge dy.$$

If $\beta = e^{x+y^2+z^3} dx \wedge dy$, then

$$\begin{aligned} de^{x+y^2+z^3} &= e^{x+y^2+z^3} \cdot d\left(x + y^2 + z^3\right) \\ &= e^{x+y^2+z^3} \cdot \left(dx + 2ydy + 3z^2 dz\right) \end{aligned}$$

and therefore,

$$\begin{aligned} d\beta &= de^{x+y^2+z^3} \wedge dx \wedge dy = e^{x+y^2+z^3} \cdot \left(dx + 2ydy + 3z^2 dz\right) \wedge dx \wedge dy \\ &= e^{x+y^2+z^3} \cdot 3z^2 dz \wedge dx \wedge dy = e^{x+y^2+z^3} \cdot 3z^2 dx \wedge dy \wedge dz. \end{aligned}$$

Definition 8.50. *A form $\omega \in \Omega^k(U)$ is **closed** if $d\omega = 0$ and it is **exact** if $\omega = d\mu$ for some $\mu \in \Omega^{k-1}(U)$.*

Note that if $\omega = d\mu$, then $d\omega = d^2\mu = 0$, so exact forms are closed but the converse is not always true.

Example 8.51. In this example, again x, y, z be the standard coordinates on $\mathbb{R}^3$ (actually any smooth function on $\mathbb{R}^3$ or $\mathbb{R}^k$ for that matter would work). If

$$\alpha = ydx + (z\cos yz + x)\,dy + y\cos yz dz$$

then

$$d\alpha = dy \wedge dx + ((\cos yz - yz\sin yz)\,dz + dx) \wedge dy + (\cos yz - zy\sin yz)\,dy \wedge dz = 0,$$

i.e. α is closed, see Definition 8.50.

Exercise 8.10. Let $\alpha = xdx - ydy$, $\beta = zdx \wedge dy + xdy \wedge dz$ and $\gamma = zdy$ on $\mathbb{R}^3$, calculate,

$$\alpha \wedge \beta, \quad \alpha \wedge \beta \wedge \gamma, \quad d\alpha, \quad d\beta, \quad d\gamma.$$

Exercise 8.11. Let (x, y) be the standard coordinates on $\mathbb{R}^2$, and define,

$$\alpha := \left(x^2 + y^2\right)^{-1} \cdot (xdy - ydx) \in \Omega^1\left(\mathbb{R}^2 \setminus \{0\}\right).$$

Show α is closed. [We will eventually see that this form is not exact.]

Exercise 8.12 (Divergence Formula). Let $f = (f_1, f_2, f_3, \dots, f_n)$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$. By Example 8.38 with $k = n$ we have

$$i_f\omega = i_F\omega = \sum_{j=1}^n (-1)^{j-1} f_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Show

$$d\,[i_F\omega] = (\nabla \cdot f)\,\omega \text{ where } \nabla \cdot f = \sum_{i=1}^n \partial_i f_i,$$

i.e. $\nabla \cdot f$ is the divergence of f from your vector calculus course.

Exercise 8.13 (Curl Formula). Let $f = (f_1, f_2, f_3) \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$,

$$\omega = dx_1 \wedge dx_2 \wedge dx_3, \text{ and}$$
$$\alpha = f \cdot (dx_1, dx_2, dx_3) := f_1 dx_1 + f_2 dx_2 + f_3 dx_3.$$

Show $d\alpha = i_{\nabla \times f}\omega$ where $\nabla \times f$ is the usual vector calculus curl of f, see Eq. (7.12) of Definition 7.19 with F replaced by $f = (f_1, f_2, f_3)$.

Theorem 8.52 (d commutes with φ^*). *Suppose that $V \subset_o \mathbb{R}^m$ and $U \subset_o \mathbb{R}^n$ and $\varphi : V \to U$ is a smooth function. Then d commutes with the pull-back, φ^*. In more detail, if $0 \le p \le m$ and $\alpha \in \Omega^p(V)$ then $d(\varphi^*\alpha) = \varphi^*(d\alpha)$.*

Proof. We may assume that $\alpha = g_0 dg_1 \wedge \cdots \wedge dg_p$ in which case

$$\begin{aligned}\varphi^*\alpha &= \varphi^*(g_0 dg_1 \wedge \cdots \wedge dg_p) \\ &= \varphi^* g_0 \left[d(\varphi^* g_1) \wedge \cdots \wedge d(\varphi^* g_p)\right]\end{aligned}$$

and so

$$d\varphi^*\alpha = d(\varphi^* g_0) \wedge d(\varphi^* g_1) \wedge \cdots \wedge d(\varphi^* g_p)$$

while from Lemma 8.47,

$$\begin{aligned}\varphi^* d\alpha &= \varphi^* dg_0 \wedge \varphi^* dg_1 \wedge \cdots \wedge \varphi^* dg_p \\ &= d(\varphi^* g_0) \wedge d(\varphi^* g_1) \wedge \cdots \wedge d(\varphi^* g_p) = d[\varphi^*\alpha].\end{aligned}$$

■

9

An Introduction of Integration of Forms

One of the main point of differential k-forms is that they may be integrated over k-dimensional manifolds. Although we are not going to define the notation of a manifold at this time, please do have a look at Chapter 6 starting on page 75 of Reyer Sjamaar's notes: Manifolds and Differential Forms, for the notion of a manifold and associated tangent spaces along with lots of pictures! (Pictures is one thing in short supply in our book.)

9.1 Integration of Forms Over "Parameterized Surfaces"

Definition 9.1 (Basic integral). *If $D \subset_o \mathbb{R}^k$ and $\alpha = f dx_1 \wedge \cdots \wedge dx_k \in \Omega^k(D)$, we define*

$$\int_D \alpha := \int_D f dm$$

provided the latter integral makes sense, i.e. provided $\int_D |f| \, dm < \infty$. Often times we will guarantee this to be the case by assuming $f \in C_c^\infty(D)$.

We now want elaborate on this basic integral.

Definition 9.2. *Let $D \subset_o \mathbb{R}^k$ and $U \subset_o \mathbb{R}^n$. We say a smooth function, $\gamma : D \to U$, is a* ***parametrized k-surface*** *in U.*

Definition 9.3. *If $\gamma : D \to U$ is a parametrized k-surface in U and $\omega \in \Omega^k(U)$ is a k-form, then we define,*

$$\int_\gamma \omega := \int_D \gamma^* \omega.$$

Example 9.4 (Line Integrals). Suppose that $\omega = \sum_{j=1}^n f_j dx_j \in \Omega^1(U)$ and $\gamma = (\gamma_1, \dots, \gamma_n)^{\mathrm{tr}} : [a,b] \to U$ is a smooth curve, then letting t be the standard coordinate on $\mathbb{R}$ (i.e. $t(a) = a$ for all $a \in \mathbb{R}$) we find,

$$\gamma^* f = \sum_{j=1}^n f_j \circ \gamma(t) \, d(x_j \circ \gamma(t)) = \sum_{j=1}^n f_j(\gamma(t)) \, d(\gamma_j(t))$$
$$= \sum_{j=1}^n f_j(\gamma(t)) \, \dot{\gamma}_j(t) \, dt$$

and hence

$$\int_\gamma f = \int_{[a,b]} \sum_{j=1}^n f_j(\gamma(t)) \, \dot{\gamma}_j(t) \, dt = \int_a^b \mathbf{f}(\gamma(t)) \cdot \dot{\gamma}(t) \, dt$$

where $\mathbf{f} = (f_1, \dots, f_n)^{\mathrm{tr}}$ thought of as a vector field on U.

Example 9.5 (Integrals over surfaces). Suppose that $D = (-1,1)^2 \subset \mathbb{R}^2$, $U = \mathbb{R}^3$, and $\gamma(x,y) = \left(x, y, 2 - x^2 - y^2\right)$ as in Figure 9.1 and let

$$\gamma^* \omega = f dx \wedge dy.$$

In this picture we have divided the base up into little square and then found

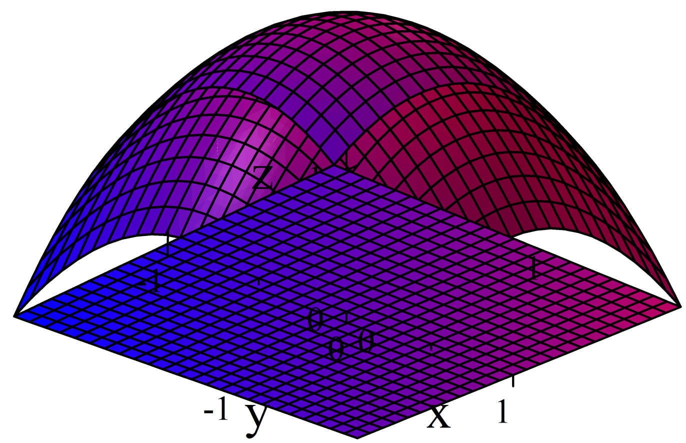


Fig. 9.1. This is the plot of graph of $\gamma(x,y) = \left(x, y, 2 - x^2 - y^2\right)$ over D.

their images under γ. It is reasonable to assign a contribution to $\int_\gamma \omega$ from a little base square, $Q_j := p_j + \varepsilon [0,1]^2$ to be approximately,

$$\omega\left(\gamma_*\left([\varepsilon e_1]_{p_j}\right), \gamma_*\left([\varepsilon e_2]_{p_j}\right)\right) = (\gamma^* \omega)\left([\varepsilon e_1]_{p_j}, [\varepsilon e_2]_{p_j}\right)$$
$$= f(p_j) \, \varepsilon^2 = f(p_j) \cdot \mathrm{Area}(Q_j)$$

and therefore we should have

$$\int_\gamma \omega \cong \sum_j f(p_j) \cdot \text{Area}(Q_j) \to \int_D f dm \text{ as } \varepsilon \downarrow 0.$$

Theorem 9.6 (Stoke's Theorem II). *Suppose that* $D = \mathbb{H} = \{(p_1, \dots, p_n) \in \mathbb{R}^n : p_1 \le 0\}$ *is the* ***"left half space",*** $U \subset_o \mathbb{R}^m$, $\gamma : D \to U$ *is a parametrized* n*-surface*[1], *and* $\mu \in \Omega^{n-1}(U)$, *then assuming that* $\gamma^* \mu$ *is the restriction of a smooth compactly supported* $n-1$*-form on* $\mathbb{R}^{n-1}$, *we have*

$$\int_\gamma d\mu = \int_{\partial\gamma} \mu$$

where $\partial\gamma : \mathbb{R}^{n-1} \to U$ *is defined by*

$$\partial\gamma(t_1, \dots, t_{n-1}) = \gamma(0, t_1, \dots, t_{n-1}).$$

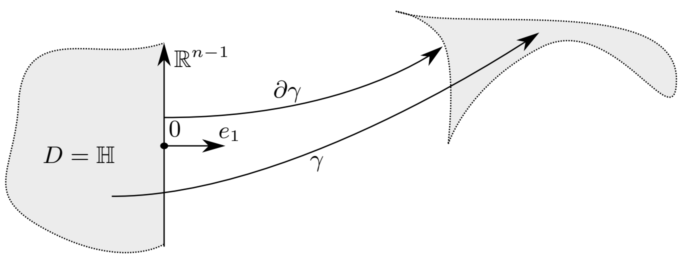


Fig. 9.2. Half space

Proof. If, as in book Exercise 3.2viii (our first version of Stoke's theorem), we let $i : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the inclusion map,

$$\iota(t_1, \dots, t_{n-1}) := (0, t_1, \dots, t_{n-1}),$$

then $\partial\gamma := \gamma \circ \iota : \mathbb{R}^{n-1} \to U$. Therefore, using pull-backs commute with d, the definitions of integration we have given along with your book Exercise 3.2viii, we find,

$$\begin{aligned}\int_\gamma d\mu &:= \int_{\mathbb{H}} \gamma^* d\mu = \int_{\mathbb{H}} d[\gamma^*\mu] = \int_{\mathbb{R}^{n-1}} \iota^* [\gamma^* \mu] \\ &= \int_{\mathbb{R}^{n-1}} (\gamma \circ \iota)^* \mu = \int_{\mathbb{R}^{n-1}} (\partial\gamma)^* \mu =: \int_{\partial\gamma} \mu.\end{aligned}$$

■

[1] We assume γ extends to a smooth function into U on an open neighborhood of $\mathbb{H}$.

9.2 The Goal (Degree Theorem):

In the end of the day we would really like to define an integral of the form, $\int_{\gamma(D)} \omega$, by which we mean we want the integral to depend only on the image of γ and not on the particular choice of parametrization of this image. For example of $f : D' \to D$ is a **diffeomorphism,** so that $\gamma(D) = \gamma \circ f(D')$, we are going to want,

$$\int_D \gamma^* \omega = \int_\gamma \omega = \int_{\gamma \circ f} \omega = \int_{D'} (\gamma \circ f)^* \omega = \int_{D'} f^* (\gamma^* \omega).$$

In other words we would like to show if $f : D' \to D$ is a diffeomorphism then the following change of variable theorem hold,

$$\int_D \alpha = \int_{D'} f^* \alpha \text{ for all } \alpha \in \Omega_c^k(D). \tag{9.1}$$

This last assertion will actually only be true up to sign ambiguity when D is connected and we will have to take care of this sign ambiguity later by introducing the notion of an orientation. Nevertheless, the next very important step in our development of integration of forms is to find how to relate $\int_{D'} f^* \alpha$ to $\int_D \alpha$. This will lead us to the deepest topic of this course, namely degree theory and the change of variables theorem.

Definition 9.7 (Compact support). *If* V *is an open subset of* $\mathbb{R}^n$ *we let* $\Omega_c^k(V)$ *denote the* "compactly supported" k*-forms on* V. *This means there is a closed and bounded subset,* $K \subset V$, *such that* $\omega_p = 0$ *if* $p \notin K$.

Here is the statement of the theorem[2] we are heading to prove.

Theorem 9.8. *Let* U *and* V *be connected open subsets of* $\mathbb{R}^n$ *and* $f : U \to V$ *be a smooth proper*[3] *map. Then there exists an integer,* $\deg(f) \in \mathbb{Z}$, *such that*

$$\int_U f^* \omega = \deg(f) \cdot \int_V \omega \text{ for all } \omega \in \Omega_c^n(V).$$

The degree function has the following properties;

1. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$,
2. $\deg(f) = 1\ (-1)$ *if* f *is an orientation preserving (reversing) diffeomorphism,*

[2] This theorem is probably is the most important and deepest theorem of this course.
[3] If U and V are bounded open sets then $\varphi : U \to V$ is proper if $\varphi(p)$ is near the boundary of V when p is near the boundary of U. The general definition of proper will be given later.

3. $\deg(f) = 0$ *if* $f(U) \subsetneqq V$,
4. if $q \in V$ *is a regular value*[4] *of* f, *then*

$$\deg(f) = \sum_{p \in f^{-1}(\{q\})} \operatorname{sgn}(\det f'(q)).$$

Example 9.9. If $U = (a,b)$ and $V = (c,d)$ are bounded open intervals in $\mathbb{R}$ and $f : U \to V$ is a smooth map. There are four possible ways for f to be a proper map, see Corollary 10.49 below. In what follows we write $f(a+) := \lim_{x \downarrow a} f(x)$ and $f(b-) := \lim_{x \uparrow b} f(x)$.

1. $f(a+) = f(b-) = c$.
2. $f(a+) = f(b-) = d$.
3. $f(a+) = c$ and $f(b-) = d$
4. $f(a+) = d$ and $f(b-) = c$.

In the first two case $\deg(f) = 0$ while in case $\deg(f) = 1$ and $\deg(f) = -1$ in case 4, see Figure 9.3 below.

Exercise 9.1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the smooth map,

$$f(x,y) = \left(x^2 - y^2, 2xy\right).$$

Show $\|f(x,y)\| \to \infty$ when $\|(x,y)\| \to \infty$ which turns out to be equivalent to the statement the f is proper, see Example 10.43 below. Further compute the $\deg(f)$.

We are going to spend a fair bit of time understanding the hypothesis and proving Theorem 9.8. To understand the hypothesis we need to discuss some basic topological notions, which we do in the next chapter. We will also need to construct lots of smooth compactly supported functions on open subsets of $\mathbb{R}^n$. The next section is the grist mill for constructing smooth compactly supported functions.

9.3 Constructing elements of C_c^∞ (open rectangles)

Here is the statement of the theorem[5] we are heading to prove.

Exercise 9.2. Let

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0, \end{cases}$$

see Figure 9.4. Show $f \in C^\infty(\mathbb{R}, [0,1])$. Here is a possible outline.

[4] A point $q \in V$ is a **regular value** of φ if for all $p \in \varphi^{-1}(\{q\})$, $\det \varphi'(p) \neq 0$.
[5] This theorem is probably is the most important and deepest theorem of this course.

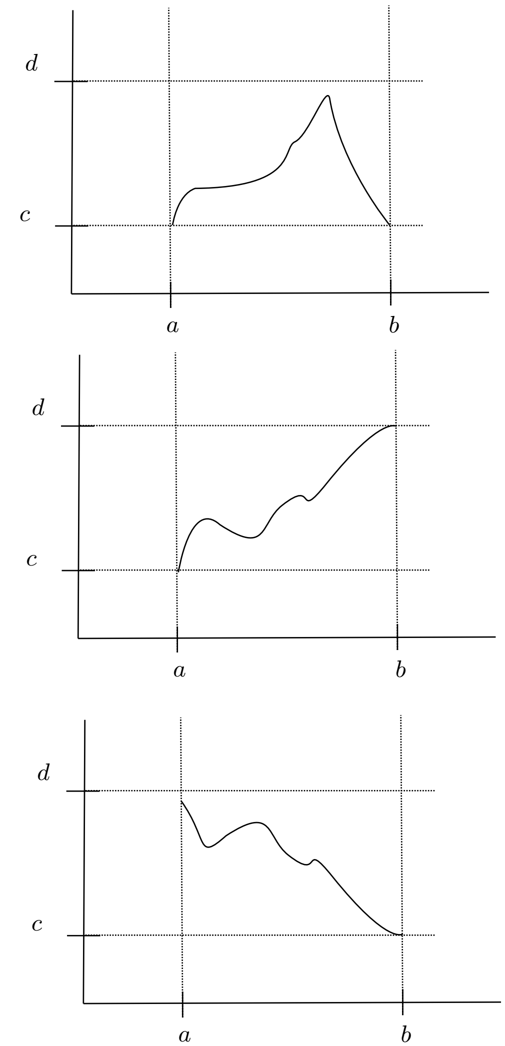


Fig. 9.3. Here are three examples of proper map one dimensional maps with their degrees which are 0, -1, and 1 respectively. Let us also note in the first case, the map is not surjective which always leads to a degree zero map. In the one dimensional case, the degree can only take the values $\{0, 1, -1\}$.

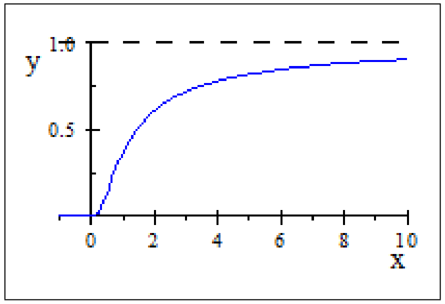


Fig. 9.4. Plot of $y = f(x)$.

1. First show for $x > 0$ and $n \in \mathbb{N}$ there is a polynomial function, $p_n(t)$, such that $f^{(n)}(x) = p_n\left(x^{-1}\right) f(x)$. For example $p_1(t) = t^2$ and $p_2(t) = t^4 - 2t^3$.
2. Show $\lim_{t\to\infty}\left(t^n e^{-t}\right) = 0$ for all $n \in \mathbb{N}$ (hint take logarithms) and then use this to show $\lim_{x\downarrow 0} f^{(n)}(x) = 0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
3. Show by the mean value theorem along with induction on n that $f \in C^n(\mathbb{R}, \mathbb{R})$ and $f^{(k)}(0) = 0$ for $0 \le k \le n$ for all $n \in \mathbb{N}$.

Notation 9.10 (Approximate Heavyside) *For $a > 0$, let*

$$H_a(x) = f(x) / (f(x) + f(a - x))$$

so that $H_a \in C^\infty(\mathbb{R}, [0,1])$ such that $H_a(x) = 0$ for $x \le 0$, $H_a(x) = 1$, see Figure 9.5. [H_a is a smooth approximation to the Heaviside function and tends to a heaviside function in the limit as $a \downarrow 0$.]

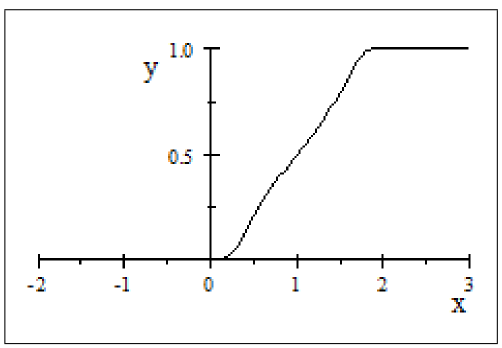


Fig. 9.5. This is a plot of the funciton, $H_2(x) = f(x) / (f(x) + f(x-2))$.

Definition 9.11. *For $-\infty < a < b < \infty$, let*

$$\varphi_{a,b}(x) = f(a + x) f(b - x),$$

see Figure 9.6.

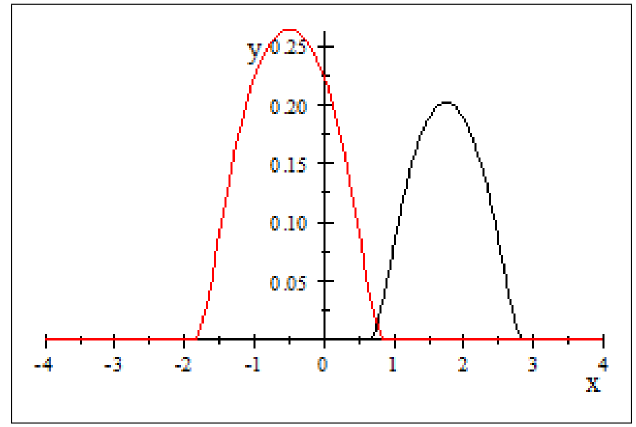


Fig. 9.6. Here is the plots for $\varphi_{0.5,3}$ (black) and $\varphi_{-2,1}$ (red).

These bump functions, $\varphi_{a,b}$, satisfy; 1) $\varphi_{a,b} \in C_c^\infty(\mathbb{R}, [0,1])$, 2) $\varphi_{a,b}(x) > 0$ for $x \in (a,b)$, 3) and $\varphi_{a,b}(x) = 0$ for $x \notin (a,b)$ and in particular $\operatorname{supp}(\varphi_{a,b}) = \overline{(a,b)} = [a,b]$.

Definition 9.12. *For $a, b \in \mathbb{R}^d$, let us write $a < b$ to mean $a_j < b_j$ for $j \in [d]$ and when $a < b$, let $R = (a,b)$ and $\bar{R} = [a,b]$ denote the open and closed rectangles respectively,*

$$R = (a,b) := (a_1, b_1) \times \cdots \times (a_d, b_d) \text{ and}$$
$$\bar{R} = [a,b] := [a_1, b_1] \times \cdots \times [a_d, b_d].$$

We further for $R = (a,b)$, let $\varphi_R = \varphi_{a_1,b_1} \otimes \cdots \otimes \varphi_{a_d,b_d}$, i.e. for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let

$$\varphi_R(x) = \varphi_{a_1,b_1}(x_1) \otimes \cdots \otimes \varphi_{a_d,b_d}(x_d).$$

Note again that 1) $\varphi_R \in C_c^\infty\left(\mathbb{R}^d, [0,1]\right)$, 2) $\varphi_R(x) > 0$ for $x \in (a,b)$, 3) and $\varphi_R(x) = 0$ for $x \notin (a,b)$ and in particular $\operatorname{supp}(\varphi_R) = \overline{(a,b)} = [a,b]$. We will use these functions a little later to construct many for compactly supported smooth functions.

10

Elements of Point Set (Metric) Topology

Definition 10.1 (Euclidean metric). *Let X be a subset of $\mathbb{R}^n$ and for $p, q \in X$, let*

$$d(p,q) := \|p-q\| = \sqrt{\sum_{j=1}^{n} (p_j - q_j)^2} \tag{10.1}$$

be the Euclidean distance between p and q.

Remark 10.2. The function d has the following basic properties for all $p, q, r \in X$

1. $d(p,q) \geq 0$ with $d(p,q) = 0$ iff $p = q$,
2. $d(p,q) = d(q,p)$ and
3. $d(p,q) \leq d(p,r) + d(r,q)$.

Definition 10.3 (Metric Space). *A metric space is a set X equipped with a function, $d : X \times X \to [0,\infty)$, satisfying the three properties in Remark 10.2 above.*

Example 10.4. There are many examples of metric spaces. For example, one might let X be the points on the surface of the earth and for $p, q \in X$, let $d(p,q)$ be the geodesic distance between p and q. This distance function generalizes to general "Riemannian manifolds."

Although we are likely to only use metric spaces of the form in Definition 10.1, I will give many of the basic definitions and properties for general metric spaces as it require no extra work. You are free to always assume that X is a subset of $\mathbb{R}^n$ and $d(p,q)$ is given by Eq. (10.1) if you prefer.

Definition 10.5 (Limits of sequences). *A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X,d) is said to be* ***convergent*** *if there exists a point $x \in X$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. In this case we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

Exercise 10.1. Show that x in Definition 10.5 is necessarily unique by showing $d(x,y) = 0$ if $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

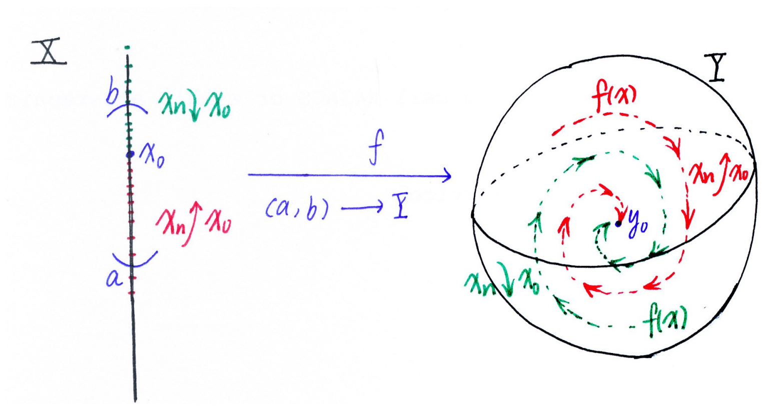
10.1 Continuity

Definition 10.6 (Continuity). *Let (X,d) and (Y,d_Y) be two metric spaces. A function $f : X \to Y$ is* ***continuous at*** *$x \in X$ if*

$$\lim_{n\to\infty} f(x_n) = f(x) \text{ for all } \{x_n\}_{n=1}^{\infty} \subset X \text{ with } \lim_{n\to\infty} x_n = x.$$

We say f is continuous on X if it is continuous at all points in X which may be stated as

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) \text{ whenever } \lim_{n\to\infty} x_n \text{ exists in } X.$$



Definition 10.7. *A function $f : (X,d) \to (Y,d_Y)$ is said to be Lipschitz (continuous) if there exists $K < \infty$ such that*

$$d_Y(f(x), f(x')) \leq K d(x,x') \text{ for all } x, x' \in X. \tag{10.2}$$

[Note f is continuous since if $x_n \to x$ then

$$d_Y(f(x_n), f(x)) \leq K d(x_n, x) \to 0 \text{ as } n \to \infty.]$$

If f satisfies Eq. (10.2) we will say that f is Lip $-K$ ***continuous.***

Example 10.8. Any function, $f:\mathbb{R}\to\mathbb{R}$ which is everywhere differentiable is Lipschitz iff $K:=\sup_{t\in\mathbb{R}}|f'(t)|<\infty$. Indeed if

$$|f(y)-f(x)|\le K|y-x| \text{ for all } x,y\in\mathbb{R}$$

then

$$|f'(x)|=\lim_{y\to x}\frac{|f(y)-f(x)|}{|y-x|}\le K \text{ for all } x\in\mathbb{R}.$$

Conversely, if $K:=\sup_{t\in\mathbb{R}}|f'(t)|<\infty$, then by the mean value theorem, for all $y>x$ there exists $c\in(x,y)$ such that

$$\left|\frac{f(y)-f(x)}{y-x}\right|=|f'(c)|\le K.$$

It turns out that every metric spaces with an infinite number of elements comes equipped with a large collection of Lipschitz functions.

Lemma 10.9 (Distance to a Set). *For any non empty subset $A\subset X$, let*

$$d_A(x):=\inf\{d(x,a)|a\in A\},$$

then

$$|d_A(x)-d_A(y)|\le d(x,y)\ \forall x,y\in X, \tag{10.3}$$

i.e. $d_A:X\to[0,\infty)$ is Lip-1 *continuous.*

Proof. Let $a\in A$ and $x,y\in X$, then

$$d_A(x)\le d(x,a)\le d(x,y)+d(y,a).$$

Take the infimum over a in the above equation shows that

$$d_A(x)\le d(x,y)+d_A(y)\quad \forall x,y\in X.$$

Therefore, $d_A(x)-d_A(y)\le d(x,y)$ and by interchanging x and y we also have that $d_A(y)-d_A(x)\le d(x,y)$ which implies Eq. (10.3). ∎

Example 10.10. For fixed $x\in X$, the function $f(y)=d(x,y)$ is Lip-1 continuous as is seen by taking $A=\{x\}$ in Lemma 10.9.

Corollary 10.11 (Optional on first read). *The function d satisfies,*

$$|d(x,y)-d(x',y')|\le d(y,y')+d(x,x').$$

Therefore $d:X\times X\to[0,\infty)$ is continuous in the sense that $d(x,y)$ is close to $d(x',y')$ if x is close to x' and y is close to y'. In particular, if $x_n\to x$ and $y_n\to y$ then

$$\lim_{n\to\infty}d(x_n,y_n)=d(x,y)=d\left(\lim_{n\to\infty}x_n,\lim_{n\to\infty}y_n\right).$$

Proof. First Proof. By Lemma 10.9 for single point sets and the triangle inequality for the absolute value of real numbers,

$$\begin{aligned}|d(x,y)-d(x',y')|&\le|d(x,y)-d(x,y')|+|d(x,y')-d(x',y')|\\&\le d(y,y')+d(x,x').\end{aligned}$$

Second Proof. By the triangle inequality,

$$d(x,y)\le d(x,x')+d(x',y)\le d(x,x')+d(x',y')+d(y',y)$$

from which it follows that

$$d(x,y)-d(x',y')\le d(x,x')+d(y',y).$$

Interchanging x with x' and y with y' in this inequality shows,

$$d(x',y')-d(x,y)\le d(x,x')+d(y',y)$$

and the result follows from the last two inequalities. ∎

Example 10.12. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(x)=\begin{cases}1 \text{ if } x\in\mathbb{Q}\\ 0 \text{ if } x\notin\mathbb{Q}.\end{cases}$$

The function f is discontinuous at all points in $\mathbb{R}$. For example, if $x_0\in\mathbb{Q}$ we may choose $x_n\in\mathbb{R}\setminus\mathbb{Q}$ such that $\lim_{n\to\infty}x_n=x_0$ while

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}0=0\ne1=f(x_0).$$

Similarly if if $x_0\in\mathbb{R}\setminus\mathbb{Q}$ we may choose $x_n\in\mathbb{Q}$ such that $\lim_{n\to\infty}x_n=x_0$ while

$$\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}1=1\ne0=f(x_0).$$

Exercise 10.2. Consider $\mathbb{N}$ as a metric space with $d(m,n):=|m-n|$ and suppose that (Y,d) is a metric space. Show that every function, $f:\mathbb{N}\to Y$ is continuous.

We will assume the reader is familiar with the basic properties of complex numbers and in particular with the following theorem.

Theorem 10.13. *If $\{w_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ are convergent sequences of complex numbers, then*

1. $\lim_{n\to\infty}(w_n+z_n)=\lim_{n\to\infty}w_n+\lim_{n\to\infty}z_n$.
2. $\lim_{n\to\infty}(w_n\cdot z_n)=\lim_{n\to\infty}w_n\cdot\lim_{n\to\infty}z_n$.

3. *if we further assume that* $\lim_{n\to\infty} z_n \neq 0$*, then*

$$\lim_{n\to\infty}\left(\frac{w_n}{z_n}\right) = \frac{\lim_{n\to\infty} w_n}{\lim_{n\to\infty} z_n}.$$

Exercise 10.3. Suppose that (X,d) is a metric space and $f,g : X \to \mathbb{C}$ are two continuous functions on X. Show;

1. $f+g$ is continuous,
2. $f\cdot g$ is continuous,
3. f/g is continuous provided $g(x) \neq 0$ for all $x \in X$.

Example 10.14. The functions $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ defined by $f(z) = 1/z$ is continuous since $g(z) = z$ is Lip-1 continuous as $|g(z) - g(w)| = |z-w|$, it follows that $f(z) = 1/g(z)$ is continuous by part 3. of Exercise 10.3.

Exercise 10.4. (You only need explain parts 1. and 5. of this problem as we have done the other parts in class.) Show the following functions from $\mathbb{C}$ to $\mathbb{C}$ are continuous.

1. $f(z) = c$ for all $z \in \mathbb{C}$ where $c \in \mathbb{C}$ is a constant.
2. $f(z) = |z|$. [**Hint:** $f(z) = d(z,0)$ where $d(z,w) := |z-w|$ is the Euclidean norm on $\mathbb{C} \cong \mathbb{R}^2$.]
3. $f(z) = z$ and $f(z) = \bar{z}$.
4. $f(z) = \operatorname{Re} z$ and $f(z) = \operatorname{Im} z$.
5. $f(z) = \sum_{m,n=0}^{N} a_{m,n} z^m \bar{z}^n$ where $a_{m,n} \in \mathbb{C}$.

Exercise 10.5. Suppose now that (X,d), (Y,d_Y), and (Z,d_Z) are three metric spaces and $f : X \to Y$ and $g : Y \to Z$. Let $x \in X$ and $y = f(x) \in Y$, show $g\circ f : X \to Z$ is continuous at x if f is continuous at x and g is continuous at y. Recall that $(g\circ f)(x) := g(f(x))$ for all $x \in X$. In particular this implies that if f is continuous on X and g is continuous on Y then $f\circ g$ is continuous on X.

Example 10.15. If $f : X \to \mathbb{C}$ is a continuous function then $|f|$ is continuous and

$$F := \sum_{m,n=0}^{N} a_{mn} f^m \cdot \bar{f}^n$$

is continuous.

Definition 10.16. *A map $f : X \to Y$ between topological spaces is called a **homeomorphism** provided that f is bijective, f is continuous and $f^{-1} : Y \to X$ is continuous. If there exists $f : X \to Y$ which is a homeomorphism, we say that X and Y are homeomorphic. (As topological spaces X and Y are essentially the same.)*

10.2 Closed and Open Sets

Definition 10.17 (Closed Sets). *A set $F \subset X$ is **closed** iff every convergent sequence $\{x_n\}_{n=1}^\infty$ which is contained in F has its limit back in F. We will write $F \sqsubset X$ to indicate F is a closed subset of X.*

Definition 10.18 (Open Sets). *A set $V \subset X$ is **open** iff V^c is closed and we write $V \subset_o X$ to indicate the V is an open subset of X.*

Notation 10.19 *To simplify notation in future arguments we will write;*

1. *$z_n \in A$ i.o. (read $z_n \in A$ **infinitely often**) to mean $\#\{n : z_n \in A\} = \infty$ and*
2. *$z_n \in A$ a.a. (read $z_n \in A$ **almost always**) to mean $\#\{n : z_n \notin A\} < \infty$. [Equivalently, $z_n \in A$ a.a. iff there exists $N < \infty$ such that $z_n \in A$ for all $n \geq N$.]*

Theorem 10.20. *The closed subsets of (X,d) have the following properties;*

1. *X and $\emptyset$ are closed.*
2. *If $\{C_\alpha\}_{\alpha\in I}$ is a collection of closed subsets of X, then $\cap_{\alpha\in I} C_\alpha$ is closed in X.*
3. *If A and B are closed sets then $A\cup B$ is closed.*

Proof. 3. Let $\{z_n\}_{n=1}^\infty \subset A\cup B$ such that $\lim_{n\to\infty} z_n =: z$ exists. Then $z_n \in A$ i.o. or $z_n \in B$ i.o. For sake of definiteness say $z_n \in A$ i.o. in which case we may choose a subsequence, $w_k := z_{n_k} \in A$ for all k. Since $\lim_{k\to\infty} w_k = z$ and A is closed it follows that $z \in A$ and hence $z \in A\cup B$. Thus we have shown $A\cup B$ is closed. ■

Exercise 10.6. Prove item 2. of Theorem 10.20. If $\{C_\alpha\}_{\alpha\in I}$ is a collection of closed subsets of X, then $\cap_{\alpha\in I} C_\alpha$ is closed in X.

Example 10.21. Let $X = \mathbb{R}$ and $-\infty < a < b < \infty$.

1. The sets, $[a,b]$, $[a,\infty)$ and $(-\infty,a]$ are all closed sets. For example if $\{x_n\}_{n=1}^\infty \subset [a,b]$ and $x = \lim_{n\to\infty} x_n$, then $a \leq x_n \leq b$ for all n and therefore by the "sandwich lemma," $a \leq x \leq b$.
2. The sets (a,b), (a,∞), and $(-\infty,a)$ are all open sets. For example, $(a,b)^c = (-\infty,a]\cup[b,\infty)$ is the union of closed sets and hence closed.
3. The sets $(a,b]$ and $[a,b)$ are neither closed nor open.

Exercise 10.7. Let (X,d) be a metric space and $C := \{x_1,\dots,x_n\}$ be a finite subset of X. Show C is closed and hence $X\setminus C$ is an open.

Corollary 10.22. *Let (X,d) be a metric space. Then the collection of open subsets, τ_d, of X satisfy;*

1. *X and $\emptyset$ are in τ_d.*
2. *τ_d is closed under taking arbitrary unions. i.e. if $\{U_\alpha\}_{\alpha\in I}$ is a collection of open sets then $\cup_{\alpha\in I}U_\alpha$ is open..*
3. *τ_d is closed under taking finite intersections, i.e. if U and V are open sets then then $U\cap V$ is open as well.*

Proof. 1) Since $X^c=\emptyset$ and $\emptyset^c=X$ are both closed it follows that $X,\emptyset$ are open, i.e. in τ_d. 2) If $\{U_\alpha\}_{\alpha\in I}$ are open sets then $\{U_\alpha^c\}_{\alpha\in I}$ are closed sets and therefore $\cap_{\alpha\in I}U_\alpha^c$ is closed and so$[\cap_{\alpha\in I}U_\alpha^c]^c=\cup_{\alpha\in I}U_\alpha$ is open. 3. If if U and V are open then U^c and V^c are closed. Therefore, $U^c\cup V^c$ is closed and hence $[U^c\cup V^c]^c=U\cap V$ is open. ■

Theorem 10.23. *Let (X,d) and (Y,d_Y) be two metric spaces and $f:X\to Y$ be a continuous function. We then have;*

1. *if $C\subset Y$ is closed then $f^{-1}(C)$ is closed in X, and*
2. *if $V\subset Y$ is open then $f^{-1}(V)$ is open in X.*

Proof. If $\{x_n\}_{n=1}^\infty\subset f^{-1}(C)$ and $x=\lim_{n\to\infty}x_n$ exists in X, then $f(x_n)\in C$ for all n. So by the continuity of f and C being closed it follows that

$$f(x)=f\left(\lim_{n\to\infty}x_n\right)=\lim_{n\to\infty}f(x_n)\in C\implies x\in f^{-1}(C)$$

which shows $f^{-1}(C)$ is closed. If V is open in Y then V^c is closed in Y and therefore, $f^{-1}(V)^c=f^{-1}(V^c)$ is closed which implies $f^{-1}(V)$ is open. ■

Lemma 10.24. *If $f:X\to\mathbb{R}$ is a continuous function and $k\in\mathbb{R}$, then the following sets are closed,*

$$A:=\{x\in X:f(x)\le k\},\quad B:=\{x\in X:f(x)=k\},\text{ and}$$
$$C:=\{x\in X:f(x)\ge k\}.$$

Proof. First note that $A=f^{-1}((-\infty,k])$, $B=f^{-1}(\{k\})$, and $C=f^{-1}([k,\infty))$. Since $(-\infty,k]$, $\{k\}$, and $[k,\infty)$ are all closed subsets of $\mathbb{R}$ the result follows from Theorem 10.23. ■

Example 10.25. Using Exercise 10.4 along with Lemma 10.24 shows the following subsets of $\mathbb{C}$ are closed;

1. $\{z\in\mathbb{C}:a\le\operatorname{Im}z\le b\}$ for all $a\le b$ in $\mathbb{R}$.
2. $\{z\in\mathbb{C}:a\le\operatorname{Re}z\le b\}$ for all $a\le b$ in $\mathbb{R}$.
3. $\{z\in\mathbb{C}:\operatorname{Im}z=0\text{ and }a\le\operatorname{Re}z\le b\}$ for all $a\le b$ in $\mathbb{R}$.

Definition 10.26 (Open/Closed Balls). *The **open ball** $B(x,\delta)\subset X$ centered at $x\in X$ with radius $\delta>0$ is the set*

$$B(x,\delta):=\{y\in X:d(x,y)<\delta\}.\tag{10.4}$$

We will often also write $B(x,\delta)$ as $B_x(\delta)$.

*The **closed ball** centered at $x\in X$ with radius $\delta>0$ as the set*

$$C_x(\delta)=C(x,\delta):=\{y\in X:d(x,y)\le\delta\}.\tag{10.5}$$

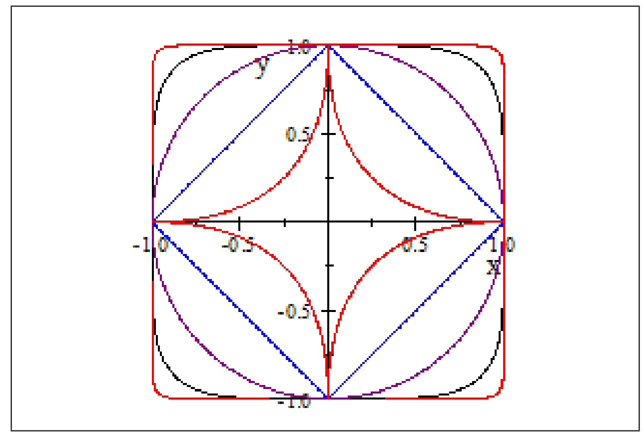


Fig. 10.1. Balls in $\mathbb{R}^2$ corresponding to the 1 – norm, 2– norm, 5 – norm, and $\frac{1}{2}$ – "norm."

Lemma 10.27. *Closed balls are closed and open balls are open.*

Proof. Let $\delta>0$. As we have seen $f(y):=d(x,y)$ is continuous and therefore

$$C_x(\delta):=\{y\in X:d(x,y)\le\delta\}=\{y\in X:f(y)\le\delta\}$$

is a closed set by Lemma 10.24. [Notice that $\{x\}=C_x(0)$ is a closed set for all $x\in X$.] Similarly

$$B_x(\delta)^c:=X\setminus B_x(\delta)=\{y\in X:d(x,y)\ge\delta\}=\{y\in X:f(y)\ge\delta\}$$

is closed and so $B_x(\delta)$ is open. ■

Proposition 10.28. *Let U be a subset of a metric space (X,d). Show the following are equivalent;*

1. *U is open,*
2. *for all $z \in U$ there exists $r > 0$ such that $B_z(r) \subset U$.*
3. *U can be written as a union of open balls.*

Proof. 1 $\implies$ 2. We will show (not 2 $\implies$ not 1). If there exists a $z \in U$ such that $B_z(r) \nsubseteq U$ for all $r > 0$ then for each $n \in \mathbb{N}$ there exists $z_n \in B_z(1/n) \setminus U$. We then have $z_n \in U^c$ for all n and $\lim_{n\to\infty} z_n = z \in U$, i.e. $z \notin U^c$. This shows that U is not closed. [**Alternatively:** by Lemma A.29 below, if $z \in U$ then $r := d_{U^c}(z) > 0$ and therefore $B_z(r) \subset U$.]

2 $\implies$ 3. Given 2. for all $z \in U$ there exists $r_z > 0$ such that $B_z(r_z) \subset U$. We then have that $U = \cup_{z\in U} B_z(r)$ which shows that U may be written as a union of open balls.

3 $\implies$ 1. From Lemma 10.27 we know that open balls are open. Thus if U is written as a union of open balls it must be open as well by Corollary 10.22. ■

Exercise 10.8. Give an example of a collection of closed subsets, $\{A_n\}_{n=1}^\infty$, of $\mathbb{C}$ such that $\cup_{n=1}^\infty A_n$ is not closed.

Lemma 10.29 (Approximating open sets from the inside by closed sets). *Let $U \subset X$ be an open set and*

$$F_\varepsilon := \{x \in X | d_{U^c}(x) \geq \varepsilon\} \subset X.$$

Then F_ε is closed for all $\varepsilon > 0$ and $F_\varepsilon \uparrow U$ as $\varepsilon \downarrow 0$.

Proof. The set F_ε is closed by Lemma 10.24 and the fact that d_{U^c} is continuous. It is clear that $d_{U^c}(x) = 0$ for $x \in U^c$ so that $F_\varepsilon \subset U$ for each $\varepsilon > 0$ and hence $\cup_{\varepsilon>0} F_\varepsilon \subset U$. Now suppose that $x \in U \subset_o X$. By Proposition 10.28, there exists an $\varepsilon > 0$ such that $B_x(\varepsilon) \subset U$, i.e. $d(x,y) \geq \varepsilon$ for all $y \in U^c$. Hence $x \in F_\varepsilon$ and we have shown that $U \subset \cup_{\varepsilon>0} F_\varepsilon$. Finally it is clear that $F_\varepsilon \subset F_{\varepsilon'}$ whenever $\varepsilon' \leq \varepsilon$. ■

It turns out that metric spaces always have lots of continuous functions.

Lemma 10.30 (Urysohn's Lemma for Metric Spaces). *Let (X,d) be a metric space and suppose that A and B are two disjoint closed subsets of X. Then*

$$f(x) = \frac{d_B(x)}{d_A(x) + d_B(x)} \text{ for } x \in X \tag{10.6}$$

defines a continuous function, $f : X \to [0,1]$, such that $f(x) = 1$ for $x \in A$ and $f(x) = 0$ if $x \in B$.

Proof. By Lemma 10.9, d_A and d_B are continuous functions on X. Since A and B are closed, $d_A(x) > 0$ if $x \notin A$ and $d_B(x) > 0$ if $x \notin B$. Since $A \cap B = \emptyset$, $d_A(x) + d_B(x) > 0$ for all x and $(d_A + d_B)^{-1}$ is continuous as well. The remaining assertions about f are all easy to verify. ■

Sometimes Urysohn's lemma will be use in the following form. Suppose $F \subset V \subset X$ with F being closed and V being open, then there exists $f \in C(X,[0,1]))$ such that $f = 1$ on F while $f = 0$ on V^c. This of course follows from Lemma 10.30 by taking $A = F$ and $B = V^c$.

10.3 Compactness

Throughout this section, let (X,d) be a metric space.

Definition 10.31 (Open covers). *Let A be a subset of a metric space, (X,d). An **open cover** of A is a collection of, $\mathcal{U}$, of open subsets of X such that $A \subset \cup_{V\in\mathcal{U}} V$. We further say that A has a **finite subcover** if there exists a finite subcollection, $\mathcal{U}_0 \subset_f \mathcal{U}$ such that $\mathcal{U}_0$ is still a cover of A.*

There are two (equivalent) notions of a compact subset, $K \subset X$.

Definition 10.32 (Open cover compactness). *The subset $K \subset X$ is **open cover compact**[1] if every open cover of K has finite a sub-cover. (We will write $K \sqsubset\sqsubset X$ to denote that $K \subset X$ and K is compact.)*

Definition 10.33 (Sequential compactness). *As subset $K \subset X$ is **(sequentially) compact** if every sequence $\{z_n\}_{n=1}^\infty \subset K$ has a convergent subsequence, $\{w_k := z_{n_k}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} w_k \in K$.*

Fortunately these two notions of compactness above are the same as the following theorem states.

Theorem 10.34. *If (X,d) is a metric space, then X is sequentially compact iff it is open cover compact. In the future we will just refer to compact sets with out the any extra adjectives.*

Proof. This theorem should be proved in Math 140. A proof may be found in Theorem A.13 of Appendix A. ■

Exercise 10.9. Suppose that K and F are compact subsets of a metric space, (X,d). Show $K \cup F$ is compact in X as well.

The following theorem is typically also prove in an undergraduate real analysis class.

[1] We will usually simply say A is compact in this case.

Theorem 10.35 (Bolzano–Weierstrass / Heine–Borel theorem). *A subset $K \subset \mathbb{R}^D$ is compact iff it is closed and bounded.*

Proof. A proof is sketched at the start of Appendix A. ■

Example 10.36 (Warning!). It is **not** true that a closed and bounded subset of an **arbitrary** metric space (X, d) is necessarily compact.

The results in the next few exercises explain the importance of compact sets. As these results are so important I will supply solutions in the text.

Exercise 10.10. If (X, d) is a metric space and $K \subset X$ is compact. Show subset, $C \subset K$, which is closed is compact as well.

Proof. Let $\{z_n\}_{n=1}^\infty \subset C$, then $\{z_n\}_{n=1}^\infty \subset K$ and therefore has a convergent subsequence, $w_k := z_{n_k}$. As C is close $\lim_{k\to\infty} w_k \in C$ and so every sequence in $\{z_n\}_{n=1}^\infty \subset C$ has a convergent subsequence to an element in C, i.e. C is compact. ■

Exercise 10.11. Let (X, d) and (Y, ρ) be metric spaces, $K \subset X$ be a compact set, and $f : K \to Y$ be a continuous function. Show $f(K)$ is compact in Y. In particular, for $C \subset K$ closed, we have $f(C)$ is closed and in fact compact in Y.

Proof. Let $\{w_n\}_{n=1}^\infty \subset f(K)$ be a given sequence. By definition of $f(K)$ this implies there exists $\{z_n\}_{n=1}^\infty \subset K$ such that $f(z_n) = w_n$. Since K is compact, there is a subsequence, $z'_k := z_{n_k}$ of $\{z_n\}_{n=1}^\infty$ such that $\lim_{k\to\infty} z'_k = z \in K$. Then $w'_k := f(z'_k)$ is a subsequence of $\{w_n\}_{n=1}^\infty$ such that $\lim_{k\to\infty} w'_k = \lim_{k\to\infty} f(z'_k) = f(z) \in f(K)$. This shows that $f(K)$ is compact. Now if $C \subset K$ is closed then it is compact by Exercise 10.10. It then follows that $f(C)$ is compact and hence closed. ■

Exercise 10.12. If $K \subset \mathbb{R}$ is compact then $\sup(K) \in K$ and $\inf(K) \in K$, i.e. $\sup(K) = \max(K)$ and $\inf(K) = \min(K)$.

Proof. Choose $x_n \in K$ such that $x_n \uparrow \sup(K)$. Since K is compact, there exists a subsequence, $y_k := x_{n_k}$ such that $\lim_{k\to\infty} y_k$ exists in K. But then $\sup(K) = \lim_{k\to\infty} y_k \in K$. The proof that $\inf(K) = \min(K)$ is analogous. ■

Exercise 10.13 (Extreme value theorem). Let K be compact subset of X and $f : K \to \mathbb{R}$ be a continuous function. Show $-\infty < \inf_{x\in K} f(x) \le \sup_{x\in K} f(x) < \infty$ and there exists $a, b \in K$ such that $f(a) = \inf_{x\in K} f(x)$ and $f(b) = \sup_{x\in K} f(x)$. **Hint:** first argue that there exists $\{z_n\}_{n=1}^\infty \subset K$ such that $f(z_n) \uparrow \sup_{x\in K} f(x)$ as $n \to \infty$.

Proof. By Exercise 10.11, $f(K)$ is a compact subset of $\mathbb{R}$ and hence

$$\inf_{x\in K} f(x) = \inf(f(K)) = \min(f(K)) \text{ and}$$
$$\sup_{x\in K} f(x) = \sup(f(K)) = \max(f(K)).$$

So if $M = \max(f(K))$, then $M \in f(K)$ which means $M = f(b)$ for some $b \in K$. Similarly $\inf_{x\in K} f(x) = f(a)$ for some $a \in K$. ■

Exercise 10.14 (Compacts are closed and Bounded). Let K be compact subset of an arbitrary metric space (X, d), then K is closed and bounded. [This proves the easy direction in Theorem 10.35.]

Proof. (K is closed.) Let $\{x_n\}_{n=1}^\infty \subset K$ such that $x_n \to x \in X$. By compactness, there exists a subsequence, $\{x_{n_k}\}_{k=1}^\infty$, which converges to a point $y \in K$. But by uniqueness of limits we must have $x = y \in K$ and so K is closed.

(K is bounded.) Let $o \in X$ be fixed and then apply the extreme valued theorem with $f(x) = d(x, o) = d_{\{o\}}(x)$ to find $R = \max(f(K)) < \infty$, i.e. $K \subset C_o(R)$ which shows K is bounded. ■

10.3.1 *More optional results

You may jump to Section 10.4 and skip this subsection on first reading.

Exercise 10.15 (Uniform Continuity). Let K be compact subset of X and $f : K \to \mathbb{C}$ be a continuous function. Show that f is uniformly continuous, i.e. if $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ if $w, z \in K$ with $d(w, z) < \delta$. **Hint:** prove the contrapositive.

Proof. If not, there would exist $\varepsilon > 0$ and sequences w_n and z_n in K such that $d(w_n, z_n) \to 0$ while $|f(z_n) - f(w_n)| \ge \varepsilon$ for all n. Using sequentially compactness of K, we may assume, by passing to subsequences if necessary, that $w_n \to w \in K$ and $z_n \to z \in K$. Since $d(w_n, z_n) \to 0$ we must have $z = w$ and hence we arrive at the contradiction:

$$\varepsilon \le \lim_{n\to\infty} |f(z_n) - f(w_n)| = |f(z) - f(w)| = 0.$$

■

This variant of Exercise 10.13 is left to the reader.

Exercise 10.16 (Extreme value theorem II). Suppose that $f : X \to \mathbb{R}$ is a continuous function on a metric space (X, d). Further assume there exists a compact subset $K \subset X$ and $x_0 \in K$ such that $f(x_0) \le f(x)$ for all $x \in X \setminus K$. Show there exists $k \in K$ such that $\inf_{x\in X} f(x) = f(k)$.

Theorem 10.37 (Fundamental Theorem of Algebra). *Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial on $\mathbb{C}$ with $a_n \neq 0$ and $n > 0$. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. Since

$$\lim_{|z|\to\infty} \frac{|p(z)|}{|z|^n} = \lim_{|z|\to\infty} \left| \sum_{k=0}^{n-1} \frac{a_k}{z^{n-k}} + a_n \right| = |a_n|,$$

if $R > 0$ is sufficiently large, then

$$|p(z)| \geq |a_n| |z|^n / 2 \geq |a_n| R^n / 2 \geq |a_0| = |p(0)| \text{ for } |z| > R.$$

Applying Exercise 10.16 with $K = \{z \in \mathbb{C} : |z| \leq R\}$ and $x_0 = 0$ there exists $z_0 \in K \subset \mathbb{C}$ such that $|p(z_0)| \leq |p(z)|$ for all $z \in \mathbb{C}$. It now follows from Lemma 10.38 below that $p(z_0) = 0$.

■

Lemma 10.38. *Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial on $\mathbb{C}$ with $a_n \neq 0$ and $n > 0$. If $|p(z)|$ has a minimum at $z_0 \in \mathbb{C}$, then $|p(z_0)| = 0$.*

Proof. For sake of contradiction, let us suppose that $p(z_0) \neq 0$ and set

$$q(z) := p(z_0 + z) = \sum_{k=0}^{n} a_k (z + z_0)^k = \sum_{k=0}^{n} \alpha_k z^k.$$

Then $0 < |q(0)| = |\alpha_0| \leq |q(z)|$ for all $z \in \mathbb{C}$ and $\alpha_n = a_n \neq 0$. Let $l \geq 1$ be the first index such that $\alpha_l \neq 0$ so that

$$\begin{aligned} q(z) &= \alpha_0 + \alpha_l z^l + \cdots + \alpha_n z^n \\ &= \alpha_0 \left[1 + \frac{\alpha_l}{\alpha_0} z^l + \cdots + \frac{\alpha_n}{\alpha_0} z^n \right] \end{aligned}$$

Evaluating this at $z = re^{i\theta}$ with θ chosen so that $\frac{\alpha_l}{\alpha_0} e^{il\theta} = -\left|\frac{\alpha_l}{\alpha_0}\right| = -\varepsilon$ implies for $r > 0$ small that

$$\begin{aligned} |\alpha_0| \leq \left| q\left(re^{i\theta}\right) \right| &= |\alpha_0| \left| 1 - \varepsilon r^l + c_{l+1} r^{l+1} + \cdots + c_n r^n \right| \\ &\leq |\alpha_0| \left[1 - \varepsilon r^l + \left| c_{l+1} r^{l+1} + \cdots + c_n r^n \right| \right] \\ &\leq |\alpha_0| \left[1 - \varepsilon r^l + C r^{l+1} \right] = |\alpha_0| \left[1 - r^l \left[\varepsilon - C r \right] \right] \end{aligned}$$

where $\{c_k\}$ and C are certain appropriate constants. Choosing $r \in (0, \varepsilon/C)$ then gives the $|\alpha_0| < |\alpha_0|$ which is absurd and we have reached the desired contradiction. ■

Remark 10.39. The fundamental theorem of algebra does not hold for polynomials in z and $\bar{z}$ or in x and y. For example consider the polynomial

$$p(z, \bar{z}) = 1 + z\ \bar{z} = 1 + x^2 + y^2.$$

The point is that Lemma 10.38 does not hold for these more general classes of polynomials.

10.4 Proper maps

Again let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function.

Example 10.40. If $f : \mathbb{R} \to \mathbb{R}$ is the function, $f(x) = 0$ for all $x \in \mathbb{R}$, then f is continuous, $K = \{0\}$ is a compact subset of $\mathbb{R}$, while $f^{-1}(K) = \mathbb{R}$ is **not** compact. This shows in general that inverse images of compact sets under continuous maps need not be compact. It is however always true that $f^{-1}(K)$ is closed in X for all compact $K \subset Y$.

Definition 10.41 (Proper Maps). *A continuous function, $f : X \to Y$, is **proper** if $f^{-1}(K)$ is compact in X for all compact subsets of $K \sqsubset\sqsubset Y$.*

Example 10.42. If $f : X \to Y$ is a homeomorphism, then f is proper. Indeed, in this case $g = f^{-1}$ is a continuous map and hence $f^{-1}(\text{compact}) = g(\text{compact})$ is compact.

Example 10.43. Suppose that $f : \mathbb{R}^m \to \mathbb{R}^n$ is a continuous map such that $\lim_{\|x\|\to\infty} \|f(x)\| = \infty$, then f is proper. Indeed if $K \subset \mathbb{R}^n$ is a compact set then it is closed and bounded and $f^{-1}(K)$ is closed because f is continuous. If $f^{-1}(K)$ were not bounded, there would exist$\{x_n\} \subset f^{-1}(K)$ such that $\lim_{n\to\infty} \|x_n\| = \infty$ and then by assumption, $\lim_{n\to\infty} \|f(x_n)\| = \infty$. Since $f(x_n) \in K$ for all n this would imply K is not bounded which violates K being compact.

Let us now suppose that U and V are open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a continuous function. We are going to state and prove a necessary and sufficient criteria for f to be proper. Before doing so let us record a few more properties involving compact subsets of V.

Definition 10.44. *Given a non-empty open subset, $V \subset \mathbb{R}^n$ and $\varepsilon > 0$, let*

$$K_\varepsilon^V = \left\{ y \in V : d_{V^c}(y) \geq \varepsilon \text{ and } d(y, 0) \leq \frac{1}{\varepsilon} \right\} \text{ and} \tag{10.7}$$

$$W_\varepsilon^V := \left\{ y \in V : d_{V^c}(y) > \varepsilon \text{ and } d(y, 0) < \frac{1}{\varepsilon} \right\}. \tag{10.8}$$

See Figure 10.2

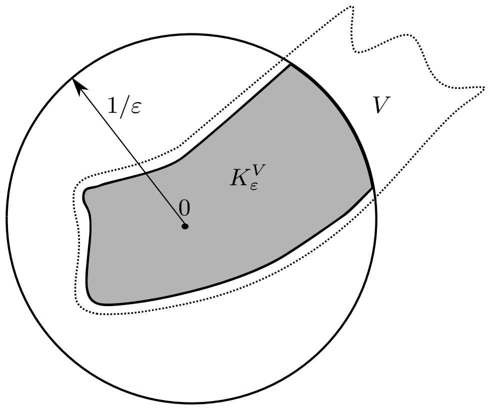


Fig. 10.2. The shaded area indicates K_ε^V where as W_ε^V is the region K_ε^V with its boundary removed.

Remark 10.45. Let V be an open subset of $\mathbb{R}^n$ and let $d(x,y) = \|x-y\|$ for all $x, y \in \mathbb{R}^n$.

1. K_ε^V is a closed bounded subset of $\mathbb{R}^n$ (hence compact), W_ε^V is an open set such that $W_\varepsilon^V \subset K_\varepsilon^V \subset V$ and $W_\varepsilon^V \uparrow V$ as $\varepsilon \downarrow 0$.
2. If K is any compact subset of V, then there exists an $\varepsilon > 0$ such that $K \subset K_\varepsilon^V$. Indeed $\left\{W_\varepsilon^V\right\}_{\varepsilon>0}$ is an open cover of K and hence $K \subset W_\varepsilon^V \subset K_\varepsilon^V$ for some $\varepsilon > 0$.
3. If K is a compact subset of V, then

$$d(K, V^c) = \inf_{x\in K} d_{V^c}(x) = \min_{x\in K} d_{V^c}(x) > 0$$

because $d_{V^c}(\cdot)$ is continuous, $d_{V^c}(x) > 0$ for all $x \in K$, and K is compact.

Theorem 10.46. *Suppose that U and V are open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a continuous function. Then f is proper iff for all $\varepsilon > 0$ there exits a $\delta = \delta(\varepsilon) > 0$ such that*

$$f\left(U \setminus K^U_{\delta(\varepsilon)}\right) \subset V \setminus K_\varepsilon^V.$$

Proof. We are first going to prove the following claim.

Claim: f is proper iff for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $f^{-1}\left(K_\varepsilon^V\right) \subset K_\delta^U$.

Proof of claim. ($\Longrightarrow$) If f is proper, then $f^{-1}\left(K_\varepsilon^V\right)$ is a compact subset of U and so by item 2. of Remark 10.45 there is a $\delta = \delta(\varepsilon) > 0$ such that $f^{-1}\left(K_\varepsilon^V\right) \subset K_\delta^U$.

($\Longleftarrow$) We now suppose that for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $f^{-1}\left(K_\varepsilon^V\right) \subset K_\delta^U$ and let K be a compact of $\mathbb{R}^n$ contained in V. By item 2. of Remark 10.45, there exists $\varepsilon > 0$ such that $K \subset K_\varepsilon^V$ and then by the given assumption there exists $\delta = \delta(\varepsilon) > 0$ so that

$$f^{-1}(K) \subset f^{-1}\left(K_\varepsilon^V\right) \subset K_\delta^U \subset U.$$

This shows that $f^{-1}(K)$ is bounded in $\mathbb{R}^n$ and so $f^{-1}(K)$ will be compact if we can show $f^{-1}(K)$ is closed and also closed in $\mathbb{R}^n$. To this end, suppose that $\{x_n\}_{n=1}^\infty \subset f^{-1}(K)$ and $\lim_{n\to\infty} x_n = x$ exists in $\mathbb{R}^n$. As K_δ^U is closed we know that $x \in K_\delta^U \subset U$ and therefore $f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) \in K$, i.e. $x \in f^{-1}(K)$ and so $f^{-1}(K)$ is closed in $\mathbb{R}^n$ and the proof of the claim is complete.

The theorem now follows from the claim the observations that iff

$$\begin{aligned} f^{-1}\left(K_\varepsilon^V\right) \subset K_\delta^U &\iff U \setminus K_\delta^U \subset V \setminus f^{-1}\left(K_\varepsilon^V\right) = f^{-1}\left(V \setminus K_\varepsilon^V\right) \\ &\iff f\left(U \setminus K^U_{\delta(\varepsilon)}\right) \subset V \setminus K_\varepsilon^V. \end{aligned}$$

■

Remark 10.47 (Meaning of Proper). Since

$$U \setminus K_\delta^U = \{x \in U : d_{U^c}(x) < \delta\} \cup \left\{x \in U : d(x,0) > \frac{1}{\delta}\right\}$$

we see that $x \in U \setminus K_\delta^U$ iff x is δ-close to the boundary of U or $1/\delta$-far from 0. With this in mind, Theorem 10.46 states that f is proper iff whenever $x \in U$ is close to U^c or x is far from 0, then $f(x) \in V$ has to be either close to V^c or far from 0.

Corollary 10.48. *If U and V are bounded open subsets of $\mathbb{R}^n$ and $f : U \to V$ is a continuous map, then f is proper iff $d_{V^c}(f(x)) \to 0$ as $x \in U$ with $d_{U^c}(x) \to 0$.*

Corollary 10.49. *Let $-\infty \le a < b \le \infty$, $-\infty \le c < d \le \infty$, and $f : (a,b) \to (c,d)$ be a continuous map. Then f is proper iff $f(b-) := \lim_{x\uparrow b} f(x)$ and $f(a+) = \lim_{x\downarrow a} f(x)$ both exist (in the extended sense, i.e. we also allow for $\{\pm\infty\}$ as possible limits) and $f(a+), f(b-) \in \{c, d\}$.*

Proof. Suppose that f is proper. If $\lim_{x\uparrow b} f(x)$ does not exists in the extended sense then there exists $c < c_0 < d_0 < d$ and a sequence $\{x_n\}_{n=1}^\infty \subset (a,b)$ such that $x_n \uparrow b$ as $n \to \infty$ and $f(x_n) \le c_0$ for n odd and $f(x_n) \ge d_0$ for n even. By the intermediate value theorem it follows that there exists $\{y_n\}_{n=1}^\infty \subset f^{-1}(\{c_0\})$ such that $y_n \uparrow b$. On the other hand $f^{-1}(\{c_0\})$ is compact and hence $b = \lim_{n\to\infty} y_n \in f^{-1}(\{c_0\}) \subset (a,b)$ which is impossible. Thus it follows that $f(b-) = \lim_{x\uparrow b} f(x)$ exists and by a similar argument,

$f(a+) = \lim_{x \downarrow a} f(x)$ exists as well. We leave it to the reader to use Theorem 10.47 (see Remark 10.47) to show that both of these limits must be in $\{c, d\}$.

Conversely, suppose that $f(b-) := \lim_{x \uparrow b} f(x)$ and $f(a+) = \lim_{x \downarrow a} f(x)$ both exist and $f(a+), f(b-) \in \{c, d\}$. From this it follows that $f(x)$ near $\{c, d\}$ when $x \in (a, b)$ is near $\{a, b\}$. [Here for example, we say x is near $\{a, \infty\}$ iff either x is near a or x is very large and positive.] ■

Example 10.50. There are four possible case for the limits in Corollary 10.49.

1. $f(a+) = f(b-) = c$.
2. $f(a+) = f(b-) = d$.
3. $f(a+) = c$ and $f(b-) = d$
4. $f(a+) = d$ and $f(b-) = c$.

In the first two case $\deg(f) = 0$ while in case $\deg(f) = 1$ and $\deg(f) = -1$ in case 4, see Figure 10.3 below.

Exercise 10.17. Explain why;

1. $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = 1/x$ is a proper map.
2. $f : (0, \infty) \to (-\infty, \infty)$ defined by $f(x) = 1/x$ is **not** a proper map.

Exercise 10.18. Let $V := \mathbb{R}^2 \setminus \{0\}$ and $f : V \to V$ be defined by $f(x) = \frac{x}{\|x\|^2}$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

1. Explain why f is a proper map.
2. Show $f \circ f(x) = x$ so that f is in fact a homeomorphism.
3. Compute the $\deg(f)$. [Hint: evaluate $f'(p)$ at your favorite point in V.]

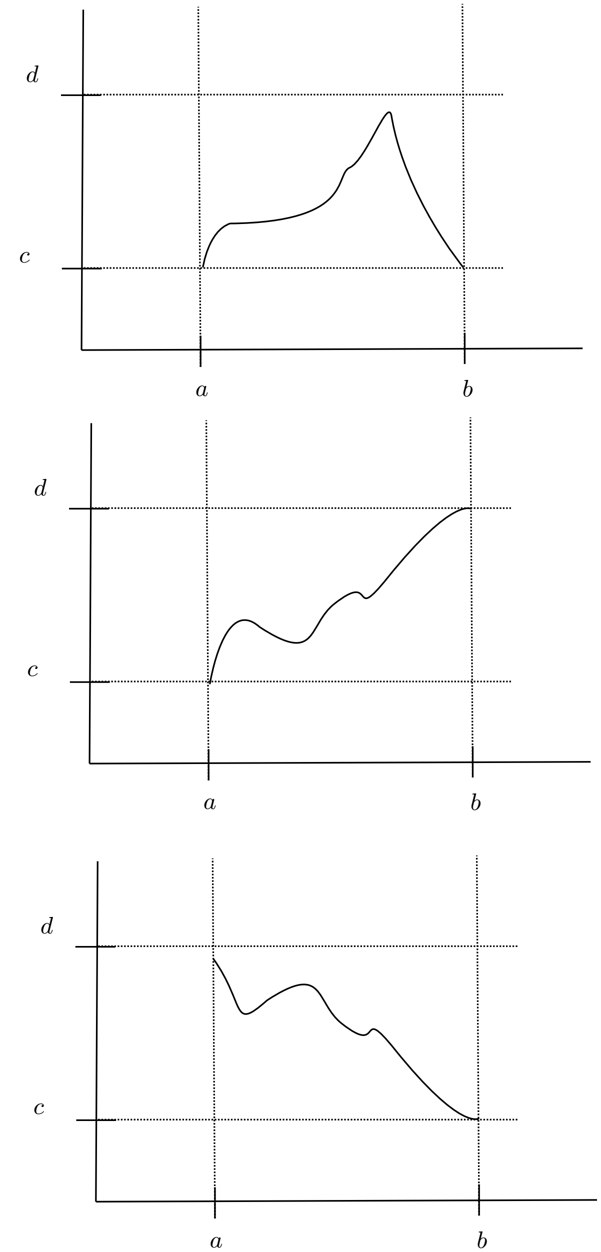


Fig. 10.3. Here are three examples of proper map one dimensional maps with their degrees which are 0, -1, and 1 respectively. Let us also note in the first case, the map is not surjective which always leads to a degree zero map. In the one dimensional case, the degree can only take the values $\{0, 1, -1\}$.

11

A Poincaré Lemma

The following main theorem of this chapter follows as an easy consequence of Theorem 11.8 below.

Theorem 11.1. *If $U \subset_o \mathbb{R}^n$ is connected and $\omega \in \Omega_c^n(U)$ with $\int_U \omega = 0$, then $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(U)$.*

The outline of this chapter is as follows.

1. Section 11.1 gives a necessary conditions in order to differentiate past the integral.
2. In Section 11.2 we will prove Theorem 11.1 when U is an open rectangle.
3. Section 11.3 is devoted to smooth "partitions of unity." This is a localization technique that is used repeatedly in differential geometry and analysis.
4. Lastly in Section 11.4 we combine the results in Sections 11.2 and 11.3 to complete the proof of Theorem 11.1.

11.1 Differentiating past the integral

Corollary 11.2 (Differentiation Under the Integral). *Suppose that $J \subset \mathbb{R}$ is an open interval and $f : J \times \mathbb{R}^d \to \mathbb{R}$ is a function such that*

1. *$x \to f(t,x)$ is measurable for each $t \in J$.*
2. *$f(t_0,\cdot) \in L^1(m)$ for some $t_0 \in J$.*
3. *$\frac{\partial f}{\partial t}(t,x)$ exists for all (t,x).*
4. *There is a function $g \in \mathrm{L}^1(m)$ such that $\left|\frac{\partial f}{\partial t}(t,\cdot)\right| \le g$ for each $t \in J$.*

Then $f(t,\cdot) \in \mathrm{L}^1(m)$ for all $t \in J$ (i.e. $\int_{\mathbb{R}^d} |f(t,x)|\, dm(x) < \infty$), $t \to \int_{\mathbb{R}^d} f(t,x)dm(x)$ is a differentiable function on J, and

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)dm(x) = \int_{\mathbb{R}^d} \frac{\partial f}{\partial t}(t,x)dm(x).$$

Proof. By the mean value theorem,

$$|f(t,x) - f(t_0,x)| \le g(x)\,|t-t_0| \text{ for all } t \in J \tag{11.1}$$

and hence

$$|f(t,x)| \le |f(t,x) - f(t_0,x)| + |f(t_0,x)| \le g(x)\,|t-t_0| + |f(t_0,x)|.$$

This shows $f(t,\cdot) \in \mathrm{L}^1(m)$ for all $t \in J$. Let $G(t) := \int_{\mathbb{R}^d} f(t,x)dm(x)$, then

$$\frac{G(t)-G(t_0)}{t-t_0} = \int_{\mathbb{R}^d} \frac{f(t,x)-f(t_0,x)}{t-t_0} dm(x).$$

By assumption,

$$\lim_{t\to t_0} \frac{f(t,x)-f(t_0,x)}{t-t_0} = \frac{\partial f}{\partial t}(t,x) \text{ for all } x \in \mathbb{R}^d$$

and by Eq. (11.1),

$$\left|\frac{f(t,x)-f(t_0,x)}{t-t_0}\right| \le g(x) \text{ for all } t \in J \text{ and } x \in \mathbb{R}^d.$$

Therefore, we may apply the dominated convergence theorem to conclude

$$\begin{aligned}\lim_{n\to\infty} \frac{G(t_n)-G(t_0)}{t_n-t_0} &= \lim_{n\to\infty}\int_{\mathbb{R}^d} \frac{f(t_n,x)-f(t_0,x)}{t_n-t_0} dm(x) \\ &= \int_{\mathbb{R}^d} \lim_{n\to\infty} \frac{f(t_n,x)-f(t_0,x)}{t_n-t_0} dm(x) \\ &= \int_{\mathbb{R}^d} \frac{\partial f}{\partial t}(t_0,x)dm(x)\end{aligned}$$

for **all** sequences $t_n \in J \setminus \{t_0\}$ such that $t_n \to t_0$. Therefore, $\dot{G}(t_0) = \lim_{t\to t_0} \frac{G(t)-G(t_0)}{t-t_0}$ exists and

$$\dot{G}(t_0) = \int_{\mathbb{R}^d} \frac{\partial f}{\partial t}(t_0,x)dm(x).$$

■

11.2 A Local Poincaré Lemma

Theorem 11.3 (A Local Poincaré Lemma). *Let $n \in \mathbb{N}$, $-\infty < a_i < b_i < \infty$ for $i \in [n]$, and R be the open rectangle,*

$$R = (a, b) = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n.$$

If $\omega \in \Omega_c^n(R)$ *satisfies,* $\int_R \omega = 0$*, then there exits* $\mu \in \Omega_c^{n-1}(R)$ *such that* $\omega = d\mu$*.*

Proof. Let $\mathrm{Vol} = dx_1 \wedge \cdots \wedge dx_n$ so that $\omega = f\,\mathrm{Vol}$ for some $f \in C_c^\infty(R)$. Further let $g = (g_1, g_2, \ldots, g_n)$ with $g_i \in C_c^\infty(R)$ and then set

$$\mu = i_g \mathrm{Vol} = \sum_{j=1}^n (-1)^{j-1} g_j \cdot dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

so, as you have proved, $d\mu = (\nabla \cdot g)\,\mathrm{Vol}$. Thus we may reformulate the problem into given $f \in C_c^\infty(R)$ such that $\int_R f dm = 0$, we want to find $g_i \in C_c^\infty(R)$ such that $f = \nabla \cdot g$. The proof will be by induction on n.

For $n = 1$ this is trivially done using

$$g(x) = \int_{-\infty}^x f(t)\,dt.$$

Note that $g(x) = 0$ when $x < b$ but sufficiently close to b, since then

$$g(x) = \int_{-\infty}^x f(t)\,dt = g(x) = \int_a^b f(t)\,dt = 0.$$

For the inductive step, let us write $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and then set

$$h(x, y) := \int_{-\infty}^x f(t, y)\,dt.$$

We then have $\partial_1 h(x, y) = f(x, y)$ as desired, however

$$h(\infty, y) := \int_{\mathbb{R}} f(t, y)\,dt \neq 0.$$

To fix this we choose $\rho \in C_c^\infty((a_1, b_1), [0, \infty))$ so that $\int_{\mathbb{R}} \rho(t)\,dt = 1$ and then let

$$\hat{f}(x, y) := f(x, y) - \rho(x)\,h(\infty, y)$$

which is still compactly supported in R. We then let

$$g_1(x, y) = \int_{-\infty}^x \hat{f}(t, y)\,dt = h(x, y) - \int_{-\infty}^x \rho(t)\,dt \cdot h(\infty, y)$$

which is again compactly supported in R. Moreover,

$$\partial_1 g(x, y) = f(x, y) - \rho(x)\,h(\infty, y).$$

Since

$$\int_{\mathbb{R}^{n-1}} h(\infty, y)\,dy = \int_{\mathbb{R}^n} f(x, y)\,dxdy = 0,$$

we may use the induction hypothesis to find $\tilde{g}_2(y), \ldots, \tilde{g}_n(y)$ with compact support in $\prod_{i=2}^n (a_i, b_i)$ so that

$$h(\infty, y) = \sum_{j=2}^n \partial_j \tilde{g}_j(y).$$

It then follows that taking $g_j(x, y) = \rho(x)\,\tilde{g}_j(x, y)$ gives g so that

$$\begin{aligned}\sum_{j=1}^n \partial_j g_j(x, y) &= f(x, y) - \rho(x)\,h(\infty, y) + \rho(x) \sum_{j=2}^n \partial_j \tilde{g}_j(y) \\ &= f(x, y) - \rho(x)\,h(\infty, y) + \rho(x)\,h(\infty, y) = f(x, y)\end{aligned}$$

as desired. ■

11.3 Smooth Uryshon's Lemma and Partitions of Unity

Corollary 11.4 (C^∞ − Uryshon's Lemma). *Let $U \subset \mathbb{R}^n$ be an open set and $K \subset \mathbb{R}^n$ be a compact set such that $K \subset U$. Then there exists $f \in C_c^\infty(\mathbb{R}^n, [0, 1])$ such that* $\mathrm{supp}(f) \subset U$ *and $f = 1$ on K.*

Proof. Since K is compact it may be covered by finitely many open rectangles, $\{R_j\}_{j=1}^m$ satisfying, $K \subset \cup_{j=1}^m R_j \subset \cup_{j=1}^m \bar{R}_j \subset U$. Now let $\varphi_{R_j} \in C_c^\infty(\mathbb{R}^n, [0, 1])$ be the bump function constructed in Definition 9.12 so that $\varphi_{R_j}(x) > 0$ for $x \in R_j$ and $\varphi_{R_j}(x) = 0$ for $x \notin R_j$. Then the function, $\psi(x) := \sum_{j=1}^m \varphi_{R_j}(x)$ is smooth, supported on the compact subset, $K' = \cup_{j=1}^m \bar{R}_j$, which is contained in U, and is positive on K. By the extreme value theorem it follows that $a := \min_{x \in K} \psi(x) > 0$ and so the function, $f(x) := H_a(\psi(x))$ (with H_a as in Notation 9.10) satisfies the required properties. ■

The next proposition is a powerful localization technique which allows us to reduce many global problems in integration and differentiation theory to local problems.

Proposition 11.5 (Smooth Partitions of Unity over Compacts). *Suppose that X is an open subset of $\mathbb{R}^n$, $K \subset X$ is a compact set and $\mathcal{U} = \{U_j\}_{j=1}^m$ is an open cover of K. Then there exists a smooth (i.e. $h_j \in C^\infty(X, [0, 1])$) partition of unity $\{h_j\}_{j=1}^m$ of K such that $h_j \prec U_j$ for all $j = 1, 2, \ldots, m$.*

Proof. For all $x \in K$ choose an bounded open rectangle, R_x (or a more general open precompact neighborhood, V_x if you prefer) of x such that $\overline{R}_x \subset U_j$. Since K is compact, there exists a finite subset, Λ, of K such that $K \subset \bigcup_{x\in\Lambda} R_x$. Let

$$F_j = \cup\left\{\bar{R}_x : x \in \Lambda \text{ and } \bar{R}_x \subset U_j\right\}.$$

Then F_j is compact, $F_j \subset U_j$ for all j, and $K \subset \cup_{j=1}^m F_j$. By Urysohn's Lemma (Corollary 11.4) there exists $f_j \in C_c^\infty(U_j, [0,1])$ such that $f_j = 1$ on F_j for $j = 1, 2, \ldots, m$. By convention we let $f_{m+1} \equiv 1$. We will now give two methods to finish the proof.

Method 1. Let $h_1 = f_1$, $h_2 = f_2(1-h_1) = f_2(1-f_1)$,

$$h_3 = f_3(1-h_1-h_2) = f_3(1-f_1-(1-f_1)f_2) = f_3(1-f_1)(1-f_2)$$

and continue on inductively to define

$$h_k = f_k(1-h_1-\cdots-h_{k-1}) = f_k \cdot \prod_{j=1}^{k-1}(1-f_j)\,\forall\, k = 2, 3, \ldots, m \tag{11.2}$$

while at the same time showing

$$h_{m+1} = (1-h_1-\cdots-h_m)\cdot 1 = 1\cdot\prod_{j=1}^m (1-f_j). \tag{11.3}$$

From these equations it clearly follows that $h_j \in C_c(X,[0,1])$ and that $\operatorname{supp}(h_j) \subset \operatorname{supp}(f_j) \subset U_j$. Since $\prod_{j=1}^m (1-f_j) = 0$ on K, $\sum_{j=1}^m h_j = 1$ on K and $\{h_j\}_{j=1}^m$ is the desired partition of unity.

Method 2. Let $g := \sum_{j=1}^m f_j \in C_c^\infty(X)$. Then $g \geq 1$ on K and hence $K \subset \{g > \frac{1}{2}\}$. Choose $\varphi \in C_c(X,[0,1])$ such that $\varphi = 1$ on K and $\operatorname{supp}(\varphi) \subset \{g > \frac{1}{2}\}$ and define $f_0 := 1 - \varphi$. Then $f_0 = 0$ on K, $f_0 = 1$ if $g \leq \frac{1}{2}$ and therefore,

$$f_0 + f_1 + \cdots + f_m = f_0 + g > 0$$

on X. The desired partition of unity may be constructed as

$$h_j(x) = \frac{f_j(x)}{f_0(x) + \cdots + f_m(x)}.$$

Indeed $\operatorname{supp}(h_j) = \operatorname{supp}(f_j) \subset U_j$, $h_j \in C_c(X,[0,1])$ and on K,

$$h_1 + \cdots + h_m = \frac{f_1 + \cdots + f_m}{f_0 + f_1 + \cdots + f_m} = \frac{f_1 + \cdots + f_m}{f_1 + \cdots + f_m} = 1.$$

■

11.4 Poincaré Lemma for connected sets

Lemma 11.6. *Suppose that R and Q are two open rectangles such that $R \cap Q \neq \emptyset$, $\omega \in \Omega_c^n(R)$ and $\alpha \in \Omega_c^n(Q)$ are such that $\int \omega = \int \alpha$. Then there exists $\mu \in \Omega_c^{n-1}(R \cup Q)$ such that $\omega - \alpha = d\mu$.*

Proof. Choose $\omega_0 \in \Omega_c^n(R \cap Q)$ such that $\int \omega_0 = \int \omega = \int \alpha$. Then we know there exits $\mu_1 \in \Omega_c^{n-1}(R)$ and $\mu_2 \in \Omega_c^{n-1}(Q)$ so that $\omega - \omega_0 = d\mu_1$ and $\alpha - \omega_0 = d\mu_2$. Subtracting these equations shows

$$\omega - \alpha = d(\mu_1 - \mu_2) = d\mu$$

where $\mu = \mu_1 - \mu_2 \in \Omega_c^{n-1}(R \cup Q)$. ■

Lemma 11.7. *Suppose that U is a connected open subset of $\mathbb{R}^n$ and R and Q be open rectangles such that $\bar{R} \cup \bar{Q} \subset U$. If $\omega \in \Omega_c^n(R)$ and $\alpha \in \Omega_c^n(Q)$ are such that $\int \omega = \int \alpha$, then there exists $\mu \in \Omega_c^{n-1}(U)$ such that $\omega - \alpha = d\mu$.*

Proof. Choose a chain of rectangles $\{R_j\}_{j=0}^m$ (see Figure 3.3.1 on page 87 of the book) such that $R_0 = R$ and $R_m = Q$ and $R_j \cap R_{j-1} \neq \emptyset$ for $1 \leq j \leq m$. Then choose $\omega_j \in \Omega_c^n(R_j)$ for $1 \leq j < n$ such that $\int \omega_j = \int \omega$ for all j. We further let $\omega_0 = \omega$ and $\omega_n = \alpha$. Then by Lemma 11.6 there exists

$$\mu_j \in \Omega_c^{n-1}(R_{j-1} \cup R_j) \subset \Omega_c^{n-1}(U) \text{ for } 1 \leq j < n,$$

such that $\omega_j - \omega_{j-1} = d\mu_j$. Summing this last equation on j shows

$$\alpha - \omega = \sum_{j=1}^n (\omega_j - \omega_{j-1}) = \sum_{j=1}^n d\mu_j = d\mu$$

where $\mu = \sum_{j=1}^n \mu_j \in \Omega_c^{n-1}(U)$. ■

Theorem 11.8. *If $U \subset_o \mathbb{R}^n$ is connected and $\omega \in \Omega_c^n(U)$ with $\int_U \omega = 0$, then $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(U)$.*

Proof. Fix $\omega_0 \in \Omega_c^n(Q)$ such that $\int \omega_0 = 1$ and $\bar{Q} \subset U$. We are going to show for each $\omega \in \Omega_c^n(U)$ there exists $\mu \in \Omega_c^{n-1}(U)$ so that

$$\omega = \left(\int \omega\right) \cdot \omega_0 + d\mu \tag{11.4}$$

which certainly suffices to prove the theorem.

Let $\mathcal{R} := \{R_j\}_{j=0}^N$ be a covering of $K = \operatorname{supp}(\omega) \subset U$ by open rectangles such that $\bar{R}_j \subset U$ for all j and then let $\{\varphi_j\}_{j=0}^N$ be a partition of unity subordinate to $\mathcal{R}$ such that $\sum_{j=0}^N \varphi_j = 1$ on K. Also let $\omega_j := \varphi_j \omega$ and $c_j = \int \omega_j$ for each j. Then by multiple uses of Lemma 11.7, there exists $\mu_j \in \Omega_c^{n-1}(U)$ such that $\omega_j - c_j\omega_0 = d\mu_j$ for $0 \leq j \leq N$. Summing this identity on j then shows

$$\omega = \sum_j \omega_j = \sum_j (c_j\omega_0 + d\mu_j) = \left(\sum_j c_j\right)\omega_0 + d\mu$$

where

$$\mu := \sum_{j=0}^{N} \mu_j \in \Omega_c^{n-1}(U) \text{ and}$$

$$\sum_j c_j = \sum_j \int \omega_j = \int \sum_j \omega_j = \int \omega.$$

Altogether, the last displayed equations prove Eq. (11.4). ■

Part IV

Appendices

A

*More metric space results

This appendix is optional for sure!

A.1 Compactness in Euclidean Spaces

Definition A.1 (Pre-compact). *A subset $A \subset X$ is* ***precompact*** *if $\bar{A}$ is compact.*

Example A.2. Suppose that $F \subset X$ is an unbounded set, i.e. for all $n \in \mathbb{N}$ there exists $z_n \in F$ such that $d(x, z_n) \geq n$. If there were a subsequence, $\{w_k := z_{n_k}\}_{k=1}^{\infty}$ such that $w = \lim_{k\to\infty} w_k$ existed in X, then we would have $n_k \leq d(x, w_k) \to d(x, w) < \infty$ which is clearly impossible. This shows that compact sets must be bounded.

Example A.3. Suppose that $F \subset X$ is not closed. Then there exists $\{z_n\}_{n=1}^{\infty} \subset F$ such that $z := \lim_{n\to\infty} z_n \notin F$. Moreover, although every subsequence of $\{z_n\}_{n=1}^{\infty}$ is convergent, they all still converge to $z \notin F$. This shows that a compact set must be closed.

Lemma A.4 (Bolzano–Weierstrass property for $\mathbb{C}^D$). *Let $D \in \mathbb{N}$. Every bounded sequence, $\{z(n)\}_{n=1}^{\infty} \subset \mathbb{C}^D$, has a convergent subsequence.*

Proof. By assumption there exists $M < \infty$ such that $\|z(n)\| = d(z(n), 0) \leq M$ for all $n \in \mathbb{N}$. Writing $z(n) = (z_1(n), \dots, z_D(n)) \in \mathbb{C}^D$. Since $|z_i(n)| \leq \|z(n)\|$ it follows that $\{z_i(n)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{C}$. Hence by the Bolzano–Weierstrass property for $\mathbb{C}$ we replace $z(n)$ by a subsequence $z(n_k)$ such that $\lim_{k\to\infty} z(n_k) = z_1$ exists. We may now replace the original z by this new subsequence and then find a further subsequence $z(n_k)$ such that $\lim_{k\to\infty} z_i(n_k) = z_i$ exists for $i = 1, 2$. We may continue this way inductively to find a subsequence such that $\lim_{k\to\infty} z_i(n_k) = z_i$ exists for all $1 \leq i \leq D$. It then follows that $\lim_{k\to\infty} \|z - z(n_k)\| = 0$ as desires where $z := (z_1, \dots, z_D)$. ■

Proof of Theorem 10.35. In light of Examples A.2 and A.3 we are left to show that closed and bounded sets are sequentially compact. So let $K \subset \mathbb{C}^D$ be a closed and bounded set and $\{z_n\}_{n=1}^{\infty}$ be any sequence in K. According toe Lemma A.4, $\{z_n\}_{n=1}^{\infty}$ has a convergent subsequence, $\{w_k := z_{n_k}\}_{k=1}^{\infty}$. Since $w_k \in K$ for all k and K is closed it necessarily follows that $\lim_{k\to\infty} w_k \in K$ which shows K is sequentially compact. ■

Example A.5 (Warning!). It is **not** true that a closed and bounded subset of an arbitrary metric space (X, d) is necessarily sequentially compact. For example let Z denote the vector space of continuous functions on $[0,1]$ with values in $\mathbb{R}$ and for $f \in Z$ let $\|f\| = \sup_{t\in[0,1]} |f(t)|$. Then the set $C := \{f_n\}_{n=0}^{\infty}$ where

$$f_n(t) = 2^{(n+2)} \begin{cases} 0 & \text{if } t \in [0, 2^{-(n+1)}] \cup [2^{-n}, 1] \\ t - 2^{-(n+1)} & \text{if } 2^{-(n+1)} \leq t \leq 3 \cdot 2^{-(n+2)} \\ 2^{-n} - t & \text{if } \ 3 \cdot 2^{-(n+2)} \leq t \leq 2^{-n}. \end{cases}$$

[So $f_n(t)$ is a shark tooth over the interval $\left[2^{-(n+1)}, 2^{-n}\right]$.] Notice that $\|f_n\| = 1$ for all n so that C is bounded. Moreover $\|f_n - f_m\| = 1$ for all $m \neq n$, therefore there are no convergent subsequence of C. The reader should use this fact to see that C is closed and bounded but not sequentially compact!

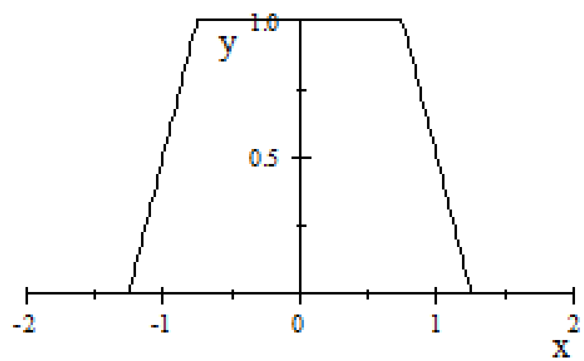


Fig. A.1. Here are the plots of f_0 and f_1.

A.2 Open Cover Compactness

Example A.6. Suppose that A is an unbounded subset of X. Pick $a_1 \in A$ and then choose $\{a_n\}_{n=2}^{\infty}$ inductively so that $d_{\{a_1,\dots,a_n\}}(a_{n+1}) \geq 1$ for all n. This sequence then has the property that $d(a_k, a_l) \geq 1$ for all $k \neq l$ and from this it follows that $F := \{a_1, a_2, \dots\}$ is a closed set. We then define an open cover of A by taking,

$$\mathcal{U} = \{F^c, B_{a_1}(1/3), B_{a_2}(1/3), B_{a_3}(1/3), \dots\}.$$

This cover has no finite subcover. Therefore A can not be open cover compact.

Alternatively. Given $x \in X$, the collection $\mathcal{U} := \{B_x(n) : n \in \mathbb{N}\}$ is an open cover of X. So if K is an open cover compact subset of X, there must exist $n_1 < n_2 < \dots < n_l$ so that

$$K \subset B_x(n_1) \cup B_x(n_2) \cup \dots \cup B_x(n_l) = B_x(n_l).$$

This shows K is bounded.

Lemma A.7. *Suppose that $K \subset X$ is an open cover compact set, then K is closed.*

Proof. We will shows that K^c is open. To this end suppose $x \in K^c$. Then let $\varepsilon_k := \frac{1}{3} d(x, k) > 0$ for all $k \in K$. It then follows that $B_x(\varepsilon_k) \cap B_k(\varepsilon_k) = \emptyset$ for all $k \in K$. As $\{B_k(\varepsilon_k)\}_{k \in K}$ is an open cover K, there exists $\Lambda \subset_f K$ such that $K \subset \cup_{k \in \Lambda} B_k(\varepsilon_k)$. If we now let $\delta := \min_{k \in \Lambda} \varepsilon_k > 0$, then

$$B_x(\delta) \cap B_k(\varepsilon_k) \subset B_x(\varepsilon_k) \cap B_k(\varepsilon_k) = \emptyset \text{ for all } k \in \Lambda$$

and therefore

$$B_x(\delta) \cap K \subset B_x(\delta) \cap [\cup_{k \in K} B_k(\varepsilon_k)] = \emptyset.$$

■

Proposition A.8 (Optional). *Suppose that $K \subset X$ is an open cover compact set and $F \subset K$ is a closed subset. Then F is open cover compact. If $\{K_i\}_{i=1}^n$ is a finite collections of open cover compact subsets of X then $K = \cup_{i=1}^n K_i$ is also an open cover compact subset of X.*

Proof. Let $\mathcal{U} \subset \tau$ be an open cover of F, then $\mathcal{U} \cup \{F^c\}$ is an open cover of K. The cover $\mathcal{U} \cup \{F^c\}$ of K has a finite subcover which we denote by $\mathcal{U}_0 \cup \{F^c\}$ where $\mathcal{U}_0 \subset_f \mathcal{U}$. Since $F \cap F^c = \emptyset$, it follows that $\mathcal{U}_0$ is the desired subcover of F. For the second assertion suppose $\mathcal{U} \subset \tau$ is an open cover of K. Then $\mathcal{U}$ covers each compact set K_i and therefore there exists a finite subset $\mathcal{U}_i \subset_f \mathcal{U}$ for each i such that $K_i \subset \cup \mathcal{U}_i$. Then $\mathcal{U}_0 := \cup_{i=1}^n \mathcal{U}_i$ is a finite cover of K. ■

Exercise A.1. Suppose $f : X \to Y$ is continuous and $K \subset X$ is open cover compact, then $f(K)$ is an open cover compact subset of Y. Give an example of continuous map, $f : X \to Y$, and an open cover compact subset K of Y such that $f^{-1}(K)$ is not open cover compact.

Exercise A.2 (Extreme value theorem III). Let (X, d) be an open cover compact metric space and $f : X \to \mathbb{R}$ be a continuous function. Show $-\infty < \inf f \leq \sup f < \infty$ and there exists $a, b \in X$ such that $f(a) = \inf f$ and $f(b) = \sup f$. **Hint:** use Exercise A.1 and Theorem **??**.

Exercise A.3 (Uniform Continuity). Let (X, d) be an open cover compact metric space, (Y, ρ) be a metric space and $f : X \to Y$ be a continuous function. Show that f is uniformly continuous, i.e. if $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f(y), f(x)) < \varepsilon$ if $x, y \in X$ with $d(x, y) < \delta$.

Exercise A.4. Suppose $f : X \to Y$ is continuous and $K \subset X$ is open cover compact, then $f(K)$ is an open cover compact subset of Y. Give an example of continuous map, $f : X \to Y$, and an open cover compact subset K of Y such that $f^{-1}(K)$ is not open cover compact.

Exercise A.5 (Dini's Theorem). Let X be an open cover compact metric space and $f_n : X \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \downarrow 0$ as $n \to \infty$ for each $x \in X$. Show that in fact $f_n \downarrow 0$ uniformly in x, i.e. $\sup_{x \in X} f_n(x) \downarrow 0$ as $n \to \infty$. **Hint:** Given $\varepsilon > 0$, consider the open sets $V_n := \{x \in X : f_n(x) < \varepsilon\}$.

Definition A.9. *A collection $\mathcal{F}$ of closed subsets of a metric space (X, d) has the **finite intersection property** if $\cap \mathcal{F}_0 \neq \emptyset$ for all $\mathcal{F}_0 \subset_f \mathcal{F}$.*

The notion of open cover compactness may be expressed in terms of closed sets as follows.

Proposition A.10. *A metric space X is open cover compact iff every family of closed sets $\mathcal{F} \subset 2^X$ having the **finite intersection property** satisfies $\bigcap \mathcal{F} \neq \emptyset$.*

Proof. The basic point here is that complementation interchanges open and closed sets and open covers go over to collections of closed sets with empty intersection. Here are the details.

($\Rightarrow$) Suppose that X is open cover compact and $\mathcal{F} \subset 2^X$ is a collection of closed sets such that $\bigcap \mathcal{F} = \emptyset$. Let

$$\mathcal{U} = \mathcal{F}^c := \{C^c : C \in \mathcal{F}\} \subset \tau,$$

then $\mathcal{U}$ is a cover of X and hence has a finite subcover, $\mathcal{U}_0$. Let $\mathcal{F}_0 = \mathcal{U}_0^c \subset_f \mathcal{F}$, then $\cap \mathcal{F}_0 = \emptyset$ so that $\mathcal{F}$ does not have the finite intersection property.

($\Leftarrow$) If X is not open cover compact, there exists an open cover $\mathcal{U}$ of X with no finite subcover. Let

$$\mathcal{F} = \mathcal{U}^c := \{U^c : U \in \mathcal{U}\},$$

then $\mathcal{F}$ is a collection of closed sets with the finite intersection property while $\bigcap \mathcal{F} = \emptyset$. ■

A.3 Equivalence of Sequential and Open Cover Compactness in Metric Spaces

Definition A.11. *A metric space (X, d) is ε − **bounded** $(\varepsilon > 0)$ if there exists a finite cover of X by balls of radius ε. We further say (X, d) is **totally bounded** if it is ε − bounded for all $\varepsilon > 0$.*

Remark A.12 (Totally bounded means almost finite). An equivalent way to state that (X, d) is ε − bounded is to say there exists a finite subset $\Lambda = \Lambda_\varepsilon \subset_f X$ such that $d_\Lambda(x) < \varepsilon$ for all $x \in X$. In other words, (X, d) is ε − bounded iff X is a finite subset within an ε − error.

Theorem A.13. *Let (X, d) be a metric space. The following are equivalent.*

(a) X is open cover compact.
(b) X is sequentially compact.
(c) X is totally bounded and complete.

Proof. The proof will consist of showing that $a \Rightarrow b \Rightarrow c \Rightarrow a$.

$(a \Rightarrow b)$ We will show that **not** $b \Rightarrow$ **not** a. Suppose there exists $\{x_n\}_{n=1}^\infty \subset X$ which has no convergent subsequence. In this case the set $S := \{x_n \in X : n \in \mathbb{N}\}$ must be an infinite set as we have already seen finite sets are sequentially compact. For every $x \in X$ we must have

$$\varepsilon(x) := \liminf_{n\to\infty} d(x_n, x) > 0$$

since otherwise there would be a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $\lim_{k\to\infty} d(x_{n_k}, x) = 0$, i.e. $\lim_{k\to\infty} x_{n_k} = x$. We now let $V_x := B_x\left(\frac{1}{2}\varepsilon(x)\right)$ and observe that x_n can be in V_x for only finitely many n – otherwise we would conclude that $\liminf_{n\to\infty} d(x_n, x) \le \varepsilon(x)/2$. From these observations, $\mathcal{U} := \{V_x : x \in X\}$ is an open cover of X with no finite subcover. Indeed, if $\Lambda \subset_f X$, then we must still have $x_n \in \cup_{x\in\Lambda} V_x$ for only finitely many n and in particular $\cup_{x\in\Lambda} V_x$ can not cover S.

$(b \Rightarrow c)$ Suppose $\{x_n\}_{n=1}^\infty \subset X$ is a Cauchy sequence. By assumption there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ which is convergent to some point $x \in X$. Since $\{x_n\}_{n=1}^\infty$ is Cauchy it follows that $x_n \to x$ as $n \to \infty$ showing X is complete.

Now for sake of contradiction suppose that X is not totally bounded. There there exists $\varepsilon > 0$ for which X is not ε − bounded. In particular, $\mathcal{U} := \{B_x(\varepsilon) : x \in X\}$ is an open cover of X with no finite subcover. We now use this to construct a sequence $\{x_n\}_{n=1}^\infty \subset X$. Choose $x_1 \in X$ at random, then choose $x_2 \in X \setminus B_{x_1}(\varepsilon)$, then $x_3 \in X \setminus [B_{x_1}(\varepsilon) \cup B_{x_2}(\varepsilon)]$ ...

$$x_n \in X \setminus \left[B_{x_1}(\varepsilon) \cup B_{x_2}(\varepsilon) \cup \cdots \cup B_{x_{n-1}}(\varepsilon)\right], \ldots .$$

The process may be continued indefinitely as $\mathcal{U}$ has no finite subcover. By construction we have chosen $\{x_n\}_{n=1}^\infty$ such that $d_{\{x_1,\ldots,x_{n-1}\}}(x_n) \ge \varepsilon$ for all n and therefore $d(x_k, x_l) \ge \varepsilon$ for all $k \ne l$. Every subsequence will share this property, i.e. not be Cauchy, and hence can not be convergent.

$(c \Rightarrow a)$ For sake of contradiction, assume there exists an open cover $\mathcal{V} = \{V_\alpha\}_{\alpha\in A}$ of X with no finite subcover. Since X is totally bounded for each $n \in \mathbb{N}$ there exists $\Lambda_n \subset_f X$ such that

$$X = \bigcup_{x\in\Lambda_n} B_x(1/n) \subset \bigcup_{x\in\Lambda_n} C_x(1/n).$$

Choose $x_1 \in \Lambda_1$ such that no finite subset of $\mathcal{V}$ covers $K_1 := C_{x_1}(1)$. Since $K_1 = \cup_{x\in\Lambda_2} K_1 \cap C_x(1/2)$, there exists $x_2 \in \Lambda_2$ such that $K_2 := K_1 \cap C_{x_2}(1/2)$ can not be covered by a finite subset of $\mathcal{V}$, see Figure A.2. Continuing this way inductively, we construct sets $K_n = K_{n-1} \cap C_{x_n}(1/n)$ with $x_n \in \Lambda_n$ such that no K_n can be covered by a finite subset of $\mathcal{V}$. Now choose $y_n \in K_n$ for each n. Since $\{K_n\}_{n=1}^\infty$ is a decreasing sequence of closed sets such that $\operatorname{diam}(K_n) \le 2/n$, it follows that $\{y_n\}$ is a Cauchy and hence convergent with

$$y = \lim_{n\to\infty} y_n \in \cap_{m=1}^\infty K_m.$$

Since $\mathcal{V}$ is a cover of X, there exists $V \in \mathcal{V}$ such that $y \in V$. Since $K_n \downarrow \{y\}$ and $\operatorname{diam}(K_n) \to 0$, it now follows that $K_n \subset V$ for some n large. But this violates the assertion that K_n can not be covered by a finite subset of $\mathcal{V}$.

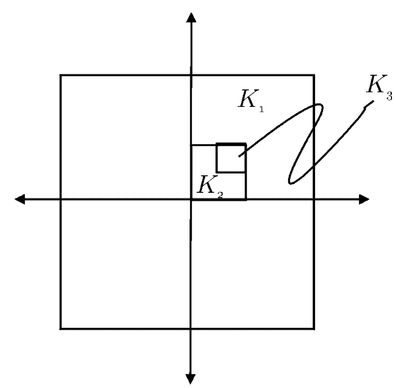


Fig. A.2. Nested Sequence of cubes.

■

Corollary A.14. *The compact subsets of $\mathbb{R}^n$ are the closed and bounded sets.*

Proof. If K is closed and bounded then K is complete (being the closed subset of a complete space) and K is contained in $[-M,M]^n$ for some positive integer M. For $\delta>0$, let

$$\Lambda_\delta = \delta\mathbb{Z}^n \cap [-M,M]^n := \{\delta x : x\in\mathbb{Z}^n \text{ and } \delta|x_i|\le M \text{ for } i=1,2,\dots,n\}.$$

We will show, by choosing $\delta>0$ sufficiently small, that

$$K\subset[-M,M]^n\subset\cup_{x\in\Lambda_\delta}B(x,\varepsilon) \tag{A.1}$$

which shows that K is totally bounded. Hence by Theorem A.13, K is compact. Suppose that $y\in[-M,M]^n$, then there exists $x\in\Lambda_\delta$ such that $|y_i-x_i|\le\delta$ for $i=1,2,\dots,n$. Hence

$$d^2(x,y)=\sum_{i=1}^n (y_i-x_i)^2\le n\delta^2$$

which shows that $d(x,y)\le\sqrt{n}\delta$. Hence if choose $\delta<\varepsilon/\sqrt{n}$ we have shows that $d(x,y)<\varepsilon$, i.e. Eq. (A.1) holds. ■

A.4 Connectedness

Definition A.15. *Let (X,d) be a metric space. Two subset A and B of X are* ***separated***[1] *if $\bar{A}\cap B=\emptyset=A\cap\bar{B}$. A set $E\subset X$ is* ***disconnected*** *if $E=A\cup B$ where A and B are two* ***non-empty*** *separated sets, otherwise E is said to be* ***connected****.*

Theorem A.16 (The Connected Subsets of $\mathbb{R}$). *The connected subsets of $\mathbb{R}$ are intervals.*

Proof. We will break the proof into two parts. First we show if E is connected then E is an interval. Then we show if E is disconnected then E is not an interval.

1) Suppose that $E\subset\mathbb{R}$ is a connected subset and that $a,b\in E$ with $a<b$. If there exists $c\in(a,b)$ such that $c\notin E$, then $A:=(-\infty,c)\cap E$ and $B:=(c,\infty)\cap E$ would be two non-empty separated subsets such that $E=A\cup B$. Hence $(a,b)\subset E$. Let $\alpha:=\inf(E)$ and $\beta:=\sup(E)$ and choose $\alpha_n,\beta_n\in E$ such that $\alpha_n<\beta_n$ and $\alpha_n\downarrow\alpha$ and $\beta_n\uparrow\beta$ as $n\to\infty$. By what we have just shown, $(\alpha_n,\beta_n)\subset E$ for all n and hence $(\alpha,\beta)=\cup_{n=1}^\infty(\alpha_n,\beta_n)\subset E$. From this it follows that $E=(\alpha,\beta)$, $[\alpha,\beta)$, $(\alpha,\beta]$ or $[\alpha,\beta]$, i.e. E is an interval.

2) Now suppose that E is a disconnected subset of $\mathbb{R}$. Then $E=A\cup B$ where A and B are non-empty separated sets and let $a\in A$ and $b\in B$. After relabelling A and B if necessary we may assume that $a<b$. Let $p=\sup([a,b]\cap A)$. Since $p\in\bar{A}\cap[a,b]$ it follows that $p\notin B$ and $a\le p\le b$. Since $b\in B$, $p\ne b$ and hence $a\le p<b$.

i) if $p\notin A$ then $p\notin E$ and $a<p<b$ which shows E is not an interval.

ii) if $p\in A$, then $p\notin\bar{B}$ and there exists[2] p_1 such that $a\le p<p_1<b$ and $p_1\notin\bar{B}\supset B$. Again $p_1\notin A$ (for otherwise $p\ge p_1$) and so $p_1\notin E$ and hence again E is not an interval.

■

Lemma A.17. *Suppose that $f:X\to Y$ is a continuous function between two metric spaces (X and Y) and A and B are separated subsets of Y. Then $\alpha:=f^{-1}(A)$ and $\beta:=f^{-1}(B)$ are separated subsets of X. In particular, if $E\subset X$ is connected, then $f(E)$ is connected in Y.*

Proof. Since $\alpha:=f^{-1}(A)\subset f^{-1}(\bar{A})$ and $f^{-1}(\bar{A})$ is the closed being the inverse image under a continuous function of the closed set $\bar{A}$, it follows that $\bar{\alpha}\subset f^{-1}(\bar{A})$. Thus if $x\in\bar{\alpha}\cap\beta$, then $f(x)\in\bar{A}\cap B=\emptyset$ and hence $\bar{\alpha}\cap\beta=\emptyset$. Similarly one shows $\alpha\cap\bar{\beta}=\emptyset$ as well.

If $f(E)$ disconnected, there exists non-empty separated subsets, A and B of $f(E)$, so that $f(X)=A\cup B$. The sets $\alpha:=f^{-1}(A)$ and $\beta:=f^{-1}(B)$ are now non-empty separated subsets of X. It now follows that $\alpha_0:=\alpha\cap E$ and $\beta_0:=\beta\cap E$ are non-empty sets such that

$$\bar{\alpha}_0\cap\beta_0\subset\bar{\alpha}\cap\beta=\emptyset \text{ and } \alpha_0\cap\bar{\beta}_0\subset\alpha\cap\bar{\beta}=\emptyset$$

which would imply E is disconnected. Thus if E is connected we must have that $f(E)$ is connected. ■

Theorem A.18 (Intermediate Value Theorem). *Suppose that (X,d) is a connected metric space and $f:X\to\mathbb{R}$ is a continuous map. Then f satisfies the intermediate value property. Namely, for every pair $x,y\in X$ such that $f(x)<f(y)$ and $c\in(f(x),f(y))$, there exists $z\in X$ such that $f(z)=c$.*

Proof. By Lemma A.17, $f(X)$ is a connected subset of $\mathbb{R}$. So by Theorem A.16, $f(X)$ is a subinterval of $\mathbb{R}$ and this completes the proof. ■

Lemma A.19. *If $E\subset X$ is a connected set and $E\subset A\cup B$ where A and B are two separated sets, then $E\subset A$ or $E\subset B$.*

Proof. Let $\alpha:=E\cap A$ and $\beta:=E\cap B$, then $\bar{\alpha}\cap\beta\subset\bar{A}\cap B=\emptyset$ and similarly $\alpha\cap\bar{\beta}=\emptyset$. Thus $E=\alpha\cup\beta$ with α and β being separated sets. Since E is connected we must have $\alpha=\emptyset$ or $\beta=\emptyset$, i.e. $E\subset B$ or $E\subset A$. ■

[1] Notice that separated sets are disjoint. The sets $A=(0,1)$ and $B=[1,\infty)$ are disjoint but not separated.

[2] This is because $\bar{B}^c$ is an open subset of $\mathbb{R}$.

Proposition A.20. *Suppose that F and G are connected subsets of X such that $\bar{F} \cap G \neq \emptyset$, then $E = F \cup G$ is connected in X as well.*

Proof. Suppose that $E = A \cup B$ where A and B are separated sets. Then from Lemma A.19 we know that $F \subset A$ or $F \subset B$ and similarly $G \subset A$ or $G \subset B$. If both F and G are in the same set (say A), then $E \subset A \subset E$ and B must be empty. On the other hand if $F \subset A$ and $G \subset B$, then $\emptyset \neq \bar{F} \cap G \subset \bar{A} \cap B$ which would violate A and B being separated. Thus we have shown there does not exists two non-empty separated sets A and B such that $E = A \cup B$, i.e. E is connected. ∎

Definition A.21. *A subset E of a metric space X is **path connected** if to every pair of points $\{x_0, x_1\} \subset E$ there exists $\sigma \in C([0,1], E)$, such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. We refer to σ as a path joining x_0 to x_1.*

Proposition A.22. *Every path connected subset, E, of a metric space X is connected.*

Exercise A.6. Prove Proposition A.22, i.e. if $E \subset X$ is path connected then E is connected.. **Hint:** for sake of contradiction suppose that A and B are two non-empty separated subsets of X such that $E = A \cup B$ and choose a path connecting a point in A to a point in B.

Definition A.23. *A subset, C, of a vector space, X, is **convex** if for all $a, b \in C$ the path,*

$$\sigma(t) = a + t(b - a) = (1 - t)a + tb \text{ for } 0 \leq t \leq 1$$

is contained in C.

Example A.24. Every convex subset, C, of a normed vector space $(X, \|\cdot\|)$ is path connected and hence connected.

Exercise A.7. Suppose that $(X, \|\cdot\|)$ is a normed space, $x \in X$, and $R > 0$. Show the open and closed balls in X, $B_x(R)$ and $C_x(R)$, are both convex sets and hence path connected.

Definition A.25. *A metric space X is **locally path connected** if for each $x \in X$, there is an open neighborhood $V \subset X$ of x which is path connected.*

Proposition A.26. *Let X be a metric space.*

1. *If X is connected and locally path connected, then X is path connected.*
2. *If X is any connected open subset of $\mathbb{R}^n$, then X is path connected.*

Exercise A.8. Prove item 1. of Proposition A.26, i.e. if X is connected and locally path connected, then X is path connected. **Hint:** fix $x_0 \in X$ and let W denote the set of $x \in X$ such that there exists $\sigma \in C([0,1], X)$ satisfying $\sigma(0) = x_0$ and $\sigma(1) = x$. Then show W is both open and closed.

Exercise A.9. Prove item 2. of Proposition A.26, i.e. if X is any connected open subset of $\mathbb{R}^n$, then X is path connected.

A.4.1 Connectedness Problems

Exercise A.10. In this exercise we will work inside the metric space $\mathbb{Q}$ with $d(x,y) := |x - y|$ for all $x, y \in \mathbb{Q}$. Let $a, b \in \mathbb{Q}$ with $a < b$ and let

$$J := [a, b] \cap \mathbb{Q} = \{x \in \mathbb{Q} : a \leq x \leq b\}.$$

Show J is disconnected in $\mathbb{Q}$.

Exercise A.11. Suppose $a < b$ and $f : (a, b) \to \mathbb{R}$ is a non-decreasing function. Show if f satisfies the intermediate value property (see Theorem A.18), then f is continuous.

Exercise A.12. Suppose $-\infty < a < b \leq \infty$ and $f : [a, b) \to \mathbb{R}$ is a strictly increasing continuous function. Using the intermediate value theorem, one sees that $f([a, b))$ is an interval and since f is strictly increasing it must of the form $[c, d)$ for some $c \in \mathbb{R}$ and $d \in \bar{\mathbb{R}}$ with $c < d$. Show the inverse function $f^{-1} : [c, d) \to [a, b)$ is continuous and is strictly increasing. In particular if $n \in \mathbb{N}$, apply this result to $f(x) = x^n$ for $x \in [0, \infty)$ to construct the positive n^{th} – root of a real number.

Exercise A.13. Let

$$X := \left\{(x, y) \in \mathbb{R}^2 : y = \sin(x^{-1}) \text{ with } x \neq 0\right\} \cup \{(0, 0)\}$$

equipped with the relative topology induced from the standard topology on $\mathbb{R}^2$. Show X is connected but not path connected.

Remark A.27 (Structure of open sets in $\mathbb{R}$). Let $V \subset \mathbb{R}$ be an open set. For $x \in V$, let $a_x := \inf\{a : (a, x] \subset V\}$ and $b_x := \sup\{b : [x, b) \subset V\}$. Since V is open, $a_x < x < b_x$ and it is easily seen that $J_x := (a_x, b_x) \subset V$. Moreover if $y \in V$ and $J_x \cap J_y \neq \emptyset$, then $J_x = J_y$. The collection, $\{J_x : x \in V\}$, is at most countable since we may label each $J \in \{J_x : x \in V\}$ by choosing a rational number $r \in J$. Letting $\{J_n : n < N\}$, with $N = \infty$ allowed, be an enumeration of $\{J_x : x \in V\}$, we have $V = \coprod_{n<N} J_n$ as desired.

A.5 *More on the closure and related operations

This section is optional reading.

Definition A.28 (Closure). *Given a set A contained in a metric space X, let $\bar{A} \subset X$ be the **closure of** A defined by*

$$\bar{A} := \{x \in X : \exists \{x_n\} \subset A \ni x = \lim_{n\to\infty} x_n\}.$$

*That is to say $\bar{A}$ contains all **limit points** of A.*

Lemma A.29 (Optional). *For any* $A \subset X$, *then*

1. $A = \bar{A}$ *if* A *is closed.*
2. $\bar{A} = \{x : d_A(x) = 0\}$ *and* $\bar{A}$ *is closed.*
3. $\bar{A} = \{x \in X : A \cap B_x(r) \neq \emptyset \ \forall \ r > 0\}$.
4. $d_A(x) > 0$ *for all* $x \in A^c$ *if* A *is closed.*

Proof. 1. We always have $A \subset \bar{A}$. If A is closed we can not leave A by taking limits and hence $\bar{A} \subset A$, i.e. $A = \bar{A}$ if A is closed.

2. Let $F := \{x : d_A(x) = 0\}$ which is a closed set since d_A is continuous. If $x \in F$ (i.e. $d_A(x) = 0$), there exists $x_n \in A$ such that $d(x, x_n) \leq 1/n$ for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} x_n = x$ and so $x \in \bar{A}$. This shows $F \subset \bar{A}$. Conversely if $x \in \bar{A}$, there exists $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = x$ and so

$$d_A(x) = d_A\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} d_A(x_n) = \lim_{n\to\infty} 0 = 0$$

which shows $x \in F$.

2. $\iff$ 3. Since $A \cap B_x(r) \neq \emptyset$ happens iff $d_A(x) < r$ we see that $A \cap B_x(r) \neq \emptyset \ \forall \ r > 0$ iff $d_A(x) < r$ for all $r > 0$, i.e. iff $d_A(x) = 0$.

4. If A is closed then

$$A = \bar{A} = \{x \in X : d_A(x) = 0\}$$

and therefore $A^c = \{x \in X : d_A(x) > 0\}$. ■

Proposition A.30. *If* $A \subset X$, *then*

$$\bar{A} = \cap\{F : A \subset F \subset X \text{ with } F \text{ closed}\}. \tag{A.2}$$

That is to say $\bar{A}$ *is the smallest closed set containing* A *and in particular* $\bar{A}$ *is closed.*

Proof. **$\bar{A}$ is closed.** By Lemma A.29 to see that $\bar{A} := \{x \in X : d_A(x) = 0\}$ and hence $\bar{A}$ is closed because d_A is continuous. **Alternatively** without using Lemma A.29**,** suppose $\{x_n\} \subset \bar{A}$ is a convergent sequence and $x = \lim_{n\to\infty} x_n \in X$. By definition of $\bar{A}$, there exists $y_n \in A$ such that $d(x_n, y_n) \leq n^{-1}$ for all n. Therefore by the triangle inequality,

$$d(x, y_n) \leq d(x, x_n) + d(x_n, y_n) \leq d(x, x_n) + \frac{1}{n} \to 0 \text{ as } n \to \infty$$

which shows $x = \lim_{n\to\infty} y_n \in \bar{A}$ and hence $\bar{A}$ is closed.

Equation (A.2) holds. If F is any closed set containing A and $\{x_n\} \subset A$ is a sequence converging to a point $x \in \bar{A}$, it follow that $x \in F$ because F is closed. Thus $\bar{A} \subset F$ and hence Eq. (A.2) holds as $\bar{A}$ is closed. ■

Exercise A.14. Suppose that A and B are subsets of a metric space, show $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Exercise A.15. Given an example showing that $\overline{\cup_{n=1}^\infty A_n}$ need not be equal to $\cup_{n=1}^\infty \overline{A_n}$.

Exercise A.16. If A is a non-empty subset of X, then $d_A = d_{\bar{A}}$.

Definition A.31 (Boundary of a set). *The* ***boundary*** *of* A *is the set* $\mathrm{bd}(A) = \bar{A} \cap \overline{A^c}$.

Proposition A.32. *If* A *is a subset of a metric space,* (X, d), *then* $\mathrm{bd}(A)$ *may be computed using either;*

1. $\mathrm{bd}(A) = \bar{A} \cap \overline{A^c}$,
2. $\mathrm{bd}(A) = \{x \in X : d_A(x) = 0 = d_{A^c}(x)\}$,
3. $\mathrm{bd}(A) = \{x \in X : B_x(r) \cap A \neq \emptyset \neq B_x(r) \cap A^c \text{ for all } r > 0\}$, *or*
4. $\mathrm{bd}(A) = \{x \in X : \exists \{x_n\} \subset A \text{ and } \{y_n\} \subset A^c \ni \lim_{n\to\infty} x_n = x = \lim_{n\to\infty} y_n\}$.[3]

Proof. Item 1. is the definition of $\mathrm{bd}(A)$. Item 2. now follow from item 1. and the fact that $\bar{A} = \{x \in X : d_A(x) = 0\}$. I leave it to the reader to check that 2. $\implies$ 3. $\implies$ 4. $\implies$ 1. ■

Definition A.33 (Dense / Separable). *We say* A *is* ***dense in*** X *if* $\bar{A} = X$, *i.e. every element* $x \in X$ *is a limit of a sequence of elements from* A. *A metric space is said to be* ***separable*** *if it contains a countable dense subset,* D.

Definition A.34. *Let* (X, d) *be a metric space and* A *be a subset of* X.

1. *The* ***closure*** *of* A *is the smallest closed set* $\bar{A}$ *containing* A, *i.e.*
$$\bar{A} := \cap \{F : A \subset F \sqsubset X\}.$$
(Because of Proposition A.30 this is consistent with Definition A.28 for the closure of a set in a metric space.)
2. *The* ***interior*** *of* A *is the largest open set* A^o *contained in* A, *i.e.*
$$A^o = \cup \{V \in \tau : V \subset A\}.$$
3. $A \subset X$ *is a* ***neighborhood of a point*** $x \in X$ *if* $x \in A^o$.
4. *The* ***accumulation points*** *of* A *is the set*
$$\mathrm{acc}(A) = \{x \in X : V \cap [A \setminus \{x\}] \neq \emptyset \text{ for all } V \in \tau_x\}.$$

[3] So the boundary points of A are those points in x which are “on the boundary of A.”

5. *The **boundary** of A is the set* $\mathrm{bd}(A) := \bar{A} \setminus A^o$.
6. *A is **dense** in X if $\bar{A} = X$ and X is said to be **separable** if there exists a countable dense subset of X.*

Lemma A.35. *Let (X,d) be a metric space and A be a subset of X, then*

$$A^o = \{x \in X : B_x(r_x) \subset A \text{ for some } r_x > 0\}.$$

Proof. Let $V := \{x \in X : B_x(r) \subset A \text{ for some } r = r_x > 0\}$. If $B_x(r) \subset A$ and $y \in B_x(r)$ and $\delta = r - d(x,y)$, then $B_y(\delta) \subset B_x(r) \subset A$ which shows that $y \in V$, i.e. $B_x(r) \subset V$. Thus we may write

$$V = \cup\{B_x(r) : B_x(r) \subset A\}.$$

This shows that V is an open subset of A. Moreover if W is another open subset of A, then

$$W = \cup\{B_x(r) : B_x(r) \subset W\} \subset \cup\{B_x(r) : B_x(r) \subset A\} = V$$

so that V is the largest open subset contained in A. This completes the proof. ■

Remark A.36. The relationships between the interior and the closure of a set are:

$$(A^o)^c = \bigcap\{V^c : V \in \tau \text{ and } V \subset A\} = \bigcap\{C : C \text{ is closed } C \supset A^c\} = \overline{A^c}$$

and similarly, $(\bar{A})^c = (A^c)^o$.

Definition A.37. *A subset $A \subset X$ is a **neighborhood** of x if there exists an open set $V \subset_o X$ such that $x \in V \subset A$. We will say that $A \subset X$ is an **open neighborhood** of x if A is open and $x \in A$.*

Example A.38. Let $x \in X$ and $\delta > 0$, then $C_x(\delta)$ and $B_x(\delta)^c$ are closed subsets of X. For example if $\{y_n\}_{n=1}^\infty \subset C_x(\delta)$ and $y_n \to y \in X$, then $d(y_n, x) \le \delta$ for all n and using Corollary 10.11 it follows $d(y,x) \le \delta$, i.e. $y \in C_x(\delta)$. A similar proof shows $B_x(\delta)^c$ is closed, see Exercise **??**.

Exercise A.17 (Completeness). Let (X,d) be a complete metric space. Let $A \subset X$ be a subset of X viewed as a metric space using $d|_{A\times A}$. Show that $(A, d|_{A\times A})$ is complete iff A is a closed subset of X.

Exercise A.18. Let $(X, \|\cdot\|)$ be a normed space and $d(x,y) := \|y - x\|$. Show;

1. $C_x(r)^o = B_x(r)$,
2. $\overline{B_x(r)} = C_x(r)$,
3. $\mathrm{bd}(C_x(r)) = \mathrm{bd}(C_x(r)) = \{y \in X : \|y - x\| = r\}$.

Example A.39 (Words of Caution). Let (X,d) be a metric space. It is always true that $\overline{B_x(\varepsilon)} \subset C_x(\varepsilon)$ since $C_x(\varepsilon)$ is a closed set containing $B_x(\varepsilon)$. However, it is not always true that $\overline{B_x(\varepsilon)} = C_x(\varepsilon)$. For example let $X = \{1,2\}$ and $d(1,2) = 1$, then $B_1(1) = \{1\}$, $\overline{B_1(1)} = \{1\}$ while $C_1(1) = X$. For another counterexample, take

$$X = \left\{(x,y) \in \mathbb{R}^2 : x = 0 \text{ or } x = 1\right\}$$

with the usually Euclidean metric coming from the plane. Then

$$\begin{aligned} B_{(0,0)}(1) &= \left\{(0,y) \in \mathbb{R}^2 : |y| < 1\right\}, \\ \overline{B_{(0,0)}(1)} &= \left\{(0,y) \in \mathbb{R}^2 : |y| \le 1\right\}, \text{ while} \\ C_{(0,0)}(1) &= \overline{B_{(0,0)}(1)} \cup \{(1,0)\}. \end{aligned}$$

Exercise A.19. If D is a dense subset of a metric space (X,d) and $E \subset X$ is a subset such that to every point $x \in D$ there exists $\{x_n\}_{n=1}^\infty \subset E$ with $x = \lim_{n\to\infty} x_n$, then E is also a dense subset of X. If points in E well approximate every point in D and the points in D well approximate the points in X, then the points in E also well approximate all points in X.

Exercise A.20. Suppose (X,d) is a metric space which contains an uncountable subset $\Lambda \subset X$ with the property that there exists $\varepsilon > 0$ such that $d(a,b) \ge \varepsilon$ for all $a, b \in \Lambda$ with $a \ne b$. Show that (X,d) is **not** separable.

Exercise A.21 (Intermediate value theorem). Suppose that $-\infty < a < b < \infty$ and $f : [a,b] \to \mathbb{R}$ is a continuous function such that $f(a) \le f(b)$. Show for any $y \in [f(a), f(b)]$, there exists a $c \in [a,b]$ such that $f(c) = y$.[4] **Hint:** Let $S := \{t \in [a,b] : f(t) \le y\}$ and let $c := \sup(S)$.

Exercise A.22 (Inverse Function Theorem I). Let $f : [a,b] \to [c,d]$ be a strictly increasing (i.e. $f(x_1) < f(x_2)$ whenever $x_1 < x_2$) continuous function such that $f(a) = c$ and $f(b) = d$. Then f is bijective and the inverse function, $g := f^{-1} : [c,d] \to [a,b]$, is strictly increasing and is continuous.

[4] The same result holds for $y \in [f(b), f(a)]$ if $f(b) \le f(a)$ – just replace f by $-f$ in this case.